

# SPIKE DETECTION IN AXONAL-SYNAPTIC CHANNELS WITH MULTIPLE SYNAPSES

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## ABSTRACT

The human nervous systems is a complex network, which consists of neurons interacting with each other through a hybrid electro-chemical communication called neuro-spike communications. A first building-block in network analysis of the neuro-spike nano-network is multiple-input-single-output (MISO) diversity scheme. Thus, in this paper, we consider a MISO transmission in a axonal-synaptic channels. In addition, for the first time, we model axonal noise as a binary X-channel. We derive error probability in a system consisting of axonal and synaptic noises, random channel and random vesicle release in a network with multiple transmitting synapses.

## 1. INTRODUCTION

Neuro-spike communication, also known as neuro-synaptic communication, is a biological communication system in human nervous system including brain and sensory neurons. This nano-scale communication is a hybrid model consisting of molecular communication by releasing chemical messengers called neuro-transmitters in synapses and electric signal transmission via transmission of impulses in axons. In this paper, we analyze the error probability of this biological system and offer insight on its performance.

Based on neuroscience studies (see, e.g., [1, 2]), the performance of the optimum detector is affected by axonal noise. Although what nature does may not be optimum, the real synaptic detector is still adversely affected by axonal and synaptic noises. The existence of axonal noise is more prevail in thinner and longer neurons. Exploiting biophysical theory and stochastic simulations demonstrated that in central neuron system, axons of 0.1-0.5mm diameter, axonal noise is more prevail in action potential propagation [1]. Thus, the variability in postsynaptic responses that outcomes from the axonal channel noise is raised in the longer and thinner axons. In addition, based on realistic data collected from neocortex, authors in [3] showed that single cortical synapses cannot transmit information reliably. Therefore, in central nervous system synapses, multiple synapses should collectively transmit the same information, and in this way, the information capacity of axonal-synaptic channel is significantly improved.

In this paper, we consider a axonal-synaptic channel consisting of multiple transmitting synapses and a single

receiving synapse. We assume that transmitted spike-like signals are impaired by several degradations such as stochastic vesicle release, variable amplitude, and axonal and synaptic noises. Unlike the recent work in [4] that considered a single-input single-output (SISO) scenario, in this work, we assume multiple synapses that collectively transmit the data. In addition, our works differs from [4] in modeling of axonal noise, and here, we model both events of erroneous spikes and removal of spikes. Next, we analyze the performance of the system by deriving analytic expressions for probability of error at the receiving neuron. More specifically, we derive a closed-form formula for the probability density function (PDF) of MISO axonal-synaptic channel. Then, we derive analytical expressions for likelihood function and error probability of an optimum detector at the receiving neurons.

## 2. AXONAL-SYNAPTIC CHANNEL MODEL

We consider a mathematical channel model for the axonal-synaptic communication nervous system, which is located between the transmitting neuron and receiving neuron in central neural synapses. The generated spike signal is denoted by  $x(t)$ , which consist of random wide-band narrow spikes. Due to refractory effect, similar to [3–5], the spike train can be modeled as binary communication. The binary random variable  $S$  is defined, in which  $S = 0$  represents the event that there is no spike in the given time, and  $S = 1$  otherwise. The signal  $x(t)$  with spike-like pulses is generated by encoding the binary random process.

Similar to [4] and [6], we model the axonal noise as a binary random process. It is assumed that at a given time, one erroneous spike occurs with probability of  $p_a$ . Furthermore, here, we generalize the modeling of the axonal noise to consider the removal of a spike with probability of  $p_b$ , as well as the occurrence of an erroneous spike with probability  $p_a$ . That is, we consider the case that a spike disappears with probability of  $p_b$  due to negative voltage caused by random closing of ion channels. Thus, the axonal noise in this paper is modeled as X-channel model.

The output of axonal noise block at a given time can be denoted by a binary random variable  $U$ . In addition, the contaminated spike train with axonal noise is called  $u(t)$ . Due to refractory effect, similar to [3–5], the spike train  $x(t)$  can be divided into bins of size  $T_f$ . The event  $U = 1$  means that

there is one spike in a bin with probability of

$$\begin{aligned} \mathbf{P}\{U = 1\} &= \mathbf{P}\{S = 1\}(1 - p_b) + p_a \mathbf{P}\{S = 0\} \\ &= 1 - p_b - p_{\text{prior}} + p_b p_{\text{prior}} + p_a p_{\text{prior}}. \end{aligned} \quad (1)$$

where  $p_{\text{prior}} = \mathbf{P}\{S = 0\}$  is the prior probability of the input spike.

### 3. NEURO-SPIKE SYSTEM WITH MULTIPLE SYNAPSES

Next, we consider multiple synapses at the transmitter side. When presynaptic signals are related to the same information source, they are summing up constructively, and it is called cooperative synaptic communications. Thus, we consider the case that we have multiple synapses cooperatively transmitting their data toward a single postsynaptic terminal. Note that this is a practical assumption since the value of the vesicle release probability  $p_v$  is low in practice. Hence, to transfer the information reliably, the cooperative transmission by neurons is inevitable.

To get the spatial diversity gain due to the cooperative multiple synaptic case, we calculate the optimal decision strategy. The postsynaptic membrane voltage of the cooperative synaptic communication when a spike is transmitted, i.e.,  $S = 1$ , becomes

$$y_{M|S=1}(t) = \sum_{m=1}^M U_m^b V_m h_m w(t) + e(t), \quad (2)$$

where  $M$  is the number of independent parallel synapses,  $V_m$  is a Bernoulli random variable representing the vesicle release process at the  $m$ -th synapse with the probability of  $p_v$  for  $V_m = 1$ , and  $e(t)$  is additive white Gaussian noise at the receiving neuron. Moreover,  $U_m^b$  is a Bernoulli random variable representing the conditional axonal noise where  $U_m^b = 0$  is the event that one spike has disappeared at the  $m$ -th synapse with probability of  $p_b$ . Note that we assumed that the same type of receptors is used for all synaptic connections. Hence, the excitatory postsynaptic potential (EPSP) pulse-shape  $w(t)$  is the same for all summation terms in (2) and is represented by [3]

$$w(t) = \frac{w_{\max} t}{T_{\max}} \exp(1 - t/T_{\max}), \quad (3)$$

for  $t > 0$ , where  $w_{\max}$  and  $T_{\max}$  represent the maximum value of the pulse shape and its corresponding time, respectively. In addition, the random variable  $h_m$  in (2) is the EPSP variable quantal amplitude of the  $m$ -th synaptic connection. It is shown in [7] that the amplitude distribution is optimally fitted with a  $k$ th-order Gamma distribution.

**Proposition 1.** *Considering a set of independent  $k$ -th order Gamma random variables  $\mathcal{H} = \{h_1, \dots, h_M\}$  with mean*

*and variance of  $\lambda$  and  $\sigma^2$ , respectively, and a set of independent Bernoulli random variables  $\mathcal{V} = \{V_1, \dots, V_M\}$  with parameter  $p_v$  and  $\mathcal{U}^b = \{U_1^b, \dots, U_M^b\}$  with parameter  $1 - p_b$ , the PDF of the summation of independent Gamma distributed random variables, i.e.,  $H_M^b = \sum_{m=1}^M U_m^b V_m h_m$  is given by*

$$\begin{aligned} f_M^b(h) &= (1 - p_v + p_v p_b)^M \delta(h) + e^{-\mu h} \\ &\times \sum_{m=1}^M \binom{M}{m} \frac{(1 - p_b)^m p_v^m (1 - p_v + p_v p_b)^{M-m} \mu^{km}}{(km - 1)!} h^{km-1}. \end{aligned} \quad (4)$$

where  $\mu = \lambda/\sigma^2$ ,  $k = \lambda^2/\sigma^2$ .

*Proof.* The proof is given in Appendix I.  $\square$

Note that the discrete mass probability in  $f_M^b(h)$  at  $h = 0$  is due to the fact that we combined the continuous random variable  $h_m$  with discrete channels of axonal noise and vesicle release.

### 4. ANALYSIS OF THE SIGNAL DETECTION

Now, we formulate the binary detection problem at the receiving neuron by the following hypothesis test:

$$\begin{cases} \mathbf{H}_0 : y_{M|S=0}(t) = \sum_{m=1}^M U_m^a V_m h_m w(t) + e(t), \\ \mathbf{H}_1 : y_{M|S=1}(t) = \sum_{m=1}^M U_m^b V_m h_m w(t) + e(t), \end{cases} \quad (5)$$

where  $\mathbf{H}_0$  and  $\mathbf{H}_1$  refer to the hypotheses of no spike transmission event, i.e.,  $S = 0$ , and spike transmission event, i.e.,  $S = 1$ , respectively. Note that in (5),  $U_m^a$  is a Bernoulli random variable representing the conditional axonal noise where  $U_m^a = 1$  is the event that one spike is created at the  $m$ -th synapse with probability of  $p_a$  when  $S = 0$ .

Since the aggregated amplitude  $H_M^b = \sum_{m=1}^M U_m^b V_m h_m$  of the transmitted signal under  $\mathbf{H}_1$  is random, we employ the composite hypothesis testing used in detection of signals with unknown parameters [8]. Conditioned on  $\{h_1, \dots, h_M\}$ ,  $\{V_1, \dots, V_M\}$ ,  $\{U_1^b, \dots, U_M^b\}$  and  $\{U_1^a, \dots, U_M^a\}$ , the received signal  $y_M(t)$  has a Gaussian distribution under both hypotheses.

Now, by defining the likelihood ratio, denoted by  $\Lambda_S(y)$ , as the ratio of the PDF of  $y(t)$  conditioned on  $\mathbf{H}_1$  and  $\mathbf{H}_0$ , respectively, and by using [8, p. 65], we have

$$\Lambda_S(y) = \frac{\mathbb{E} \left\{ \exp \left( \frac{-\int_0^T y^2(t) dt + 2H_M^b \int_0^T w(t) y(t) dt - H_M^b{}^2 E_w}{N_0} \right) \right\}}{\mathbb{E} \left\{ \exp \left( \frac{-\int_0^T y^2(t) dt + 2H_M^a \int_0^T w(t) y(t) dt - H_M^a{}^2 E_w}{N_0} \right) \right\}}, \quad (6)$$

where  $\mathbb{E}\{\cdot\}$  denotes the expectation operation,  $N_0$  is the variance of  $e(t)$  and  $H_M^a = \sum_{m=1}^M U_m^a V_m h_m$  is defined as the aggregated amplitude of the transmitted signal under  $\mathbf{H}_0$ . In

addition, in (6),  $T_f$  denotes the refractory period between adjacent spikes [3] and  $E_w = \int_0^{T_f} w^2(t) dt$  is the energy of the EPSP response  $w(t)$ . From (3), and by considering  $w(t)$  is zero outside the refractory period of length  $T_f$ , we have

$$E_w \cong \frac{e^2 w_{\max}^2}{T_{\max}^2} \int_0^{+\infty} e^{-\frac{2t}{T_{\max}}} t^2 dt = \frac{e^2}{4} T_{\max} w_{\max}^2. \quad (7)$$

Next, the likelihood ratio in (6) can be written as

$$A_S(y) = \frac{\int_0^\infty f_M^b(h_b) \exp\left(\frac{2h_b c(y) - h_b^2 E_w}{N_0}\right) dh_b}{\int_0^\infty f_M^a(h_a) \exp\left(\frac{2h_a c(y) - h_a^2 E_w}{N_0}\right) dh_a} \quad (8)$$

where  $c(y) = \int_0^{T_f} w(t)y(t)dt$ , and  $f_M^a(h_a)$  stands for the PDF of  $H_M^a$ . Similar to Proposition 1, the PDF of  $H_M^a$  can be derived as

$$f_M^a(h) = (1 - p_v p_a)^M \delta(h) + e^{-\mu h} \times \sum_{m=1}^M \binom{M}{m} \frac{p_a^m p_v^m (1 - p_v p_a)^{M-m} \mu^{km}}{(km-1)!} h^{km-1}. \quad (9)$$

Combining (4), (8), and (9), we have

$$A_S(y) = \frac{\Phi_M^b + \sum_{m=1}^M \frac{\Psi_m^b}{(km-1)!} \int_0^\infty h^{km-1} e^{-\mu h} e^{\frac{2h c(y) - h^2 E_w}{N_0}} dh}{\Phi_M^a + \sum_{m=1}^M \frac{\Psi_m^a}{(km-1)!} \int_0^\infty h^{km-1} e^{-\mu h} e^{\frac{2h c(y) - h^2 E_w}{N_0}} dh}, \quad (10)$$

where  $\Phi_M^a = (1 - p_v p_a)^M$ ,  $\Phi_M^b = (1 - p_v + p_v p_b)^M$ , and

$$\Psi_m^a = \binom{M}{m} p_a^m p_v^m (1 - p_v p_a)^{M-m} \lambda^{km},$$

$$\Psi_m^b = \binom{M}{m} (1 - p_b)^m p_v^m (1 - p_v + p_v p_b)^{M-m} \lambda^{km}.$$

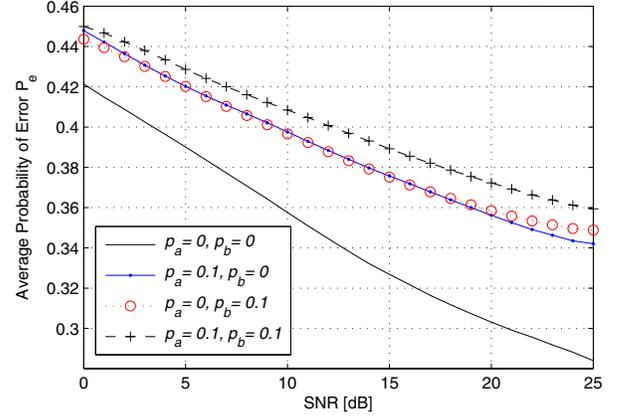
From [9, Eq. 3.462], the closed-form solution for the integrals in (10) can be obtained as

$$A_S(y) = \frac{\Phi_M^b + \sum_{m=1}^M \Theta_m^b e^{\frac{(\mu N_0 - 2c(y))^2}{8N_0 E_w}} C_{-km}\left(\frac{\mu N_0 - 2c(y)}{\sqrt{2N_0 E_w}}\right)}{\Phi_M^a + \sum_{m=1}^M \Theta_m^a e^{\frac{(\mu N_0 - 2c(y))^2}{8N_0 E_w}} C_{-km}\left(\frac{\mu N_0 - 2c(y)}{\sqrt{2N_0 E_w}}\right)} \quad (11)$$

where  $\Theta_m^b = \Psi_m^b \left(\frac{2E_w}{N_0}\right)^{-km/2}$ ,  $\Theta_m^a = \Psi_m^a \left(\frac{2E_w}{N_0}\right)^{-km/2}$ , and  $C_k(\cdot)$  is the parabolic cylinder function of order  $k$  [9, Eq. 9.240].

The variable amplitude  $h$  with distribution of  $f(h)$  in (4) has the parameter  $k$  [3]. For the case of  $k = 1$  and  $M = 1$ , this distribution has the highest variability, and  $A_S(y)$  in (11) can be simplified to

$$A_S(y) = \frac{\Phi_1^b + (1 - p_b)p_v \mu \sqrt{\frac{N_0 \pi}{E_w}} e^{\frac{(\mu N_0 - 2c(y))^2}{4N_0 E_w}} Q\left(\frac{\mu N_0 - 2c(y)}{2\sqrt{N_0 E_w}}\right)}{\Phi_1^a + p_a p_v \mu \sqrt{\frac{N_0 \pi}{E_w}} e^{\frac{(\mu N_0 - 2c(y))^2}{4N_0 E_w}} Q\left(\frac{\mu N_0 - 2c(y)}{2\sqrt{N_0 E_w}}\right)} \quad (12)$$



**Fig. 1.** The probability of error curves versus the SNR ratio for different values of axonal noise parameters in a SISO channel with  $k = 1$ ,  $M = 1$ , and  $p_v = 0.4$ .

where  $Q(y) = 1/\sqrt{2\pi} \int_y^\infty e^{-\alpha^2/2} d\alpha$  is the q-function.

Now, we investigate the probability of error of spike detection at the receiving neuron.

The average probability of error of the hypothesis test in (5) can be written as

$$P_{\text{error}} = p_{\text{prior}} P_{\text{false}} + (1 - p_{\text{prior}}) P_{\text{miss}} \quad (13)$$

where  $p_{\text{prior}} = P\{S = 0\}$  is the prior probability of hypothesis  $H_0$ . The probability  $P_{\text{false}}$  of selecting hypothesis  $H_1$  when  $H_0$  is correct is called false detection probability and is calculated as

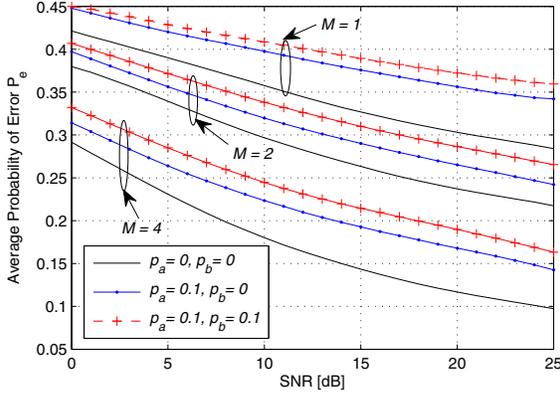
$$P_{\text{false}} = P\{R = 1|S = 0\} = P\{A_S(y) > A_0|S = 0\} \quad (14)$$

where  $R = 1$  denoted the event that spike is detected at the optimal detector output, and the critical threshold  $A_0$  is given by  $A_0 = p_{\text{prior}}/(1 - p_{\text{prior}})$ . In addition, the probability  $P_{\text{miss}}$  of selecting hypothesis  $H_0$  when  $H_1$  is correct is called miss-detection probability and is calculated as

$$P_{\text{miss}} = P\{R = 0|S = 1\} = P\{A_S(y) \leq A_0|S = 1\} \quad (15)$$

where  $R = 0$  denoted the event that spike is not detected at the optimal detector output.

The likelihood ratio  $A_S(y)$  in (11) is a function of random variable  $c(y) = \int_0^{T_f} w(t)y(t)dt$ . Hence, the cumulative distribution function of  $A_S(y)$  can be expressed in term of the cumulative distribution function of  $c(y)$ . If  $S = 0$ , we have  $c(y) = E_w H_M^a + e_{\text{out}}$ , where  $e_{\text{out}}$  becomes a white Gaussian noise and the distribution of  $H_M^a$  is given in (9). In addition, if  $S = 1$ , we have  $c(y) = E_w H_M^b + e_{\text{out}}$ , which is the summation of the random variable with the PDF in (4) and a Gaussian random variable with zero-mean.



**Fig. 2.** The probability of error curves versus the SNR ratio for different number of synapses and different values of axonal noise parameters in a channel with  $k = 1$  and  $p_v = 0.4$ .

## 5. NUMERICAL RESULTS

In this section, we present numerical results to demonstrate the performance of analytic results derived in previous sections. The parameters of variable of amplitude with Gamma distribution are given as mean  $\lambda = 1$  and  $k = 1$ . Similar to [3], we assume a pulse shape given in (3) with  $w_{\max} = 2$  mV and  $T_{\max} = 1$  msec.

In Fig. 1, the performance of SISO neuro-synaptic channels under different axonal noise values, modeled by a binary X-channel, are investigated. The average probability of error curves versus the signal-to-noise (SNR), i.e.,  $E_w/N_0$ , are shown for different values of axonal shot noise parameters  $p_a$  and  $p_b$ , when synaptic release probability is fixed to  $p_v = 0.4$  and the variable quantal amplitude with Gamma distribution parameter is assumed as  $k = 1$ . As it can be seen, the axonal noise can considerably degrade the performance of the system. For instance, it is shown that at  $P_e = 0.36$ , more than 12 dB more SNR is required when there is axonal noise with probability of 0.1, compared to a system with no axonal noise, in a channel. In addition, one can observe that erroneous spike event represented by probability  $p_a$  and removal of spike event have almost the same effect on the system performance.

As it is stated in Section I, due to the small value of synaptic release probability and other impairments, in reality, multiple number of synapses are employed to transmit the data to the receiving neuron. Thus, in Fig. 2, we investigate the effect of multiple synapses transmission on the system performance. It can be seen that using more synapses ( $M = 2, 4$ ) lead to significantly lower probabilities of error for all three cases of  $[p_a = 0, p_b = 0]$ ,  $[p_a = 0.1, p_b = 0]$ ,  $[p_a = 0, p_b = 0.1]$  and under all SNR conditions. By asymptotic analysis in high SNR conditions, it can be shown that diversity order of  $M$  is

achievable when there are  $M$  cooperating synapses.

## 6. CONCLUSION

In this paper, we investigated the performance of the neuro-synaptic communication with multiple synapses under several random degradations such as stochastic vesicle release, variable amplitude, and axonal and synaptic noises. For the first time, we modeled axonal noise as binary X-channel. In addition, we extended axonal-synaptic channels to more realistic model of MISO scheme. We derived a closed-form formula for the PDF of MISO axonal-synaptic channel. Then, we derived analytical expressions for likelihood function and error probability of an optimum detector at the receiving neurons. Furthermore, the impact of axonal noise on the performance of MISO neuro-synaptic channels are investigated by simulations.

## 7. APPENDIX: PROOF OF PROPOSITION 1

The PDF of  $h_m$  is given as  $f(h) = \frac{\lambda^k}{(k-1)!} h^{k-1} \exp(-\mu h)$ . From [10], it can be shown that, for a fixed value of  $M$ , called  $M_0$ , the sum of independent Gamma-distributed random variables with the same parameter  $\mu$  and of order  $km$  is again a Gamma distributed random variable with the order  $\sum_{m=1}^{M_0} km$ . Therefore, we have

$$f_M^b(h|M_0) = \frac{\mu^{kM_0}}{(kM_0-1)!} h^{kM_0-1} \exp(-\mu h). \quad (16)$$

Since  $U_m^b$  and  $V_m$  are two independent random Bernoulli variables, their product becomes another Bernoulli variable with the following parameter

$$\mathbf{P}\{U_m^b V_m = 1\} = (1 - p_b)p_v. \quad (17)$$

Now, by defining  $M_0 = \sum_{m=1}^M U_m^b V_m$ , it is clear that it has a binomial distribution with the following probability mass distribution

$$\mathbf{P}\{M_0 = m\} = \binom{M}{m} (1 - p_b)^m p_v^m (1 - p_v + p_v p_b)^{M-m}.$$

Hence, using the law of total probability, we have

$$\begin{aligned} f_M^b(h) &= \sum_{m=0}^M \mathbf{P}\{M_0 = m\} f_M(h|m) \\ &= (1 - p_v + p_v p_b)^M \delta(h) + \sum_{m=1}^M \mathbf{P}\{M_0 = m\} f_M(h|m) \\ &= (1 - p_v + p_v p_b)^M \delta(h) + e^{-\mu h} \\ &\quad \times \sum_{m=1}^M \binom{M}{m} \frac{(1 - p_b)^m p_v^m (1 - p_v + p_v p_b)^{M-m} \mu^{km}}{(km-1)!} h^{km-1}. \end{aligned} \quad (18)$$

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