MEASURING THE SPHERICAL-HARMONIC REPRESENTATION OF A SOUND FIELD USING A CYLINDRICAL ARRAY

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ABSTRACT

This paper describes how to encode into a spherical-harmonic representation of a 3-D sound field using a cylindrical microphone array. The standard way to represent a sound field is to use higher order Ambisonics specified over a spherical region. If the sound field is instead measured using a cylindrical array, it is more naturally expressed in terms of cylindrical harmonics. In this paper, the transforms for conversion between cylindrical and spherical-harmonic representations are derived. Simulations then compare the performance of cylindrical and spherical array processing in estimating the spherical-harmonic coefficients of a sound field.

Index Terms— Microphone array, Ambisonics, cylindrical harmonics, spherical harmonics, associated Legendre transform

1. INTRODUCTION

High-performance microphone arrays are important for application in spatial audio, noise source identification and surveillance. Ten years ago, the capture of spatial sound became popular with spherical microphone arrays [1–5]. Whilst circular microphone arrays have a limited ability to discriminate sounds in elevation, spherical array directivity is the same in every look direction. However for many applications, it is not necessary for the array to provide the same directivity in all directions. When spatial sound is spatialized by a listener, the listener's azimuthal acuity is better than their elevational acuity. By designing an array with a reduced resolution in elevation, fewer microphones are required. Cylindrical microphone arrays are suitable here, because the azimuthal directivity can be chosen independent of elevational directivity.

Past works in cylindrical array processing have focussed on utilized a single ring of microphones mounted onto the cylinder [6–10]. Such arrays possess a small number of microphones, but the designs are only appropriate for discriminating sounds propagating in the horizontal plane. A circular microphone array that can be focussed onto an out-of-plane source was devised in [11].

One open-space geometry using multiple concentric microphone arrays at different heights was proposed in [12] for measuring sound field coefficients (SFCs). Reduced processing complexity is achieved here by exploiting the zeros of associated Legendre functions. The apparent aperture of the circular array can be enlarged by mounting the microphones onto a sound hard or absorbent cylinder, or a special baffle designed to air-couple the surface waves [6, 8]. This increases the travel time of waves propagating around the cylinder which improves beamformer directivity.

Cylindrical representations of a sound field have been directly incorporated into the sound field representations in an approach called mixed-order Ambisonics [13, 14]. Measuring spherical-harmonic SFCs using a cylindrical array was proposed in [15] where the SFCs are estimated by extrapolating the sound pressure to virtual microphones on an open spherical array of larger radius. One approach is to determine the cylindrical-harmonic SFCs for a small set of elevation angles using fixed beamformer directions [16]. An alternate representation of a sound field is the plane-wave decomposition, which is applicable to both cylindrical and spherical arrays [17].

In this paper we show how to convert between the cylindrical representation of a sound field and the spherical-harmonic representation used in higher order Ambisonics formats, based on the associated Legendre transform. This extends on [10] which was limited to arrays consisting of a single ring of microphones. We further show how to convert the signals measured on a cylindrical microphone array directly into the spherical-harmonic representation. We then simulate using a cylindrical microphone array, and using a spherical microphone array of similar dimensions, to encode a 3-D sound field into a spherical-harmonic representation.

2. SOUND FIELD REPRESENTATION IN SPHERICAL AND CYLINDRICAL COORDINATES

The cylindrical harmonic expansion of a sound field at a point in free space in cylindrical coordinates (R, ϕ, z) is expressed as [18]:

$$\mathcal{P}(R, \phi, z; k) = \sum_{m=-\infty}^{\infty} e^{im\phi} \int_{-\infty}^{\infty} C_m(k_z, k) J_m(k_R R) e^{ik_z z} dk_z \tag{1}$$

where $C_m(k_z, k)$ is the cylindrical sound field coefficient (SFC) at the z-component of the wavenumber vector k_z and $k = \omega/c$ is the magnitude of the wavenumber vector and c is the speed of sound in air. Here k_R is the component of the wavenumber vector lying in the x-y plane $k_R = \sqrt{k^2 - k_z^2}$.

The sound field components for which $k_z \leq k$ are waves that propagate through space, while those for which $k_z > k$ are called evanescent wave components that are known to quickly decay to zero with distance.

The spherical harmonic expansion of a sound field at a point expressed in spherical coordinates (r, θ, ϕ) can be written [19]:

$$\mathcal{P}(r, \theta, \phi; k) = \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \beta_n^m(k) j_n(kr) Y_n^m(\theta, \phi) \tag{2}$$

where $\beta_n^m(k)$ is the spherical harmonic SFC, and $Y_n^m(\theta, \phi)$ is a spherical harmonic function defined [20] as

$$Y_n^m(\theta, \phi) = \Lambda_n^m P_n^{|m|}(\cos\theta) e^{im\phi},$$

where

$$\Lambda_n^m \triangleq \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}},$$

is a normalization term used to ensure that $Y_n^m(\theta,\phi)$ are orthonormal.

Determining $C_m(k_z,k)$ from sound pressure measurements made over a cylinder, multiply both sides of (1) by $e^{-im'\phi}/2\pi$ and integrate over azimuth $\phi\in[0,2\pi]$ to determine each phase mode:

$$\begin{aligned}\breve{\mathcal{P}}_m(R,z;k) &\triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{P}(R,\phi,z;k)e^{-im\phi}\,d\phi\\ &= \int_{-\infty}^{\infty}C_m(k_z,\omega)J_m(k_R R)e^{ik_z z}dk_z.\end{aligned}$$

Multiply both sides of this equation by $e^{-ik_z z}$ and integrate over height $z\in(-\infty,\infty)$, one can show that:

$$C_m(k_z,\omega)=\frac{1}{2\pi J_m(k_R R)}\int_{-\infty}^{\infty}\breve{\mathcal{P}}_m(R,z;k)e^{-ik_z z}\,dz. \quad (3)$$

When sound pressure is sampled not in free space but over a rigid infinite-length cylindrical baffle of radius $R=a$, $J_m(kR)$ is replaced with $B_m(ka)=-ika\,H_m'(ka)$, where for a negative time convention $H_m'(\cdot)$ is the derivative of the Hankel function of the first kind.

Similarly, the spherical-harmonic SFCs may be determined from sound pressure measurements made over a sphere. Multiplying both sides of (2) by $Y_n^m(\theta,\phi)$ and integrating over a unit sphere:

$$\begin{aligned}\beta_n^m(k) = \frac{1}{j_n(kr)}\int_0^{2\pi}\int_0^{\pi}\mathcal{P}(r,\theta,\phi;k)[Y_n^m(\theta,\phi)]^*\\ \times\sin\theta\,d\theta\,d\phi.\end{aligned} \quad (4)$$

To be clear, to obtain the cylindrical-harmonic SFCs the integration is performed over the surface of an infinitely-extending cylinder where cylinder radius R is constant whilst to obtain the spherical-harmonic SFCs, the integration in (4) is performed over a spherical shell where sphere radius r is a constant.

2.1. Representation of a Plane Wave

Consider a plane wave of wave number $k=\omega/c$ propagating in direction $\boldsymbol{\varphi}=(\vartheta,\varphi)$. The angles here are expressed in spherical coordinates. The plane wave sound pressure is expressed in cylindrical coordinates for a negative time convention as [20]:

$$e^{ik\boldsymbol{x}\cdot\boldsymbol{\varphi}}=\sum_{m=-\infty}^{\infty}i^m e^{im(\phi-\varphi)}J_m(\kappa_R R)e^{i\kappa_z z} \quad (5)$$

where $\kappa_R(\vartheta)=k\sin\vartheta$ and $\kappa_z(\vartheta)=k\cos\vartheta$. Comparing with the standard cylindrical harmonic form in (1), the cylindrical SFC can be identified as consisting of a single wave number component $k_z=\kappa_z(\vartheta)$. The SFC can be written,

$$C_m(k_z)=i^m e^{-im\varphi}\delta[k_z-\kappa_z(\vartheta)]. \quad (6)$$

Wave number z-component k_z corresponds to the component of a far-field plane wave propagating in direction $\vartheta=\cos^{-1}(k_z/k)$.

The spherical-harmonic SFCs for the plane wave are obtained from the expansion:

$$e^{ik\boldsymbol{x}\cdot\boldsymbol{\varphi}}=4\pi\sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}i^n[Y_n^m(\vartheta,\varphi)]^* j_n(kr)Y_n^m(\theta,\phi),$$

which by comparison with (2), the coefficients are identified as,

$$\beta_n^m=4\pi\,i^n[Y_n^m(\vartheta,\varphi)]^*. \quad (7)$$

3. CONVERSION BETWEEN SPHERICAL AND CYLINDRICAL REPRESENTATIONS

We now derive the transforms to convert between the two SFC representations.

3.1. Cylindrical to Spherical

Converting cylindrical SFCs to spherical-harmonic SFCs, we substitute (1) into (4):

$$\begin{aligned}\beta_n^m(k) = \frac{1}{j_n(kr)}\int_0^{2\pi}\int_0^{\pi}\sum_{m=-\infty}^{\infty}e^{im\phi}\int_{-\infty}^{\infty}C_m(k_z,k)\\ \times J_m(k_R R)e^{ik_z z}dk_z[Y_n^m(\theta,\phi)]^*\sin\theta\,d\theta\,d\phi\end{aligned}$$

where $R=r\sin\theta$ and $z=r\cos\theta$ are both functions of θ. Performing the integral in ϕ simplifies the expression for $\beta_n^m(k)$:

$$\begin{aligned}\beta_n^m(k) = \frac{2\pi\Lambda_n^m}{j_n(kr)}\int_0^{\pi}\int_{-\infty}^{\infty}C_m(k_z,k)J_m(k_z R)\\ \times e^{ik_z z}dk_z\,P_n^{|m|}(\cos\theta)\sin\theta\,d\theta.\end{aligned}$$

It can be seen that $\beta_n^m(k)$ may be obtained by integrating $C_m(k_z,k)$ with a kernel function:

$$\beta_n^m(k)=\int_{-\infty}^{\infty}C_m(k_z,k)\mathcal{T}_n^m(k_z,k)\,dk_z, \quad (8)$$

where

$$\mathcal{T}_n^m(k_z,k)=\frac{2\pi\Lambda_n^m}{j_n(kr)}\int_0^{\pi}J_m(k_R R)P_n^{|m|}(\cos\theta)e^{ik_z z}\sin\theta\,d\theta$$

Making the substitution $u=\cos\theta$,

$$\mathcal{T}_n^m(k_z,k)=\frac{2\pi\Lambda_n^m}{j_n(kr)}\int_{-1}^{1}J_m(k_R R)P_n^{|m|}(u)e^{ik_z z}\,du, \quad (9)$$

where $z=ru$ and $R=r\sqrt{1-u^2}$. The physical interpretation of $\mathcal{T}_n^m(k_z,k)$ is that it represents the weight to apply to each sound field component with wave number z-component k_z, to generate the spherical-harmonic SFC with indices (n,m).

Equation 9 can be simplified by utilizing a result in [21–23]:

$$i^n P_n^{|m|}\left(\frac{k_z}{k}\right)j_n(kr)=\frac{i^m}{2}\int_{-1}^{1}P_n^{|m|}(u)J_m(k_R R)e^{ik_z z}\,du.$$

This equation implicitly pertains to the components of a sound field associated with angle of propagation $\theta=\cos^{-1}(k_z/k)$. It is valid for propagating waves $k_z<k$. It expresses the free-space spherical-harmonic basis functions $j_n(kr)Y_n^m(\theta,\phi)$ in terms of a continuous sum of cylindrical basis functions $J_m(k_R R)e^{i(k_z z+m\phi)}$. Applying this equation, the transform in (9) simplifies to:

$$\mathcal{T}_n^m(k_z,k)=4\pi i^{n-m}\Lambda_n^m P_n^{|m|}(\cos\theta)$$

for $k_z<k$ where $\cos\theta=k_z/k$.

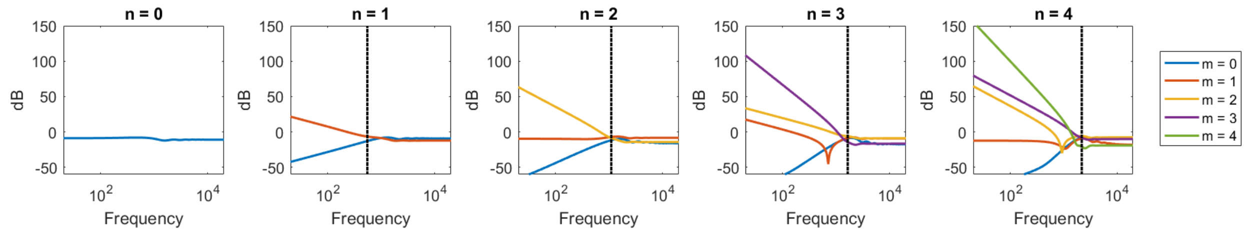


Fig. 1. Plots of the mode equalization function magnitude $|\zeta_n^m(kz;ka)|$ versus frequency for cylinder radii $a = 0.1$ m and height $z = 0.1$ m. The vertical dotted line marks the activation frequency for each sphere mode index n.

By making the approximation that the evanescent waves (for which $k_z > k$) are negligible, that is, that the sound field consists entirely of propagating wave components, the SFCs can be computed by truncating the bounds of integration in (8) as:

$$\beta_n^m(k) \approx \int_{-k}^{k} C_m(k_z,k)\mathcal{T}_n^m(k_z,k)\,dk_z$$
$$= 4\pi i^{n-m}\Lambda_n^m \int_{-k}^{k} C_m(k_z,k)P_n^{|m|}\left(\frac{k_z}{k}\right)dk_z. \qquad (10)$$

The transform in (10) can easy be shown to hold for the plane wave SFC in (6). Then because any propagating sound field in an empty region of space can be represented by a linear sum of plane waves, the transform holds for all propagating sound fields.

3.2. Spherical to Cylindrical

The transform to convert from spherical-harmonic SFCs to cylindrical SFCs can be derived as follows. The spherical-harmonic SFCs can be shown to be obtained by an associated Legendre transform defined in [24] on the cylindrical SFCs. The spherical-to-cylindrical transform can hence be written in terms of the inverse transform. Making the substitution $u = k_z/k$ in (10),

$$\beta_n^m(k) = 4\pi k i^{n-m}\Lambda_n^m \int_{-1}^{1} C_m(uk,k)\,P_n^{|m|}(u)\,du. \qquad (11)$$

The associated Legendre transform $T_n^m\{F(u)\}$ is defined as

$$f(n,m) = T_n^m\{F(u)\} = \int_{-1}^{1} \frac{P_n^m(u)}{(1-u^2)^{\frac{m}{2}}}F(u)\,du,$$

and the inverse transform is derived in [24] as:

$$F(u) = (T_n^m)^{-1}\{f(n,m)\}(u)$$
$$= \sum_{n=0}^{\infty}\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}f(n,m)(1-u^2)^{\frac{m}{2}}P_n^m(u),$$

so that $F(u)$ and $f(n,m)$ form a transform pair. We note that for Λ_n^m as defined above:

$$\frac{2n+1}{2}\frac{(n-|m|)!}{(n+|m|)!} = 2\pi[\Lambda_n^m]^2.$$

Setting $F(u) = (1-u^2)^{\frac{|m|}{2}}C_m(uk,k)$, it can be seen that

$$\beta_n^m(k) = 4\pi k i^{n-m}\Lambda_n^m T_n^{|m|}\left\{(1-u^2)^{\frac{|m|}{2}}C_m(uk,k)\right\},$$

and the inverse transform can immediately be written using the inverse associated Legendre transform:

$$C_m(uk,k) = i^{m-n}\frac{1}{4\pi k}\frac{(T_n^{|m|})^{-1}\{\beta_n^m(k)/\Lambda_n^m\}}{(1-u^2)^{\frac{|m|}{2}}},$$

for the propagating wave components $k_z = uk < k$. Substituting in the inverse transform expression, then

$$C_m(k_z,k) = \frac{i^{m-n}}{2k}\sum_{n=0}^{\infty}\Lambda_n^m P_n^{|m|}\left(\frac{k_z}{k}\right)\beta_n^m(k). \qquad (12)$$

This transform can be verified for the case of a plane wave, by inserting the plane wave SFC in (7) back into (12).

4. MEASURING SOUND FIELD COEFFICIENTS FROM SOUND PRESSURE

Using the above transforms, the spherical-harmonic SFCs can be determined directly from the sound pressure measured over a cylinder. More specifically, the SFCs can be expressed in terms of each phase mode $\breve{\mathcal{P}}_m(z;k)$. Substituting (3) into (10),

$$\beta_n^m(k) = 4\pi i^{n-m}k\int_{-\infty}^{\infty}\breve{\mathcal{P}}_m(z;k)\zeta_n^m(kz;ka)\,dz, \qquad (13)$$

where the function $\zeta_n^m(kz;ka)$ is:

$$\zeta_n^m(kz;ka) \triangleq \Lambda_n^m\int_{-1}^{1}B_m^{-1}(ka\sqrt{1-u^2})P_n^{|m|}(u)\,e^{ikz\,u}\,du \qquad (14)$$

This function represents the combined effect of the mode equalization and cylindrical-to-spherical conversion.

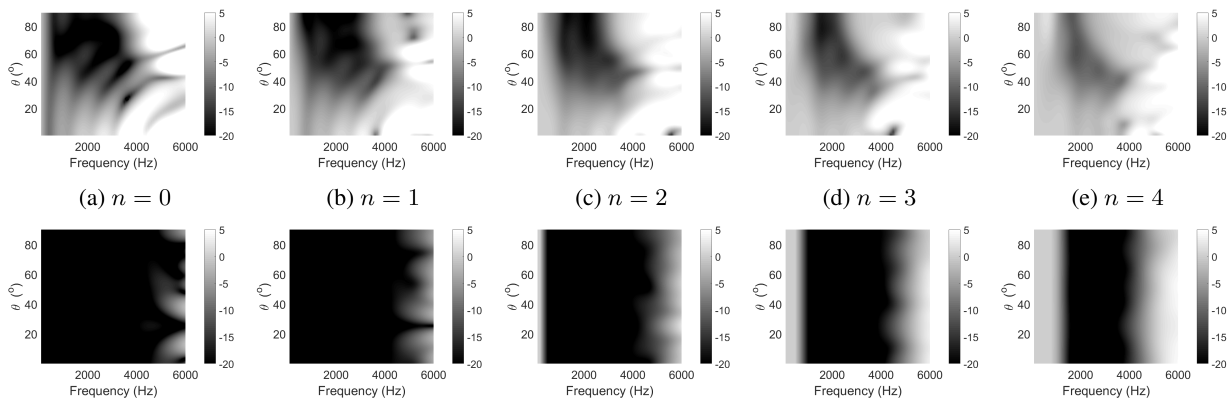
For a hard cylindrical baffle, set $B_m(ka) = i/ka\,H_m'(ka)$ to show that

$$\zeta_n^m(kz;ka) = -\Lambda_n^m\int_{-1}^{1}ika\sqrt{1-u^2}\,H_m'(ka\sqrt{1-u^2})$$
$$\times P_n^{|m|}(u)\,e^{ikzu}\,du$$

This function is plotted in Figure 1 for a baffle of radius $a = 0.1$ m at height $z = 0.1$ m upto sphere mode order $n = 4$.

When measuring phase modes using a cylinder, at low frequencies increasingly aggressive amplification is required for increasing mode order m. For a baffle of radius a, each mode has an activation property, switching on at frequency

$$f_m = \frac{mc}{2\pi a}.$$



(a) $n = 0$ (b) $n = 1$ (c) $n = 2$ (d) $n = 3$ (e) $n = 4$

(f) $n = 0$ (g) $n = 1$ (h) $n = 2$ (i) $n = 3$ (j) $n = 4$

Fig. 2. Average squared error in estimated spherical-harmonic SFCs up to 4th order, in dB for (a)-(e) the cylindrical array and (f)-(j) the spherical array. The sound field is that of a plane wave with elevational angle ϑ and azimuthal angle 0°.

In Figure 1 this mode-activation property is seen to hold, only in the sphere mode index n. Activation frequency f_n is marked on here.

Using the small argument expression for the Hankel function [18, p. 119], the mode equalization filter for a plane wave component at elevational angle ϑ is dominated by the expression

$$ika \sin\vartheta\, H'_m(ka \sin\vartheta) \approx -\frac{2^{m+1} m!}{\pi \epsilon_m} \frac{1}{(ka \sin\vartheta)^m},$$

where $\epsilon_m = 1$ for $m = 0$ and 2 for $m \geq 1$. In Figure 1, at low frequencies $\zeta_n^m(kz; ka)$ is seen to be linear in frequency in the log-log domain, and a pairing behaviour is seen to exist in the curves. It can be shown that the frequency dependence of $\zeta_n^m(kz; ka)$ at low frequencies is upper bounded by k^{-m}.

5. MICROPHONE ARRAY PROCESSING

The processing is now presented for determining the spherical-harmonic SFCs, using the signals sampled over cylindrical microphone array. Space P circular rings of microphones over the surface of a baffle of radius a, each ring positioned at height at z_p with a ring-to-ring spacing of Δd_q. Each circular array consists of Q microphones spaced equally in azimuth at angles ϕ_q.

The procedure itself is based upon (13):

1. Perform the DFT of microphone signals in azimuthal microphone index q, to calculate each phase mode m:

$$\breve{\mathcal{P}}_m(z_p; k) = \sum_{q=1}^{Q} \mathcal{P}(a, \phi_q, z_p; k) e^{-im\phi_q}\, \Delta\phi.$$

2. Approximate the transform in (13) with a Riemann sum:

$$\beta_n^m(k) = 4\pi i^{n-m} \Lambda_n^m \sum_{p=1}^{P} \breve{\mathcal{P}}_m(z_p; k) \hat{\zeta}_n^m(kz_p; ka) \Delta d_q,$$

where the functions $\hat{\zeta}_n^m(kz_p; ka)$ are regularized according to:

$$\hat{\zeta}_n^m(kz; ka) = \frac{\zeta_n^m(kz; ka)}{\lambda |\zeta_n^m(kz; ka)|^2 + 1},$$

and λ is a regularization parameter, to prevent noise amplification.

6. SIMULATION

The accuracy of the microphone array processing is evaluated, by determining the error in estimating the SFCs of a plane wave coming from direction (ϑ, φ). The performance is simulated using a cylindrical array and spherical array, both of radius $a = 0.1$ m:

- 63 microphones evenly spaced over an infinite-length solid cylinder, with $Q = 9$ and $P = 7$, with a ring spacing of $\Delta_q = 0.047$ m.
- 64 microphones mounted onto a solid sphere, arranged in a Fliege geometry [25].

Each microphone array is used to estimate the spherical-harmonic SFCs for a plane wave propagating in from an azimuthal angle of 0° and a range of elevation angles $\vartheta \in [0^\circ, 90^\circ]$. The SFCs are measured with the cylindrical array using the algorithm in Section 5, and with the spherical array using the method in [1]. Each method uses a regularization parameter $\lambda = 0.001$ for mode equalization. The average squared error in the spherical-harmonic SFCs $\beta_n^m(k)$ for mode index n is computed, using:

$$\mathrm{Error}(k; n) = \frac{\sum_{m=-n}^{n} |\hat{\beta}_n^m(k) - \beta_n^m(k)|^2}{\sum_{m=-n}^{n} |\beta_n^m(k)|^2}.$$

This error is plotted in Figure 2. The spherical array performs equally well at all elevation angles. The cylindrical array performs best for plane waves arriving from broadsides ($\vartheta = 90^\circ$) and worst for plane waves arriving from endfire ($\vartheta = 0^\circ$). Both arrays have a similar frequency range of operation for each mode index. The cylindrical array could be improved by using a least squares design.

7. CONCLUSION

A spatial sound field measured on a cylindrical array can be converted into the spherical-harmonic representation using an associated Legendre transform. Further, using the inverse transform derived, the sound field measured over a spherical array can be converted into a cylindrical-harmonic representation. When measuring the spherical-harmonic representation using a cylindrical array, performance losses occur due to the mismatch of cylindrical geometry with the spherical-harmonic basis functions.

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