

A COUPLED COMPRESSIVE SENSING SCHEME FOR UNSOURCED MULTIPLE ACCESS

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ABSTRACT

This article introduces a novel paradigm for the unsourced multiple-access communication problem. This divide-and-conquer approach leverages recent advances in compressive sensing and forward error correction to produce a computationally efficient algorithm. Within the proposed framework, every active device first partitions its data into several sub-blocks, and subsequently adds redundancy using a systematic linear block code. Compressive sensing techniques are then employed to recover sub-blocks, and the original messages are obtained by connecting pieces together using a low-complexity tree-based algorithm. Numerical results suggest that the proposed scheme outperforms other existing practical coding schemes. Measured performance lies approximately 4.3 dB away from the Polyanskiy achievability limit, which is obtained in the absence of complexity constraints.

Index Terms— Communication, forward error correction, unsourced multiple-access, compressive sensing.

1. INTRODUCTION

Unsourced multiple access communication (MAC), initially proposed by Polyanskiy [1], is a novel formulation for concurrent uplink data transfers. It is closely related to uncoordinated multiple access [2, 3]. In this new paradigm, a system contains a total of K_{tot} users; out of this group, K_a users each wish to transmit a B -bit message to the access point at any given time. The access point is tasked with recovering only the set of messages being transmitted, without regard for the identities of the corresponding sources. The total number of users K_{tot} can be very large, whereas parameters K_a and B are envisioned to remain small, typically in the hundreds.

For the regime of interest, with its characteristic short message lengths, non-asymptotic information-theoretic results apply. Along these lines, Polyanskiy [1] derives finite block-length, achievability bounds for the unsourced MAC. The findings reported therein are based on random Gaussian codebooks, and they assume that information is recovered using a maximum likelihood decoder. Algorithmically, this scheme can be very computationally demanding. In [4], Ordentlich

and Polyanskiy report that many existing multiple access strategies perform poorly in this context, especially when K_a exceeds 100. They also propose the first low-complexity coding scheme tailored to this setting. In their scheme, a transmission period is divided into sub-blocks, or slots, and the system operates in a synchronous fashion. That is, all the users are aware of slot boundaries. Within this framework, every active user transmits a codeword during a randomly chosen slot. A data block is formed with a concatenated code that is designed for a T -user real-addition Gaussian multiple access channel (T -GMAC); typical values for T range from 2 to 5. Although this proposed scheme performs significantly better than existing MAC protocols, there remains an important gap of approximately 20 dB between its performance and the achievability limit associated with the unsourced MAC [1]. In related work [5], we introduce a low-complexity coding scheme that relies on a similar slotted structure. Our previous framework consists of an improved, close-to-optimal coding strategy for the T -GMAC; coupled to the application of successive interference cancellation across slots to reduce the performance degradation caused by overcrowded slots. The combination of these two features constitutes a significant improvement over previous results [4], with a performance curve that lies only approximately 6 dB away from the above mentioned achievability limit.

Both schemes discussed above adopt a channel coding viewpoint wherein the K_a -user GMAC is reduced to multiple smaller T -GMAC channel problems. Contrastingly, in this paper, we develop an alternate compressive sensing (CS) view of the problem. To begin, we emphasize that a naive CS solution to the unsourced MAC entails resolving a 2^B -length linear problem with a K_a -sparse solution. This implies sensing matrices with 2^{100} columns, which renders the problem intractable. A key idea in establishing a pragmatic scheme is to divide the information blocks of the users into smaller sub-blocks such that each sub-block is amenable to a CS recovery. Before transmission, redundancy is added to individual sub-blocks using a systematic linear block code. The collection of sub-blocks transmitted within a slot are recovered using a CS algorithm. Once this is achieved, individual segments of the original messages need to be pieced together. This is accomplished via a low-complexity tree-based algorithm. The overall structure of this communication architecture yields better

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performance compared to other existing algorithms with comparable computational complexity.

Throughout, we employ \mathbb{R}_+ , \mathbb{Z}_+ , and \mathbb{N} to denote the non-negative real numbers, non-negative integers, and natural numbers, respectively. For any $a, b \in \mathbb{Z}_+$ with $a \leq b$, we use $[a : b]$ to denote $\{c \in \mathbb{Z}_+ : a \leq c \leq b\}$. We write $X \sim B(n, p)$ if a random variable X possesses a binomial distribution with parameters n and p . We employ $|A|$ for the cardinality of set A , and we use $\lfloor x \rfloor$ to denote the closest integer to x .

2. SYSTEM MODEL

Let \mathbf{S}_{tot} represent the collection of devices within a network, and let \mathbf{S}_a denote the subset of active devices within a communication round, $\mathbf{S}_a \subset \mathbf{S}_{\text{tot}}$. Then, we have $|\mathbf{S}_{\text{tot}}| = K_{\text{tot}}$ and $|\mathbf{S}_a| = K_a$. Every active device wishes to communicate B bits of information to a base station through an uncoordinated uplink transmission scheme. The number of channel uses dedicated to this process is N , and $W = \{\vec{w}_k : k \in \mathbf{S}_a\}$ represent the collection of B bit message vectors associated with these active devices. We assume that devices pick their message vectors independently and uniformly at random from the set of binary sequences $\{0, 1\}^B$.

The base station facilitates a slotted structure for multiple access on the uplink through coarse synchronization. As such, the signal available at the receiver assumes the form

$$\vec{y} = \sum_{k \in \mathbf{S}_a} \vec{x}_k + \vec{z},$$

where \vec{x}_k is the N -dimensional vector sent by device k and \vec{z} represents additive white Gaussian noise. The signal sent by every device is power constrained, i.e., $\|\vec{x}_k\|_2^2 \leq NP$ for $k \in \mathbf{S}_a$, a scenario akin to [1]. The energy-per-bit is then given by $\frac{E_b}{N_0} \triangleq \frac{NP}{2B}$. The receiver produces an estimate $\widehat{W}(\vec{y})$ for the list of transmitted binary vectors W with $|\widehat{W}(\vec{y})| \leq K_a$. The per-user error probability of the system is defined by

$$P_e = \frac{1}{K_a} \sum_{k \in \mathbf{S}_a} \Pr(\vec{w}_k \notin \widehat{W}(\vec{y})). \quad (1)$$

We propose an encoding and decoding scheme that achieves $P_e \leq \varepsilon$, where ε is the target error probability with manageable computational complexity.

3. PROPOSED SCHEME

The transmission strategy outlined above features two parts: a systematic linear block code based on random parity checks, which we refer to as the tree encoder, and a CS encoder. A notional diagram of the proposed system appears in Fig. 1.

Tree Encoder: Every B -bit binary message vector \vec{w} is encoded into M bits using a systematic linear block code, which has random parity check constraints. Algorithmically, a message vector is partitioned into n sub-blocks, with the

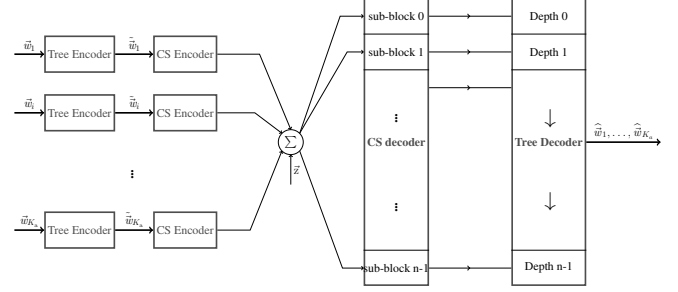


Fig. 1. This is a schematic of the proposed scheme. Original messages are split into sub-blocks, and redundancy is added to individual components. Transmitted sub-signals are then determined via a CS matrix, and sent over a MAC channel. A CS decoder recovers lists of sub-blocks, and a tree decoder reconstructs the original messages.

i^{th} sub-block consisting of m_i message bits, $\sum_{i=0}^{n-1} m_i = B$. The tree encoder appends l_i parity bits to sub-block i , except for the first block as we choose $l_0 = 0$. All the coded sub-blocks have the same length, i.e., $m_i + l_i = J \triangleq M/n$. The parity check bits in each sub-block are constructed as follows. Let $(p_0^{(i)}, p_1^{(i)}, \dots, p_{l_i-1}^{(i)})$ denote the parity bits in sub-block i . These bits are selected to satisfy random parity check constraints for all the message bits preceding their respective sub-block. To this end, we concatenate the message bits of all the sub-blocks $k \in [0 : i]$ and index them with the set $[0 : \sum_{k=0}^i m_k - 1]$. We then choose l_i subsets $\mathcal{A}_j^{(i)} \subseteq [0 : \sum_{k=0}^i m_k - 1] \forall j \in [0 : l_i - 1]$ uniformly at random without replacement. Parity check $p_j^{(i)}$ is chosen as the modulo-2 sum of all the message bits indexed by the set $\mathcal{A}_j^{(i)}$. In effect, $p_j^{(i)}$ acts as a parity check constraint for some randomly chosen message bits preceding it. In Section 4.1, we describe an optimization framework for the choice of parity length vector $\vec{l} = (l_0 = 0, l_1, \dots, l_{n-1})$.

CS Encoder: Let $\mathbf{A} = [\vec{a}_1, \dots, \vec{a}_{2^J}] \in \{\pm\sqrt{P}\}^{\tilde{N} \times 2^J}$, where $\tilde{N} \triangleq N/n$, denote a compressed sensing matrix that is designed to recover any K_a -sparse binary vector in the presence of noise with a low probability of error. The J bits in a sub-block are encoded using a bijective function $f : \{0, 1\}^J \rightarrow \{\vec{a}_j, j \in [1 : 2^J]\}$, which maps each sub-block to a column in \mathbf{A} . That is, a column of \mathbf{A} is a potentially transmitted sub-block.

3.1. Decoder

The decoding scheme consists of two components: the CS decoder operating in each sub-block, and a tree decoder operating across sub-blocks.

CS Decoder: The signal received during the i^{th} sub-block can be expressed as $\vec{y}_i = \mathbf{A} \vec{b}_i + \vec{z}_i$, where $\vec{b}_i \in \{0, 1\}^{2^J}$ is a K_a -sparse binary vector that indicates the list of i^{th} sub-

blocks transmitted by the active users. The task of the CS decoder is to provide an estimate of the sparse vector \vec{b}_i from the received signal \vec{y}_i during the corresponding time slot. This is accomplished by first applying a non-negative least squares (NNLS) algorithm to get an estimate $\vec{b}_i^{(\text{nnls})}$ of vector \vec{b}_i . Yet, this does not ensure that the entries of vector $\vec{b}_i^{(\text{nnls})}$ are binary. The desired binary estimate $\hat{\vec{b}}_i$ is obtained by setting the K largest entries of the vector $\vec{b}_i^{(\text{nnls})}$ to one and the remaining $2^J - K$ entries to zero. The number K is chosen as $K = K_a + K_\delta$, where K_δ is a small positive integer. Although the list output by the CS decoder is larger than K_a , the quantity K_δ is carefully chosen such that the erroneously decoded sub-blocks are very unlikely to satisfy the parity check constraints associated with encoding process.

Tree Decoder: The tree decoder seeks to recover the original messages transmitted by all the users by piecing together valid sequences of elements drawn from the various CS lists. Towards this end, the access point constructs a decoding tree for *each candidate message* as follows. We fix a sub-block from the list of all possible first sub-blocks supplied by the CS decoder as the root node for a tree. Once the first sub-block is determined, there are K possible choices for the second sub-block, and these are the nodes which appear in the first stage of the tree. Similarly, there are K possible choices for the third sub-block for each choice of the second sub-block and, hence, K^2 nodes in the second stage. This process continues until the $(n-1)^{\text{th}}$ stage is reached; at this point, the tree has K^{n-1} leafs. Every path connecting the root node to a leaf becomes a possible message. If there exist a single valid path at the end, the decoder outputs the corresponding message; otherwise, it reports a failure.

The number of possible paths increases exponentially with the stages of the tree and, hence, a naive search through all the leaf nodes is infeasible. In practice, invalid paths are pruned iteratively through the parity check constraints. Specifically, at stage $i \geq 1$, the decoder retains only nodes that satisfy the l_i bit parity constraints on all the message bits preceding that stage. This iterative procedure continues until the $(n-1)^{\text{th}}$ stage is reached. The complexity of this decoding scheme depends on the number of nodes surviving each stage, since parity checks have to be enforced only on the children of surviving nodes in the subsequent stages of the tree decoding process.

Remark 1 (Iterative Extension). *The successful outputs from the tree decoder can be subtracted off from their respective received signals in each sub-block. This extra step can potentially improve the estimate provided by the CS decoder compared to the previous iteration. This successive interference cancellation method can be repeated iteratively, leading to significant potential gains in performance, particularly for the first few steps.*

4. PERFORMANCE ANALYSIS

Suppose that the list output by the CS decoder contains the sub-block i transmitted by user k with a probability $1 - p_{cs}$ and, with a probability p_{cs} , this block is erroneously replaced by a vector chosen uniformly at random from the set $\{0, 1\}^J$. Let E_k denote the event that the transmitted binary message from user k is not present on the list output by the tree decoder. Similarly, let C_k be the event that all the sub-blocks corresponding to this user are present on the lists output by the CS decoder. Probability $P(E_k)$ can be computed as,

$$P(E_k) = P(E_k|C_k)P(C_k) + P(E_k|\overline{C_k})P(\overline{C_k}). \quad (2)$$

If the CS decoder fails to decode at least one of the sub-blocks that correspond to a user, then the output of the tree decoder would not contain the original message transmitted by that user. Thus, we have $P(E_k|\overline{C_k}) = 1$. The quantity $P(C_k)$ can be computed as $P(C_k) = (1 - p_{cs})^n$. We denote the event that the tree decoder declares a failure because of more than one path surviving the tree decoding process by $E_k|C_k$. We write the probability of this event as p_{tree} . When there are no iterations involved in the decoding process, the quantity P_e is the same as $P(E_k)$; they can be computed using (2) and the above observations as $P_e = 1 - (1 - p_{\text{tree}})(1 - p_{cs})^n$.

Let L_i denote the random variable corresponding to the number of erroneous paths that survive stage $i \in [1 : n-1]$ of the tree decoding process. The following results hold.

Lemma 2. *Expected values for L_i are given by*

$$\mathbf{E}[L_i] = \sum_{m=1}^i \left(K^{i-m} (K-1) \prod_{j=m}^i p_j \right) \quad (3)$$

where $p_i = 1/2^{l_i}$, $q_i = 1 - p_i$, and $i \in [1 : n-1]$.

Lemma 3. *The probability of error for the tree decoder p_{tree} is given by $p_{\text{tree}} = 1 - G_{L_{n-1}}(0)$ where*

$$G_{L_{n-1}}(z) = \prod_{i=0}^{n-2} f_{n-1-i}^{K-1}(z) \\ f_k(z) = \begin{cases} q_k + p_k f_{k+1}^K(z), & 1 \leq k \leq n-1 \\ z^{\frac{1}{K}}, & k = n, \end{cases} \quad (4)$$

and p_i, q_i are given in (3).

We define the computational complexity C of this decoder as the number of nodes on which parity checks need to be performed.

Lemma 4. *A closed-form expression for computing the expected computational complexity $\mathbf{E}[C]$ is given by*

$$\mathbf{E}[C] = K \left(n-1 + \sum_{i=1}^{n-2} \sum_{m=1}^i \left(K^{i-m} (K-1) \prod_{j=m}^i p_j \right) \right)$$

where p_i, q_i are given in (3).

K_a	25	50	75	100	125	150	175	200	225	250	275	300
J	14	14	14	14	14	15	15	15	15	15	15	15
$\varepsilon_{\text{tree}}$	0.0025	0.0045	0.006	0.01	0.0125	0.0055	0.0065	0.007	0.008	0.01	0.0125	0.0175

Table 1. Various parameters used in simulations.

The proof of Lemma 2 relies on the fact that $L_i|L_{i-1} \sim B((L_{i-1} + 1)K - 1, p_i)$. The proof of Lemma 3 is based on computing a closed-form expression for the probability generating function (PGF) $G_{L_{n-1}}(z)$ of the random variable L_{n-1} . Lemma 4 is a straightforward extension of Lemma 2. Additional details about these proofs can be found in [6].

4.1. Choice of the Parity Length Vector

We formulate the constrained optimization problem of minimizing the expected complexity subject to the probability of decoding failure being less than a carefully chosen threshold $\varepsilon_{\text{tree}}$. Since the parity lengths are non-negative integers, such a problem would be very difficult to solve. As such, we relax the problem to $(l_1, l_2, \dots, l_{n-1}) \in \mathbb{R}_+^{n-1}$. Also, we replace the constraint $p_{\text{tree}} \leq \varepsilon_{\text{tree}}$ with $\mathbf{E}[L_{n-1}] \leq \varepsilon_{\text{tree}}$ for the purpose of mathematical tractability. (By Markov's inequality, the quantity $\mathbf{E}[L_{n-1}]$ is an upper bound on p_{tree}). After these modifications, the optimization framework for the choice of parity lengths is given by

$$\begin{aligned}
& \underset{(p_1, p_2, \dots, p_{n-1})}{\text{minimize}} && \mathbf{E}[C] \\
& \text{subject to} && \mathbf{E}[L_{n-1}] \leq \varepsilon_{\text{tree}}, \\
& && \sum_{i=1}^{n-1} \log_2(1/p_i) = M - B, \\
& && p_i \in [1/2^J, 1] \quad \forall i \in [1 : n-1].
\end{aligned} \tag{5}$$

The above is a geometric program [7], and it can be solved using standard convex solvers. We choose the parity check lengths as $\hat{l}_i = \lceil \log_2(1/\hat{p}_i) \rceil$, for $i \in [1 : n-1]$ where $(\hat{p}_1, \hat{p}_2, \dots, \hat{p}_{n-1})$ is the solution to the optimization problem.

5. SIMULATION RESULTS

In this section, we study the performance of the proposed framework and we provide comparisons with existing schemes in literature. We consider a system with $K_a \in [25 : 300]$ active users, each having $B = 75$ bits of information to transmit. We divide these bits into $n = 11$ sub-blocks and the quantity J , which denotes the length of each sub-block is chosen depending on K_a ; it is given in Table 1. Similar to [5], we use sensing matrices that are constructed based on BCH codes for the compressed sensing problem. Specifically, we select a subset \mathcal{C}^0 of codewords of size $|\mathcal{C}^0| = 2^J$ from the (2047,23) BCH codebook \mathcal{C} with the following properties: (i) $\vec{c} \in \mathcal{C}^0 \implies \vec{1} \oplus \vec{c} \in \mathcal{C} \setminus \mathcal{C}^0$, where $\vec{1} \oplus \vec{c}$ denotes the one's complement of \vec{c} ; (ii) $\vec{c}_1, \vec{c}_2 \in \mathcal{C}^0 \implies \vec{c}_1 + \vec{c}_2 \in \mathcal{C}^0$;

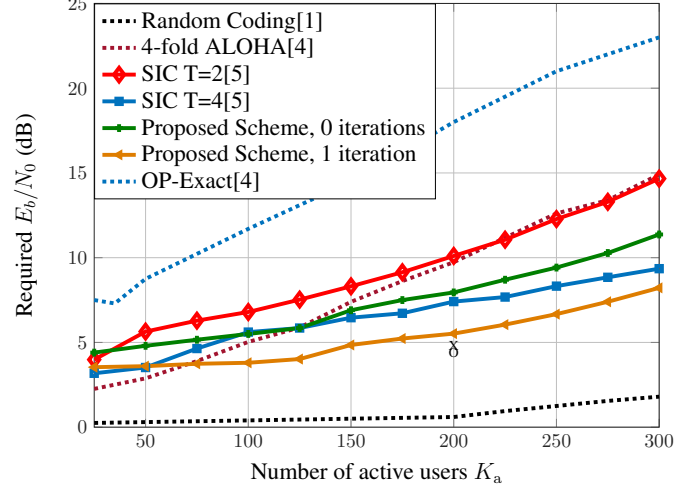


Fig. 2. Minimum E_b/N_0 required to achieve $P_e \leq 0.05$ vs. number of users for various schemes. Results for 2 and 3 iterations (see Remark 1) are represented by 'x' and 'o'. Observe that the SNR gains diminish with each iteration.

(iii) $\vec{0} \in \mathcal{C}^0$ where $\vec{0}$ denotes the all zero codeword. We then choose the sensing matrix as $\mathbf{A} = [\vec{a}_0, \vec{a}_1, \dots, \vec{a}_{2^J-1}]$, with dimension 2047×2^J where $\vec{a}_i = \sqrt{P}(2\vec{c}_i - 1)$, $\vec{c}_i \in \mathcal{C}^0$ for $i \in [0 : 2^J - 1]$. The total number of channel uses is therefore given by $N = 11 \times 2047 = 22,517$. The target error probability of the system is fixed at $\varepsilon = 0.05$. We set list size K for the NNLS CS problem to $K = K_a + 10$. For each $K_a \in [25 : 300]$, we solve the optimization problem (5) using the CVX solver [8], and the resulting solution dictates the choice of parity length vector. Choice of the quantity $\varepsilon_{\text{tree}}$ for each K_a is given in Table 1. The parameters B and N are chosen such that the rate $\frac{B}{N} = \frac{75}{22,517}$ is approximately the same as the rate resulting from the choice of parameters $B = 100$ and $N = 30,000$ in [4, 5]. This enables a fair comparison between these schemes and our proposed scheme. We emphasize that the choice of B and N for our simulations is motivated by the existence of good compressive sensing matrices based on BCH codes. When these parameters are proportionally scaled up, performance of the system can only improve, as the finite block length effects are more pronounced for lower values of B and N . In Fig. 2, the E_b/N_0 required to achieve a target error probability of 0.05 is plotted as a function of K_a for various schemes. It can be seen from Fig. 2 that our proposed scheme with just one extended round of iteration outperforms existing schemes for $K_a \in [75 : 300]$.

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