FAIRNESS IN MULTITERMINAL DATA COMPRESSION: A SPLITTING METHOD FOR THE EGALITARIAN SOLUTION

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ABSTRACT

This paper proposes a novel splitting (SPLIT) algorithm to achieve fairness in the multiterminal lossless data compression problem. It finds the egalitarian solution in the Slepian-Wolf region and completes in strongly polynomial time. We show that the SPLIT algorithm adaptively updates the source coding rates to the optimal solution, while recursively splitting the terminal set, enabling parallel and distributed computation. The result of an experiment demonstrates a significant reduction in computation time by the parallel implementation when the number of terminals becomes large. The achieved egalitarian solution is also shown to be superior to the Shapley value in distributed networks, e.g., wireless sensor networks, in that it best balances the nodes' energy consumption and is far less computationally complex to obtain.

Index Terms— Data compression, egalitarian solution, submodularity.

1. INTRODUCTION

There are many problems in signal processing involving source coding, e.g., the distributed source coding problem [1,2] that typically arises in the wireless sensor networks (WSNs), or the multimedia source coding problems in [3–5]. They are also called data compression (DC) problems since they aim to use the minimum code length to describe the correlated sources with the minimum information loss. It is well known that the achievable source coding rates for the lossless DC constitutes the Slepian-Wolf (SW) region [6, 7], which could become very large as the number of terminals/sources increases. Then, the problem is not only to reach the SW region, but also to find a solution with certain feature.

For a system where the terminals (also known as *users* or *nodes*) are equally privileged, e.g., a WSN, we always seek to attain fairness in the SW region. Several game-theoretic contributions in [8–10] showed that the Shapley value [11] is one solution within the SW region. However, computing the Shapley value is intractable in large scale systems due to the exponentially growing complexity in the number of terminals. In addition, the Shapley value distributes the source coding

rates based on the statistics of source data so that the terminals with more information are assigned higher rates. Such a method is less suitable for systems such as WSNs where balancing the battery energy consumption across all terminals is desired.

Thus, we consider another approach to attain fairness in the SW region through the egalitarian solution. We show that the egalitarian solution balances the source coding rate distribution, amongst terminals, in the SW region so that any cost proportional to the source coding rate is fairly allocated to the terminals. Finding the egalitarian solution is formulated using quadratic programming. We solve this problem with a novel splitting (SPLIT) algorithm that adaptively updates a cost allocation method until the egalitarian solution is attained. The SPLIT algorithm also recursively breaks each terminal set into two smaller subsets, allowing each of them to calculate its own optimal solution, both in a distributed and parallel manner. The result of an experiment shows that the completion time by adopting the parallel computation is significantly reduced as the number of terminals increases. In addition, the SPLIT algorithm completes in strongly polynomial time, which is much faster than the exponential time required to obtain the Shapley value.

2. SYSTEM MODEL

Let $Z_V = (Z_i : i \in V)$ be a vector of discrete random variables indexed by a finite set V. Assume that there is a distinct node/user i privately observes an n-sequence Z_i^n of the terminal Z_i that is i.i.d. generated according to the joint distribution P_{Z_V} . The users are required to encode their observations in a way such that the source sequence Z_V^n can be reconstructed at the sink¹ from the source codes. This problem is called (*multiterminal*) data compression (*DC*) or source coding [12].

For the subsets $X, Y \subseteq V$, let H(X) be the amount of randomness in Z_X measured by the Shannon entropy [12] and $H(X|Y) = H(X \cup Y) - H(Y)$ be the conditional entropy of Z_X given Z_Y . Denote $\mathbf{r}_V = (r_i : i \in V)$ a (source coding) rate vector with each dimension r_i denoting the expected

¹The sink could be a data fusion/gathering node, e.g., a cluster header in a WSN.

code length at which user *i* encodes his/her observations Z_i^n . We call \mathbf{r}_V an *achievable rate vector* if the source sequence Z_V^n can be recovered at the sink by letting the users encode Z_V^n at the rate designated by \mathbf{r}_V . Let *r* be the *sum-rate function* associated with the rate vector \mathbf{r}_V such that

$$r(X) = \sum_{i \in X} r_i, \quad \forall X \subseteq V$$

with the convention $r(\emptyset) = 0$. An achievable rate vector must satisfy the Slepian-Wolf (SW) constraints [6,7]:²

$$r(X) \ge H(X|V \setminus X), \quad \forall X \subseteq V, r(V) = H(V).$$
(1)

For a subset $X \subseteq V$, consider the constraint $r(X) \ge H(X|V \setminus X)$ in (1). Since the sum-rate is restricted to r(V) = H(V), there is necessarily an upper bound $r(V \setminus X) = H(V) - r(X) \le H(V \setminus X)$. Repeating the same conversion for all $X \subseteq V$, we have the SW region

$$\mathscr{R}_{\mathrm{DC}}(V,H) = \{ \mathbf{r}_V \in P(H,\leq) \colon r(V) = H(V) \},\$$

where $P(H, \leq) = \{\mathbf{r}_V \in \mathbb{R}^{|V|} : r(X) \leq H(X), \forall X \subseteq V\}$ is the *polyhedron* of the entropy function H [13, Section 2.2]. It is shown in [14, Section 4.2] that H is submodular so that $\mathscr{R}_{DC}(V, H)$ coincides with a submodular base polyhedron. This crucial submodularity property underpins the optimality and efficiency of the SPLIT algorithm in Section 4.³

3. FAIRNESS: EGALITARIAN VS. SHAPLEY

In general, the SW region $\mathscr{R}_{DC}(V, H)$ is not a singleton, i.e., there is usually more than one achievable rate vector. To choose a \mathbf{r}_V in $\mathscr{R}_{DC}(V, H)$ in a system where the users are equally privileged, e.g., sensors and mobile clients, a natural selection criterion is the fairness. Based on the coalitional game formulation of the DC problem in [8, 9], it is possible to reach the Shapley value [11] that attains fairness from a typical game-theoretic perspective. Let \sqcup denote the disjoint union. The Shapley value $\hat{\mathbf{r}}_V$ with each dimension being [15]

$$\hat{r}_i = \sum_{C \subseteq V \setminus \{i\}} \frac{|C|!(|V| - |C| - 1)!}{|V|!} \big(H(C \sqcup \{i\}) - H(C) \big)$$

allocates each user his/her expected marginal entropy.⁴ Since the expected source coding length is determined by the entropy [12], the Shapley value is fair in that it assigns each



Fig. 1. The polyhedron $P(H, \leq)$ and the SW region $\mathscr{R}_{DC}(V, H)$ for the DC problem in Example 1. There are two fair rate vectors in $\mathscr{R}_{DC}(V, H)$: The Shapley value $\hat{\mathbf{r}}_V = (\frac{3}{2}, \frac{3}{10}, \frac{3}{10})$ is the gravity center of the extreme points; The egalitarian solution $\mathbf{r}_V = (1, \frac{11}{20}, \frac{11}{20})$ is the minimum ℓ_2 -norm, i.e., the minimizer of min $\{\|\mathbf{r}_V\|_2 : \mathbf{r}_V \in \mathscr{R}_{DC}(V, H)\}$.

user the cost he/she introduces to the DC problem by encoding his/her observations. However, this fair rate vector may not be the desired one in all cases. See the example below.

Example 1. The three users in $V = \{1, 2, 3\}$ observe respectively

$$Z_1 = (W_a, W_b, W_c)$$
$$Z_2 = (W_c, W_d),$$
$$Z_3 = (W_b, W_d),$$

where $W_j s$ for all $j \in \{a, \ldots, d\}$ are independent random bits with $H(W_a) = 1$, $H(W_b) = H(W_c) = \frac{1}{2}$ and $H(W_d) = \frac{1}{10}$. The SW region is shown in Fig. 1. We have the Shapley value $\hat{\mathbf{r}}_V = (\frac{3}{2}, \frac{3}{10}, \frac{3}{10})$, in which user 1 is assigned the most source coding rates $r_1 = \frac{3}{2}$ since he/she has the highest expected marginal entropy. However, consider the minimum ℓ_2 -norm $\mathbf{r}_V^* = \arg\min\{\|\mathbf{r}_V\|_2: \mathbf{r}_V \in \mathscr{R}_{DC}(V, H)\} =$ $(1, \frac{11}{20}, \frac{11}{20})$. \mathbf{r}_V^* allocates the rates more evenly in $\mathscr{R}_{DC}(V, H)$ than the Shapley value. For a WSN, \mathbf{r}_V^* is better than $\hat{\mathbf{r}}_V$ in that the maximum energy consumption in source coding $\max\{r_i^*: i \in V\} < \max\{\hat{r}_i: i \in V\}$ is less and therefore the lifetime of WSN is prolonged.⁵ Here, \mathbf{r}_V^* is called the egalitarian solution [17, 18] in $\mathscr{R}_{DC}(V, H)$.⁶

Consider a more general quadratic programming problem

$$\min\{\sum_{i\in V} \frac{r_i^2}{w_i} \colon \mathbf{r}_V \in \mathscr{R}_{\mathsf{DC}}(V, H)\},\tag{2}$$

where $\mathbf{w}_V \in \mathbb{R}_{++}^{|V|}$ is a positive weight vector that could have some practical interpretation: w_i could denote the quality of wireless channel at user *i* or the remaining energy in sensor

²The interpretation of (1) is (a) the users in X must reveal $H(X|V \setminus X)$, the information that is uniquely obtained by X, to the sink; (b) the users must reveal the total information H(V) to the sink.

 $^{^{3}}$ In [14], H is shown to be a polymatroid rank function, a subset of submodular functions.

 $^{{}^{4}}H(C \sqcup \{i\}) - H(C)$ is the marginal randomness introduced by user i to user subset C; By assuming each permutation of V appears equiprobably, $\frac{|C|!(|V|-|C|-1)!}{|V|!}$ quantifies the frequency that user i will be assigned rate $H(C \sqcup \{i\}) - H(C)$. The Shapley value $\hat{\mathbf{r}}_{V}$ lies in $\mathscr{R}_{\text{DC}}(V, H)$ [15].

⁵The lifetime in a WSN is usually defined as the time to which the first sensor node runs out of battery power [16]. Assume that the energy consumption is linearly proportional to the source coding rate, switching from the rate vector $(\frac{3}{2}, \frac{3}{10}, \frac{3}{10})$ to $(1, \frac{11}{20}, \frac{11}{20})$ prolongs the lifetime by $\frac{1}{2}$.

⁶The advantage of the egalitarian solution in a peer-to-peer communication problem is also exemplified in [19].

Algorithm 1: split algorithm (SPLIT)

input : a user subset C, an oracle that returns the value of f(X) for $X \subseteq C$ and a positive weight vector \mathbf{w}_C output: $\mathbf{r}_C^* = \arg\min\{\sum_{i \in C} \frac{r_i^2}{w_i} : \mathbf{r}_C \in \mathscr{R}_{DC}(C, f)\}$ 1 begin

node *i*. The minimizer \mathbf{r}_V^* of (2) is called the *weighted*, or generalized, egalitarian solution w.r.t. \mathbf{w}_V [20,21]: It reduces to minimum ℓ_2 -norm when $\mathbf{w}_V = \mathbf{1} = (1, \ldots, 1) \in \mathbb{R}_{++}^{|V|, 7}$. For a given weight vector \mathbf{w}_V , denote $w: 2^V \mapsto \mathbb{R}_{++}$ the sum-weight function with $w(X) = \sum_{i \in X} w_i, \forall X \subseteq V$.

4. SPLIT ALGORITHM

To efficiently solve the problem (2), we exploit the submodularity of the SW region $\mathscr{R}_{DC}(V, H)$ as identified in Section 2. The authors in [22, 23] showed that the minimizer of (2) can be determined by recursively solving the submodular function minimization (SFM) problem, based on which, we propose the SPLIT algorithm in Algorithm 1. Its optimality is given by the following theorem with the proof in Section 6.

Theorem 1. The output \mathbf{r}_V^* of the call $SPLIT(V, H, \mathbf{w}_V)$ is the minimizer of (2).

Example 2. For $\mathbf{w}_V = (3, 1, 3)$, we apply the SPLIT algorithm to the system in Example 1. By calling SPLIT (V, H, \mathbf{w}_V) , we have $\lambda = \frac{3}{10}$ and $\hat{X} = \{3\}$ being the maximal minimizer of $\min\{H(X) - \lambda w(X) \colon X \subseteq V\}$. Since $\hat{X} \neq V$, we call run SPLIT $(\{3\}, H, w_3)$ and get $r_3^* = \frac{3}{5}$ returned. We

set $\mathbf{r}_{\{1,2\}}^* = \frac{H(\{3\})}{w_3} \mathbf{w}_{\{1,2\}} = (\frac{1}{5}w_1, \frac{1}{5}w_2) = (\frac{3}{5}, \frac{1}{5})$ and call SPLIT($\{1,2\}, g, \mathbf{w}_{\{1,2\}}$) where $g(X) = H(X \sqcup \{3\}) - H(\{3\})(\frac{w(X)}{w_3} + 1)$ for all $X \subseteq \{1,2\}$. We get output $\mathbf{r}_{\{1,2\}}^* = (0.525, 0.175)$, which is added to the current rates so that we have $\mathbf{r}_{\{1,2\}}^* = (\frac{3}{5}, \frac{1}{5}) + (0.525, 0.175) = (\frac{9}{8}, \frac{3}{8})$. Finally, we have $\mathbf{r}_V^* = \mathbf{r}_{\{1,2\}}^* \oplus r_3^* = (\frac{9}{8}, \frac{3}{8}, \frac{3}{5})$ at the output which is the egalitarian solution w.r.t. $\mathbf{w}_V = (3, 1, 3)$ in $\mathscr{R}_{DC}(V, H)$.

We then run SPLIT(V, H, \mathbf{w}_V) for $\mathbf{w}_V = \mathbf{1}$, where $\lambda = \frac{3}{10}$ and $\hat{X} = \{2,3\}$ is the maximal minimizer of $\min\{H(X) - \lambda | X| \colon X \subseteq V\}$. Since $\hat{X} \neq V$, we call SPLIT($\{2,3\}, H, \mathbf{w}_{\{2,3\}}$) to get $\mathbf{r}^*_{\{2,3\}} = (\frac{11}{20}, \frac{11}{20})$ and SPLIT($\{1\}, g, \mathbf{w}_{\{1\}}$) to get $r^*_1 = 1$ so that $\mathbf{r}^*_V = r^*_1 \oplus \mathbf{r}^*_{\{2,3\}} = (1, \frac{11}{20}, \frac{11}{20})$ is the egalitarian solution in Fig. 1.

4.1. Complexity

For a submodular function f, we call $\min\{f(X): X \subseteq V\}$ a submodular function minimization (SFM) problem of size |V|. We denote O(SFM(|V|)) the complexity for solving this SFM problem, which is strongly polynomial [13, Chapter VI]. While the SFM algorithms in [24–27] vary in computation complexity, the exact completion time of a SFM algorithm depends on its size |V|.⁸ It is easy to see that, due to the submodularity of H, the problem (3) in each recursion of the SPLIT algorithm is a SFM of size $|C| \le |V|$.⁹ Since the number of recursions is no greater than 2|V| - 1 [23, Theorem 9], the overall complexity is upper bounded by $O(|V| \cdot SFM(|V|))$.

4.2. Adaptive and Distributed Implementation

Steps 7 and 8 in the SPLIT algorithm indicate an adaptive rate update method: In step 7, since $\mathbf{r}_{C\setminus\hat{X}}^* \geq \frac{f(\hat{X})}{w(\hat{X})} \mathbf{w}_{C\setminus\hat{X}}$,¹⁰ we first assign $\frac{f(\hat{X})}{w(\hat{X})} \mathbf{w}_{C\setminus\hat{X}}$ to $\mathbf{r}_{C\setminus\hat{X}}^*$ and determine the remaining rates by calling SPLIT($C \setminus \hat{X}, g, \mathbf{w}_{C\setminus\hat{X}}$) in step 8. It is easy to see that $\mathbf{r}_V^* \in P(H, \leq)$ after each execution of step 7. Therefore, the SPLIT algorithm adaptively updates the rate vector in the polyhedron $P(H, \leq)$ until it finally reaches \mathbf{r}_V^* . See the examples in Fig. 2.

In addition, steps 6 to 8 in the SPLIT algorithm can be implemented in a distributed manner: The set *C* is split into two disjoint subsets \hat{X} and $C \setminus \hat{X}$; $\mathbf{r}_{\hat{X}}^*$ and $\mathbf{r}_{C\setminus\hat{X}}^*$ are independently obtained and the computation can be done in parallel. In Fig. 3, we show how to obtain the egalitarian solution $\mathbf{r}_V^* = (\frac{9}{8}, \frac{3}{8}, \frac{3}{5})$ in Example 2 in a distributed manner. Since the SPLIT algorithm recursively splits *C* into two, it usually

⁷The solution of (2) also coincides with the min-max and max-min solutions [22, Theorem 37]: $\mathbf{r}_V^* = \arg\min\max_{i \in V} \{\frac{r_i}{w_i} : \mathbf{r}_V \in \mathscr{R}_{\mathrm{DC}}(V, H)\}$ and $\mathbf{r}_V^* = \arg\max\min_{i \in V} \{\frac{r_i}{w_i} : \mathbf{r}_V \in \mathscr{R}_{\mathrm{DC}}(V, H)\}.$

 $^{^8 {\}rm For}$ example, the complexity of the SFM algorithm proposed in [24] is in the order of $|V|^5.$

⁹The minimizers of a SFM problem form a lattice, where the maximal and minimal subsets exist, and the maximal minimizer can be determined at the same time when the SFM is solved [13].

¹⁰This is based on the proof of Theorem 1 in Section 6.



Fig. 2. The rate adaptation resulted in the SPLIT algorithm in Example 2: $(0,0,0) \rightarrow (\frac{11}{20},0,0) \rightarrow (1,\frac{11}{20},\frac{11}{20})$ to the egalitarian solution w.r.t. $\mathbf{w}_V = \mathbf{1}$; $(0,0,0) \rightarrow (\frac{3}{5},\frac{1}{5},0) \rightarrow (\frac{9}{8},\frac{3}{8},\frac{3}{5})$ to the egalitarian solution w.r.t. $\mathbf{w}_V = (3,1,3)$;

results in a tree diagram where the splitting is from top to bottom. If parallel computation is allowed, the completion time in each split is dominated by the maximum size of SFM $\max\{|C \setminus \hat{X}|, |\hat{X}|\}$.

Experiment 1. Let |V| vary from 3 to 80. For each value of |V|, the following procedure is repeated for 100 times: (a) randomly generate Z_V ; (b) run the SPLIT algorithm and get the sum-size of SFM $|C \setminus \hat{X}| + |\hat{X}|$ and maximum size of SFM max{ $|C \setminus \hat{X}|, |\hat{X}|$ } and sum them up over recursions, which indicate the completion time of centralized/nonparallel and parallel implementations, respectively. We average the sum-size and maximum size of SFM over repetitions and show them in Fig. 4. It can be seen that the parallel implementation is much faster when terminal/user set V becomes large.

5. CONCLUSION

We proposed the SPLIT algorithm for determining the egalitarian solution, a fair source coding rate vector, in the SW region for the multiterminal lossless DC problem. We proved that the SPLIT algorithm gradually updates a rate vector to the egalitarian solution in strongly polynomial time. We showed how it recursively splits the terminal/user set to allow distributed and parallel computation. We confirmed experimentally that the parallel implementation greatly reduces the completion time as the number of terminals/users grows. We also showed that, compared to the Shapley value, the egalitarian solution balances energy consumed according to the source coding rates, making it a more suitable fairness metric in distributed systems. Our future works include the implementation of SPLIT in a WSN deployment for precision agriculture, and the study of the fairness in the DC problem in a given data gathering tree.

6. PROOF OF THEOREM 1

It is shown in [22, Section 3.4] that problem (2) can be solved by considering the problem $\min\{H(X) - \lambda w(X) : X \subseteq V\}$.



Fig. 3. Distributed implementation of the SPLIT algorithm in Example 2 for obtaining the egalitarian solution $\mathbf{r}_V^* = (\frac{9}{8}, \frac{3}{8}, \frac{3}{5})$ w.r.t. $\mathbf{w}_V = (3, 1, 3)$: The user set $V = \{1, 2, 3\}$ is broken into two subsets $\{1, 2\}$ and $\{3\}$ so that $\mathbf{r}_{\{1,2\}}^*$ and r_3^* can be determined in parallel.



Fig. 4. The result in Experiment 1: The average sum-size and maximum size of SFM for the nonparallel and parallel implementations, respectively, of the SPLIT algorithm.

The minimizer for all $\lambda \in \mathbb{R}$ is fully characterized by $p \leq |V|$ critical points $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_p$ and corresponding set chain $\emptyset = S_0 \subset S_1 \subset \ldots \subset S_p = V$, where S_j is the maximal minimizer of min $\{H(X) - \lambda_j w(X) : X \subseteq V\}$.¹¹ The minimizer \mathbf{r}_V^* of 2 can be obtained by $r_i^* = \lambda_j w_i, \forall i \in S_j \setminus S_{j-1}, j \in \{1, \ldots, p\}$.

The property of λ_j and S_j is derived in [23, Lemmas 5 and 6]: For $j, j' \in \{0, \dots, p\}$: j > j' and $\lambda = \phi(S_j, S_{j'}) = \frac{H(S_j) - H(S_{j'})}{w(S_j \setminus S_{j'})}$, $\lambda = \lambda_j$ if j = j' + 1; $\lambda_{j'+1} < \lambda < \lambda_j$ if j > j' + 1. It apparently suggests an recursive method to determine \mathbf{r}_V^* : For $\lambda = \phi(S_j, S_{j'})$, determine the maximal minimizer \hat{X} of min $\{f(X) - \lambda w(X) : X \subseteq V\}$; If $\hat{X} = S_j$, set $r_i^* = \lambda w_i$ for all $i \in S_j \setminus S_{j'}$ and terminate recursion; Otherwise, repeat the same procedure for $\bar{\lambda} = \phi(S_j, \hat{X})$ and $\underline{\lambda} = \phi(\hat{X}, S_{j'})$. By initiate $S_j = V, S_{j'} = \emptyset$ and f = H, this procedure determines \mathbf{r}_V^* . For S_j and $S_{j'}$, define g(X) = $f(X \sqcup S_{j'}) - f(S_{j'})(\frac{w(X)}{w(S_{j'})} + 1), \forall X \subseteq V' = S_j \setminus S_{j'}$ and $\lambda' = \frac{g(V')}{w(V')}$. It can be shown that $\lambda = \phi(S_j, S_{j'}) - \lambda w(X)$ for all $X \subseteq V'$. Therefore, $\arg \min\{g(X) : X \subseteq V'\} \sqcup$ $S_{j'} = \arg \min\{f(X) - \lambda w(X) : X \subseteq V\}$ and $\lambda \mathbf{w}_{S_j \setminus S_{j'}} =$ $\lambda' \mathbf{w}_{S_j \setminus S_{j'}} + \frac{f(S_{j'})}{w(S_{j'})} \mathbf{w}_{S_j \setminus S_{j'}}$. Theorem holds.

¹¹Also, for all $j \in \{1, ..., p\}$, S_{j-1} is the unique minimizer of $\min \{f(X) - \lambda w(X) \colon X \subseteq V\}$ if $\lambda_{j-1} < \lambda < \lambda_j$.

7. REFERENCES

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