SPARSE HEAD-RELATED TRANSFER FUNCTION REPRESENTATION WITH SPATIAL ALIASING CANCELLATION

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ABSTRACT

High-fidelity 3D audio experience requires accurate individual head-related transfer function (HRTF) representation. However, the process of measuring individual HRTFs typically involves measurements from hundreds of directions, with specialized and expensive equipment, which makes this process inaccessible for most users. In this paper, a new technique to reconstruct high resolution individual HRTFs from sparse measurements is presented. This is achieved by minimizing the spatial aliasing error in the spherical harmonics (SH) representation of the HRTFs, and by incorporating statistics calculated from a set of reference HRTFs, leading to an optimal minimum mean-square error solution. A quantitative analysis of the proposed method illustrates its benefits even for extreme cases, such as using only 25 individual HRTF measurements and a generic HRTF as a reference.

Index Terms— HRTF, Head-related transfer function, Sparse representation, Spatial aliasing, Spatial sound, Spherical harmonics, Minimum mean-square error

1. INTRODUCTION

The head-related transfer function (HRTF) provides a characterization for how a sound source from different directions is captured at each of the listener ears and is primarily useful for binaural reproduction. Accurate HRTF representation provides a high-fidelity spatial audio experience. For this reason individual HRTFs are preferable. There are two main approaches for accurately measuring HRTF. The first approach is based on acoustical measurements, which typically include microphones at the listener ears and a sound source that plays sound from different directions and at different frequencies [1–5]. The second approach is based on computer simulations using a 3D geometrical model of the head which can be obtained using a laser scanner [6], MRI scanning [2], or image-based modeling [7, 8]. These approaches are either time consuming or involve the use of expensive specialized equipment, which makes the process of measuring individual HRTFs inaccessible for the vast majority of users. One way to simplify the HRTF measurement process would be to reduce the required number of HRTF direction measurements, which may lead to a less expensive measurement setup and to a faster process. However, reducing the number of HRTF measurements may also lead to an increase in interpolation errors. Alternatively, generic HRTFs can be used. The advantages of generic HRTFs is that these functions are widely available. However, generic HRTFs provide a limited spatial audio experience with degradation in localization and externalization compared with individual HRTFs [9]. Several methods that aim to improve the performance of generic HRTFs by means of individualization were previously presented [10–14]. These approaches show improved characteristics of the individualized HRTFs relative to the non-individual HRTFs. Among the methods that rely on a small set of HRTF measurements, the most promising results were obtained using an optimal linear minimum mean square error (LMMSE) estimator for the HRTF coefficients in the SH domain, as presented by Romigh [11]. The LMMSE was evaluated for estimating the SH coefficients of *minimum-phase* HRTFs, using a small number of coefficients (less than 50).

This paper adopts the approach of mean square error minimization to estimate the HRTF coefficients in the SH domain. The approach presented in this paper, referred to as aliasing cancellation (AC), aims to reduce spatial aliasing error by means of using an optimal transformation matrix to the SH domain. A novel analysis reveals that both the AC and LMMSE estimators yield high accuracy in estimating a high order HRTF representation in the *complex* SH domain (i.e. hundreds of coefficients). In addition, unlike the LMMSE estimator, the derivation of AC does not assume that the mean HRTF coefficients are known, which facilitates the use of AC based on a small HRTF database, including the case of a single HRTF reference. Results in this paper illustrate that a full individual HRTF can be reconstructed from 25 measurements with a reconstruction error that is lower than -6 dB.

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2. HRTF AND SPARSITY ERROR

In this section the representation of HRTFs in the space domain and in the SH domain is presented. The error in these representations due to a small number of HRTF measurements, referred to as sparsity error, is then formulated.

2.1. Matrix formulation of HRTF in the SH domain

Consider the HRTF, $H(k, \Omega)$, for the left or right listener's ears, where k is the wave number, and $\Omega = (\theta, \phi)$ is the angular direction, with elevation angle θ and azimuth angle ϕ [15]. The HRTF represents a function on the surface of a sphere and can, therefore, be represented in the SH domain using complex SH basis functions $Y_n^m(\Omega)$ of order n and degree m, at the angular direction Ω [15]. Assuming that the HRTF is a function limited to order $\widetilde{N}$, the HRTF can be represented using a finite SH expansion, also known as the inverse spherical Fourier transform (SFT) [16], and can be formulated in matrix form as:

$$\mathbf{h} = \tilde{\mathbf{Y}}\tilde{\mathbf{h}}_{\mathbf{nm}}, \tag{1}$$

where the $Q \times 1$ vector $\mathbf{h} = [H(k, \Omega_1), \dots, H(k, \Omega_Q)]^T$ holds the HRTF measurements over Q directions, $\tilde{\mathbf{h}}_{\mathbf{nm}} = \left[H_{00}(k), H_{0(-1)}(k), \dots, H_{\widetilde{N}\widetilde{N}}(k)\right]^T$ is an $(\widetilde{N}+1)^2$ vector that holds the HRTF coefficients in the SH domain, and the $Q \times (\widetilde{N}+1)^2$ SH transformation matrix, $\tilde{\mathbf{Y}}$ is defined by its qth row as $\left\{\tilde{\mathbf{Y}}\right\}_q = \left[Y_0^0(\Omega_q), Y_1^{-1}(\Omega_q), \dots, Y_{\widetilde{N}}^{\widetilde{N}}(\Omega_q)\right]$.

Given a set of HRTF measurements over a sufficient number of directions $Q \geq (\widetilde{N}+1)^2$, the HRTF coefficients in the SH domain can be calculated from the HRTF measurements by multiplying (1) with the pseudo inverse of the SH transformation matrix $\tilde{\mathbf{Y}}^\dagger = (\tilde{\mathbf{Y}}^H\tilde{\mathbf{Y}})^{-1}\tilde{\mathbf{Y}}^H$, which forms the discrete SFT [16], with $(\cdot)^H$ denotes the Hermitian property,

$$\tilde{\mathbf{h}}_{\mathbf{nm}} = \tilde{\mathbf{Y}}^\dagger\mathbf{h}. \tag{2}$$

2.2. Sparsity error in HRTF representation

The number of HRTF coefficients in the SH domain that can be estimated using the SFT in (2) is constrained by the HRTF measurement scheme, following $Q \geq \lambda(N+1)^2$, where N represents the measurement scheme order and $\lambda \geq 1$ is scheme dependent. As long as the HRTF order $\widetilde{N}$ is not larger than the sampling scheme order N, i.e. $N \geq \widetilde{N}$ is satisfied, the HRTF coefficients can be accurately calculated from the Q HRTF measurements using (2). However, as the frequency increases, the HRTF becomes more spatially complicated and the HRTF order, $\widetilde{N}$, increases, roughly following the relation $\widetilde{N} \sim kr$ [17], where r denotes an average head radius. At high frequencies where $(\widetilde{N}+1)^2 > Q$, the number of measurements is insufficient for using (2) and the SFT is therefore reformulated to compute only the first $(N+1)^2$ HRTF coefficients according to the sampling scheme order

$$\hat{\mathbf{h}}_{\mathbf{nm}}^N = \mathbf{Y}^\dagger\mathbf{h} = \left(\mathbf{Y}^H\mathbf{Y}\right)^{-1}\mathbf{Y}^H\mathbf{h}, \tag{3}$$

where the $Q \times (N+1)^2$ matrix $\mathbf{Y}$ is composed of the first $(N+1)^2$ columns of matrix $\tilde{\mathbf{Y}}$.

An attempt to use SFT to estimate the HRTF coefficients up to order N with an insufficient number of measurements may lead to spatial aliasing error. The spatial aliasing error can be explicitly formulated by splitting matrix $\tilde{\mathbf{Y}}$ and the HRTF coefficients vector into two parts, $\tilde{\mathbf{Y}} = \left[\mathbf{Y}, \tilde{\mathbf{Y}}_{\boldsymbol{\Delta}}\right]$ and $\tilde{\mathbf{h}}_{\mathbf{nm}} = \left[\mathbf{h}_{\mathbf{nm}}^T, \mathbf{h}_{\boldsymbol{\Delta}}^T\right]^T$, where $\tilde{\mathbf{Y}}_{\boldsymbol{\Delta}}$ and $\mathbf{h}_{\boldsymbol{\Delta}}$ hold the elements of orders higher than N. Substituting these expressions into (1) and then into (3) leads to,

$$\hat{\mathbf{h}}_{\mathbf{nm}}^N = \mathbf{Y}^\dagger[\mathbf{Y}, \tilde{\mathbf{Y}}_{\boldsymbol{\Delta}}] \left[\begin{array}{c} \mathbf{h}_{\mathbf{nm}} \\ \mathbf{h}_{\boldsymbol{\Delta}} \end{array}\right] = \underbrace{\mathbf{h}_{\mathbf{nm}}}_{\text{desired}} + \underbrace{\mathbf{Y}^\dagger\tilde{\mathbf{Y}}_{\boldsymbol{\Delta}}\mathbf{h}_{\boldsymbol{\Delta}}}_{\text{aliasing error}}. \tag{4}$$

Equation (4) shows how high order HRTF coefficients in the vector $\mathbf{h}_{\boldsymbol{\Delta}}$ are aliased and added to the desired low order HRTF coefficients $\mathbf{h}_{\mathbf{nm}}$, which leads to spatial aliasing error in the estimated HRTF coefficients. Using these corrupted HRTF coefficients to calculate the HRTF at some L desired directions with the inverse SFT will lead to an error in the space domain as well:

$$\hat{\mathbf{h}}_L = \mathbf{Y}_L\hat{\mathbf{h}}_{\mathbf{nm}}^N, \tag{5}$$

where $\mathbf{Y}_L$ is the SH transformation matrix, as in equation (1), calculated only for the first $(N+1)^2$ columns at the L desired directions, and the HRTF at the L desired directions is given in $\hat{\mathbf{h}}_L = \left[\hat{H}(k, \Omega_1), \dots, \hat{H}(k, \Omega_L)\right]^T$.

The aliasing matrix $\mathbf{Y}^\dagger\tilde{\mathbf{Y}}_{\boldsymbol{\Delta}}$ depends on the HRTF measurement scheme and is frequency independent. Thus, at high frequencies where the magnitude of high order HRTF coefficients in $\mathbf{h}_{\boldsymbol{\Delta}}$ increases, the aliasing error is expected to increase. In addition to the distortion in low order HRTF coefficients, it is important to note that the Nth order representation using the standard SFT lacks the truncated high order HRTF coefficients. An overall error which incorporates errors due to both aliasing and truncation is defined here as sparsity error, and can be formulated as the mean-square error between the estimated HRTF and the true higher order HRTF,

$$\epsilon = \left\|\hat{\mathbf{h}}_{\mathbf{nm}} - \tilde{\mathbf{h}}_{\mathbf{nm}}\right\|^2 = \left\|\left[\begin{array}{c} \hat{\mathbf{h}}_{\mathbf{nm}}^N \\ \mathbf{0} \end{array}\right] - \tilde{\mathbf{h}}_{\mathbf{nm}}\right\|^2. \tag{6}$$

3. SPARSITY ERROR REDUCTION BY ALIASING CANCELLATION

In the previous section the sparsity error was presented for the case where the HRTF is represented in the SH domain and with $(\widetilde{N}+1)^2 > Q$. In order to reduce the sparsity error to a minimum, a new $(\widetilde{N}+1)^2 \times Q$ transformation matrix $\mathbf{C}$ is introduced, such that by multiplying it with the HRTF measurements $\mathbf{h}$ in (1), the HRTF coefficients in the SH domain

up to the high SH order $\widetilde{N}$ can be estimated:

$$\hat{\mathbf{h}}_{\mathbf{nm}} = \mathbf{Ch} = \mathbf{C}\tilde{\mathbf{Y}}\tilde{\mathbf{h}}_{\mathbf{nm}}, \qquad (7)$$

where $\hat{\mathbf{h}}_{\mathbf{nm}}$ is an $(\widetilde{N}+1)^2$ length column vector that holds the estimated HRTF coefficients in the SH domain. Equation (7) represents an under-determined system in which the number of unknowns $(\widetilde{N}+1)^2$ is larger than the number of equations Q.

One approach to minimize the sparsity error is to assume that the unknown HRTF coefficients, $\hat{\mathbf{h}}_{\mathbf{nm}}$, form a random vector, i.e. each person's HRTF coefficients are realizations from a set of possible outcomes with some probability distribution and a given *autocorrelation matrix* $\mathbf{R}_{\mathbf{hh}} = E\left[\tilde{\mathbf{h}}_{\mathbf{nm}}\tilde{\mathbf{h}}_{\mathbf{nm}}^{H}\right]$. Matrix $\mathbf{C}$ that minimizes the mean sparsity error, referred to as the AC matrix, can be calculated by solving the following minimization problem:

$$\mathbf{C}^{\text{AC}} = \arg\min_{\mathbf{C}} E\left[\epsilon\right] = \arg\min_{\mathbf{C}} E\left[\left\|\hat{\mathbf{h}}_{\mathbf{nm}} - \tilde{\mathbf{h}}_{\mathbf{nm}}\right\|^2\right]. \quad (8)$$

The error expression can be formulated using (7) as follows:

$$E\left[\left\|\hat{\mathbf{h}}_{\mathbf{nm}} - \tilde{\mathbf{h}}_{\mathbf{nm}}\right\|^2\right] = E\left[\left\|\left(\mathbf{C}\tilde{\mathbf{Y}} - \mathbf{I}\right)\tilde{\mathbf{h}}_{\mathbf{nm}}\right\|^2\right] \qquad (9)$$

$$= \sum_{l=1}^{(\widetilde{N}+1)^2} \left(\mathbf{c}_l\tilde{\mathbf{Y}} - \tilde{\mathbf{i}}_l\right)\mathbf{R}_{\mathbf{hh}}\left(\mathbf{c}_l\tilde{\mathbf{Y}} - \tilde{\mathbf{i}}_l\right)^{H} = \sum_{l=1}^{(\widetilde{N}+1)^2} |\epsilon_l|^2,$$

where $\mathbf{c}_l$ and $\tilde{\mathbf{i}}_l$ are the lth row vectors of matrix $\mathbf{C}$ and the $(\widetilde{N}+1)^2$ identity matrix $\mathbf{I}$, respectively. The error expression in (9) is composed of a sum of $(\widetilde{N}+1)^2$ positive and independent elements $|\epsilon_l|^2$. Finding the optimal coefficients in $\mathbf{c}_l$ that will minimize the corresponding error element $|\epsilon_l|^2$ for all $1 < l < (\widetilde{N}+1)^2$ will lead to the minimum value of the overall error. The optimal $\mathbf{c}_l$ is found by calculating the partial derivative of $|\epsilon_l|^2$ with respect to $\mathbf{c}_l$ and setting the result to zero, which leads to the optimal solution for the lth row, and by concatenating the solutions for all values of l, which gives the optimal transformation matrix,

$$\mathbf{C}^{\text{AC}} = \mathbf{R}_{\mathbf{hh}}\tilde{\mathbf{Y}}^{H}\left(\tilde{\mathbf{Y}}\mathbf{R}_{\mathbf{hh}}\tilde{\mathbf{Y}}^{H}\right)^{-1}. \qquad (10)$$

By substituting the transformation matrix $\mathbf{C}^{\text{AC}}$ into (7), the estimation of the HRTF coefficients is given by

$$\hat{\mathbf{h}}_{\mathbf{nm}}^{\text{AC}} = \mathbf{C}^{\text{AC}}\mathbf{h} = \mathbf{R}_{\mathbf{hh}}\tilde{\mathbf{Y}}^{H}\left(\tilde{\mathbf{Y}}\mathbf{R}_{\mathbf{hh}}\tilde{\mathbf{Y}}^{H}\right)^{-1}\mathbf{h}. \qquad (11)$$

The AC solution in (11) provides an estimation of the HRTF coefficients from a set of HRTF measurements. Given an HRTF database with a number of subjects that is significantly larger than the number of estimated HRTF coefficients $(\widetilde{N}+1)^2$, it can be assumed that $\mathbf{R}_{\mathbf{hh}}$ can be accurately estimated. In practice, the number of subjects in the database may

be limited and $\mathbf{R}_{\mathbf{hh}}$ must therefore be approximated from a relatively small sample of reference HRTFs. Therefore, two approaches to approximate the HRTF autocorrelation matrix are suggested. The first is to estimate $\mathbf{R}_{\mathbf{hh}}$ from an existing database of S subjects' HRTFs. Several databases are available online [1, 2, 18] and can be used in this case, with

$$\mathbf{R}_{\mathbf{hh}} = E\left[\tilde{\mathbf{h}}_{\mathbf{nm}}\tilde{\mathbf{h}}_{\mathbf{nm}}^{H}\right] \approx \sum_{s=1}^{S}\tilde{\mathbf{h}}_{\mathbf{nm}}^{s}\left(\tilde{\mathbf{h}}_{\mathbf{nm}}^{s}\right)^{H}, \qquad (12)$$

with $\tilde{\mathbf{h}}_{\mathbf{nm}}^{s}$ representing one HRTF from the database. The second is to use a single reference HRTF coefficients vector $\tilde{\mathbf{h}}_{\mathbf{nm}}^{\text{ref}}$ that is selected from an HRTF database:

$$\mathbf{R}_{\mathbf{hh}} \approx \tilde{\mathbf{h}}_{\mathbf{nm}}^{\text{ref}}\left(\tilde{\mathbf{h}}_{\mathbf{nm}}^{\text{ref}}\right)^{H} + \delta\mathbf{I}, \qquad (13)$$

where the purpose of the diagonal loading $\delta\mathbf{I}$ is to increase the rank of matrix $\mathbf{R}_{\mathbf{hh}}$. Such loading can also be applied when using (12) in the case of an insufficient number of subjects S.

4. SIMULATION STUDY

This section presents a simulation study that provides an objective performance evaluation of the AC approach, based on a comparison between original and estimated HRTFs. The HRTFs of 15 subjects (14 humans and a KEMAR manikin [19]) were simulated using the boundary element method that was applied on a 3D scan of the head and torso of the subjects. In this database, 2030 directions were simulated according to a Lebedev sampling scheme [20], providing an accurate HRTF representation in the SH domain of an order higher than $\widetilde{N} = 27$. A sparse version of each subject's HRTF is generated by down-sampling the original 2030 directions to a subset of $Q = 25$ directions that were chosen to approximate the extremal sampling scheme [21]. The sparse HRTF is used to estimate the HRTF coefficients in the SH domain using either the standard SFT or the AC approach. The estimated HRTF coefficients in the SH domain are then used to reconstruct the HRTF at the original $L = 2030$ directions using inverse SFT, similarly to (5). A total of three methods for estimating the HRTF coefficients are investigated in this study: the SFT in (4), and two versions of the AC in (11), one with a single reference and another with a set of references, as described in equations (12) and (13), respectively.

Fig.1 shows the left ear's HRTF magnitude of one subject over the horizontal plane and for different frequencies. The colors indicates the HRTF magnitude in decibels. Fig.1(a) shows the original HRTF that was simulated over 2030 directions. It illustrates, for example, that the HRTF magnitude is higher for sound sources from ipsilateral directions compared to contralateral directions. The following sub-figures of Fig.1 show the reconstruction of the same HRTF from a sparse measurement grid with 25 directions using the standard
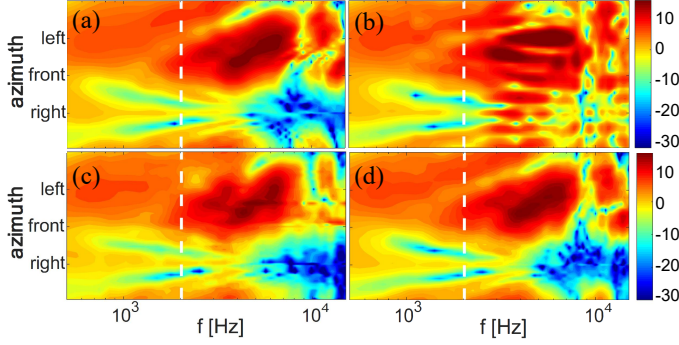
Fig. 1. Magnitude of HRTF over the horizontal plane: (a) true HRTF; (b) SFT interpolation $N = 4$; (c) AC-single reference $N = 27$; (d) AC-multiple references $N = 27$.

SFT and the AC approaches. The white dashed line represents the aliasing-free frequency limit. Comparing the reconstructed HRTFs with the true HRTF reveals that at frequencies lower than the aliasing-free limit the reconstructed HRTFs resemble the original HRTF, due to the low sparsity error at low frequencies. At higher frequencies, the sparsity error is higher and the reconstructed HRTF using SFT in Fig.1(b) is corrupted. In contrast, Fig.1(c) shows a reconstruction that is much closer to the original HRTF, which indicates the improvement gained by the AC approach with a single HRTF reference (which is the KEMAR HRTF). Interestingly, relying on all 14 available HRTF references to approximate the autocorrelation matrix, $\mathbf{R_{hh}}$, further improves the AC performance, as is evident from comparing between Fig.1(d) and the original HRTF in Fig.1(a). This result illustrates the improvement achieved by the AC approach for a specific subject over the horizontal plane.

A more general and quantitative evaluation is obtained by averaging the reconstruction error, ϵ_Ω, over the original 2030 directions, over all 14 human subjects, for a wide bandwidth up to 15kHz. ϵ_Ω is computed by

$$\epsilon_\Omega = \frac{\left\| \hat{\mathbf{h}}_L - \mathbf{h} \right\|^2}{\|\mathbf{h}\|^2}, \tag{14}$$

where $\hat{\mathbf{h}}_L$ are the estimated HRTFs in L directions similar to (5) and $\mathbf{h}$ holds the original HRTF measurements at the same directions. Both $\hat{\mathbf{h}}_L$ and $\mathbf{h}$ are complex and the reconstruction error accounts for differences in magnitude and phase of the reconstructed HRTF. The averaged reconstruction error is calculated by averaging the reconstruction error over all 14 human subjects. For the AC method with a single reference, the autocorrelation matrix is calculated using KEMAR's HRTF. For the AC method with multiple references, the autocorrelation matrix is calculated for each human subject with all other subjects as references.

Fig.2 shows the reconstruction error, averaged over 14 subjects, represented in both percentage and decibels.
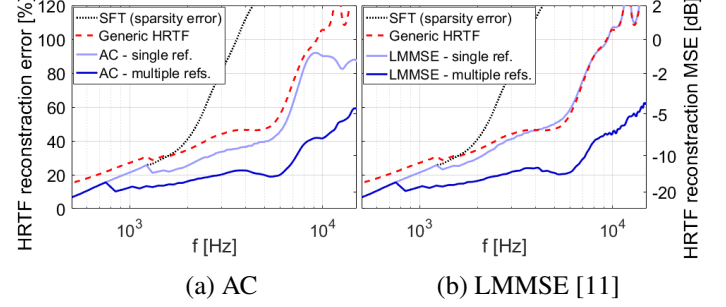

(a) AC   (b) LMMSE [11]

Fig. 2. HRTF reconstruction error using different approximation techniques

Fig.2(a) shows the errors using: SFT, AC with a single reference and AC with multiple references, represented by the dotted black, light blue and blue lines respectively. At low frequencies the error using the three methods is low. At higher frequencies the error of the SFT increases significantly. The dashed line represents the average error between KEMAR's HRTF and each of the subjects' HRTF, calculated using (14) with $\hat{\mathbf{h}}_L = \mathbf{h}^{\text{KEMAR}}$. It illustrates that the AC method with a single reference yields a noticeable improvement compared to standard SFT, and also compared to the reference itself. Using AC with multiple references further improves the estimation accuracy with an error of less than -6dB up to 12kHz. As a comparison to the results in Fig.2(a), Fig.2(b) shows the same error measures obtained using the LMMSE estimator, based on Romigh's approach [11]. Although it is possible to use the LMMSE estimator with a single reference, the improvement in the reconstruction accuracy with respect to the reference HRTF is limited. For the case of multiple references, the LMMSE reconstruction error is comparable to the AC reconstruction error. This may imply that with an increasing number of references the two approaches converge. Fig.2 demonstrates that the significant improvement achieved by the AC method holds for a wide bandwidth, over all directions, and for a variety of human HRTFs.

5. CONCLUSIONS

In this paper a new method for estimating the HRTF coefficients in the SH domain was developed by minimizing the expected sparsity error. Unlike the standard SFT that requires as many samples in space as the number of SH coefficients, it has been shown that the new method enables the estimation of a large number of HRTF SH coefficients even from a sparse HRTF measurement. This is achieved by incorporating statistics obtained from reference HRTFs with the sparsely measured HRTF. An advantage of this method over previously suggested methods is that even a single HRTF reference significantly reduces the sparsity error, and using more HRTF references may further improve the results. Informal listening tests indicate some clear perceptual benefits of the proposed method. A formal listening test is proposed for future work.

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