AN IMPROVED ITERATIVE ALGORITHM FOR BAND-LIMITED SIGNAL EXTRAPOLATION ON THE SPHERE

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ABSTRACT

We develop an algorithm for the extrapolation of band-limited signals on the sphere. The proposed algorithm improves the accuracy of the extrapolation of band-limited signal by using the information contained in the out-of-band harmonic coefficients of the signal to update the extrapolated signal at each iteration. The estimation of signals on the sphere from incomplete measurements finds applications in acoustics, cosmology and geophysics. The proposed algorithm does not only exploit the band-limited property of the signal, that is, force the harmonic coefficients outside the band-limit to zero, at each iteration as carried out in the existing algorithms but also uses the harmonic coefficients outside the harmonic domain to improve the accuracy of signal extrapolation. To demonstrate the improvement in the accuracy enabled by the proposed algorithm, we conduct numerical experiments and compare the results of the proposed algorithm with the existing iterative conjugate gradient method.

Index Terms— Extrapolation, band-limited signals, equiangular sampling, spherical harmonic transform, unit sphere.

1. INTRODUCTION

Signal processing on the sphere finds applications in the field of acoustics [1], cosmology [2], computer graphics [3] and geophysics [4], to name a few. In all these applications, processing of signals on the sphere requires harmonic analysis which is enabled by well-known spherical harmonic transform(SHT). For the exact or accurate computation of SHT, many algorithms have been proposed in literature [5–7] which require samples on the whole sphere. However, there are applications in geophysics and acoustics where the measurements cannot be taken over the whole sphere. For example, in acoustics, head related transfer function (HRTF) measurements are not reliable in the South polar cap region due to reflections from the ground. Another example is the problem of polar gap in geophysics where the inclination of satellite orbit makes the satellite measurements on poles unreliable. To address the issue of unreliable and inaccessible samples on the sphere in these applications, we consider the problem of signal extrapolation on the sphere in this work.

In literature, many algorithms have been proposed for extrapolation of band-limited signals on the sphere [1, 8–11]. An analog of Papoulis algorithm [12] for continuous signals on the sphere exploiting the band-limiting characteristics of the signal is proposed in [1] and its integral equation formulation is developed in [8]. For discrete signals on the sphere, an iterative algorithm is presented in [9], which converges to minimum norm least-squares solution. Conjugate gradient extrapolation algorithm on the sphere has been presented in [10], which in comparison to the previously proposed algorithms, enables more accurate and fast extrapolation. [13] uses the slepian functions [14] to develop an iterative algorithm for the extrapolation of band-limited signal on the sphere in the presence of noise.

In this work, we develop an iterative algorithm for the signal extrapolation over the inaccessible region on the sphere. The proposed method takes samples according to the equiangular sampling scheme which supports exact computation of SHT on the sphere. Existing schemes focus on the use of the band-limited property of the signal, that is, the signal extrapolation is carried out iteratively by forcing the harmonic coefficients outside the band-limit of the signal to zero at each iteration. In the proposed algorithm, we do not only force the harmonic coefficients to zero but also use these to improve the extrapolation of the signal over the inaccessible region at each iteration. We conduct numerical experiments and compare the accuracy of the proposed algorithm with iterative conjugate gradient algorithm proposed in [10]. We also take HRTF measurements using synthetic head model [15], extrapolate the signal on the South pole and show that the proposed algorithm enables more accurate extrapolation than the existing methods.

The rest of the paper is organized as follows. The necessary mathematical background and choice of samples using equiangular scheme are reviewed in Section 2. We pose the problem, develop the formulation and present the proposed algorithm in Section 3. Numerical experiments and accuracy analysis are carried out in Section 4. Finally, the concluding remarks are made in Section 5.

2. PRELIMINARIES

2.1. Signal on Sphere

Unit sphere is defined as $\mathbb{S}^2 \triangleq \{\hat{\boldsymbol{x}} \in \mathbb{R}^3 \colon |\hat{\boldsymbol{x}}| = 1\}$ where $\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}(\theta, \phi)$ represents a point on the unit sphere, parameterized by $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ with co-latitude $\theta \in [0, \pi]$ and longitude $\phi \in [0, 2\pi)$. We consider complex-valued square integrable functions such as f and g defined on the sphere. These functions form a complex Hilbert space, denoted by $L^2(\mathbb{S}^2)$, equipped with the inner product defined as [16]

$$\langle f, g \rangle \triangleq \int_{\mathbb{S}^2} f(\hat{\boldsymbol{x}})\overline{g(\hat{\boldsymbol{x}})}\, ds(\hat{\boldsymbol{x}}), \tag{1}$$

where $ds(\hat{\boldsymbol{x}}) = \sin\theta\, d\theta\, d\phi$ denotes the differential area element and $\overline{(\cdot)}$ represents the complex conjugate operation. The finite energy

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function $f \in L^2(\mathbb{S}^2)$ with $\|f\| \triangleq \langle f, f\rangle^{1/2} < \infty$ are referred as signals on the sphere. Using the Fredholm integral equation as [16], also define a linear integral operator $\mathcal{D}$ with kernel $D(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ as [16]

$$(\mathcal{D}f)(\hat{\boldsymbol{x}}) = \int_{\mathbb{S}^2} D(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})\, f(\hat{\boldsymbol{y}}) ds(\hat{\boldsymbol{y}}). \tag{2}$$

2.2. Harmonic Analysis

For Hilbert space $L^2(\mathbb{S}^2)$, spherical harmonic functions or spherical harmonics, denoted by $Y_\ell^m(\hat{\boldsymbol{x}})$ for degree $\ell \geq 0$ and order $|m| \leq \ell$ form a complete orthonormal set of basis functions and therefore any function f on the sphere can be expanded as

$$f(\hat{\boldsymbol{x}}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (f)_\ell^m Y_\ell^m(\hat{\boldsymbol{x}}), \tag{3}$$

where

$$(f)_\ell^m \triangleq \langle f, Y_\ell^m \rangle = \int_{\mathbb{S}^2} f(\hat{\boldsymbol{x}}) \overline{Y_\ell^m(\hat{\boldsymbol{x}})}\, ds(\hat{\boldsymbol{x}}), \tag{4}$$

is the spherical harmonic coefficient of degree ℓ and order m and equation (4) is referred to as spherical harmonic transform (SHT) (4). The signal $f \in L^2(\mathbb{S}^2)$ is considered as band-limited at degree L if $(f)_\ell^m = 0$ for $\ell \geq L$. The band-limited signals form an L^2 dimensional subspace of $L^2(\mathbb{S}^2)$, which is denoted by $\mathcal{H}_L$. For band-limited signals, the summation in (3) is truncated at $L-1$ and we express the double summation in 3 as $\sum_{\ell=0}^{L-1} \sum_{m=-\ell}^{\ell} = \sum_{\ell m}^{L-1}$.

2.3. Sampling on the Sphere

In literature, many sampling distributions have been devised for discretization of signals on the sphere. Among these sampling schemes, we adopt equiangular sampling scheme proposed in [7] as it requires least number of samples for exact computation of SHT of a band-limited signal on the sphere. We use Ω_M to denote the set of equiangular sampling points taken on M iso-latitude rings placed at the following co-latitude positions

$$\theta_t = \frac{2\pi t}{2M-1}, \quad t = 0, 1, 2, ..., M-1, \tag{5}$$

with $2M$-1 samples along longitude in each ring given by

$$\phi_p = \frac{2\pi p}{2M-1}, \quad p = 0, 1, 2, ..., 2M-2. \tag{6}$$

We note that the SHT of the signal band-limited at M can be evaluated *exactly* by taking samples of the signal over Ω_M [7].

3. PROPOSED EXTRAPOLATION ALGORITHM

3.1. Problem Formulation

Samples are taken over the whole sphere for the accurate computation of SHT or signal reconstruction. However, in some applications, samples over some region of the sphere cannot be taken due to practical limitations [1, 11]. These applications require signal processing methods or algorithms to extrapolate the signal over the inaccessible region [1, 8–11]. We consider the same problem in this work and propose an iterative algorithm for signal extrapolation which, in comparison with the existing methods, enables more accurate extrapolation.

For a band-limited signal $f \in L^2(\mathbb{S}^2)$ with maximum spherical harmonic degree L, we assume that the measurements or samples are available over some region $\mathcal{R} \subset \mathbb{S}^2$. We assume that the measurements are not available or zero over inaccessible region $\mathcal{R}^c = \mathbb{S}^2 \backslash R \subset \mathbb{S}^2$. For a spatial region $\mathcal{R}$, we define a space-limiting operator $D_\mathcal{R}$ with kernel given by

$$D_\mathcal{R}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \triangleq I_\mathcal{R}(\hat{\boldsymbol{x}}) \delta(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}), \tag{7}$$

where $\delta(x, y)$ denotes the Dirac delta function [16] and

$$I_\mathcal{R}(\hat{\boldsymbol{x}}) \triangleq \begin{cases} 1 & \hat{\boldsymbol{x}} \in \mathcal{R}, \\ 0 & \hat{\boldsymbol{x}} \in \mathcal{R}^c, \end{cases} \tag{8}$$

is an indicator function of the region R. Using the operator $D_\mathcal{R}$, we define the problem under consideration is to extrapolate the signal $f \in \mathcal{H}_L$ when only $f_\mathcal{R}(\hat{\boldsymbol{x}}) \triangleq (D_\mathcal{R} f)(\hat{\boldsymbol{x}})$ is known and $f_{\mathcal{R}^c}(\hat{\boldsymbol{x}}) \triangleq (D_{\mathcal{R}^c} f)(\hat{\boldsymbol{x}})$ is unreliable or not known. With these definitions, we can express f as

$$f(\hat{\boldsymbol{x}}) = f_\mathcal{R}(\hat{\boldsymbol{x}}) + f_{\mathcal{R}^c}(\hat{\boldsymbol{x}}), \tag{9}$$

with representation in harmonic domain given by

$$(f)_\ell^m = (f_\mathcal{R})_\ell^m + (f_{\mathcal{R}^c})_\ell^m. \tag{10}$$

We note that the signals $f_\mathcal{R}$ and $f_{\mathcal{R}^c}$ are not band-limited due to the space-limiting operation.

3.2. Proposed Signal Extrapolation - Formulation

To reconstruct the original signal $f(\hat{\boldsymbol{x}})$, we assume that the signal is sampled over the sampling grid Ω_M where we assume $M > L$[1]. Since the original signal f is band-limited at L, we have $(f)_\ell^m = 0$ for all $\ell \geq L$, equation (10) implies

$$(f_{\mathcal{R}^c})_\ell^m = -(f_\mathcal{R})_\ell^m, \quad L \leq \ell < M,\ |m| \leq \ell. \tag{11}$$

We also define function $h(\hat{\boldsymbol{x}})$ as

$$h(\hat{\boldsymbol{x}}) \triangleq f_{\mathcal{R}^c}(\hat{\boldsymbol{x}}) I_\mathcal{R}(\hat{\boldsymbol{x}}), \tag{12}$$

which can be written in the harmonic domain as

$$(h)_\ell^m = \sum_{\ell' m'}^{M-1} (f_{\mathcal{R}^c})_{\ell'}^{m'} (Z)_{\ell',\ell}^{m',m}, \tag{13}$$

where

$$\begin{aligned} (Z)_{\ell',\ell}^{m',m} &= (I_\mathcal{R}(\hat{\boldsymbol{x}}) Y_{\ell'}^{m'}(\hat{\boldsymbol{x}}))_\ell^m, \\ &= \sum_{\ell'' m''}^{M-1} (I_{\mathcal{R}^c})_{\ell''}^{m''} T_{\ell,m}^{\ell',m',\ell'',m''}. \end{aligned} \tag{14}$$

[1]Due to the fact that the known signal $f_\mathcal{R}$ is not band-limited.

Here

$$T_{\ell,m}^{\ell',m',\ell'',m''} = \int_{\mathbb{S}^2} Y_{\ell'}^{m'}(\hat{\boldsymbol{x}}) Y_{\ell''}^{m''}(\hat{\boldsymbol{x}}) \overline{Y_{\ell}^{m}(\hat{\boldsymbol{x}})}\, ds(\hat{\boldsymbol{x}}). \tag{15}$$

3.3. Proposed Signal Extrapolation - Algorithm

Since $f_{\mathcal{R}^c} = 0$ by definition, we have $h(\hat{\boldsymbol{x}}) = 0$ as defined in (12), which implies $(h)_\ell^m = 0$. Consequently, we have the following system

$$\sum_{\ell' m'}^{M-1} (f_{\mathcal{R}^c})_{\ell'}^{m'} (Z)_{\ell',\ell}^{m',m} = 0, \tag{16}$$

which can be equivalently expressed as

$$\mathbf{Z}\mathbf{f} = 0, \tag{17}$$

where $\mathbf{Z}$ is an $M^2 \times M^2$ matrix containing all the spherical harmonic coefficients of the expression derived in (14) and $\mathbf{f}$ is vector of length M^2 containing spherical harmonic coefficients $(f)_\ell^m$. During the construction of $\mathbf{f}$ in our proposed algorithm, we use (11) to replace the unknown coefficients with the negative of the known coefficients in each iteration. In order to solve the system in (17), we divide the matrix $\mathbf{Z}$ and vector $\mathbf{f}$ into two partitions of different sizes namely $\mathbf{Z}_a, \mathbf{Z}_b$ and $\mathbf{f}_a, \mathbf{f}_b$ respectively. The system now is

$$[\mathbf{Z}_a | \mathbf{Z}_b] \left[\frac{\mathbf{f}_a}{\mathbf{f}_b}\right] = 0. \tag{18}$$

Note that we ensure the system proposed in (18) is always overdetermined by selecting a suitable band-limit which is $M = \lceil \sqrt{2} L \rceil$. Using (11) and (18), the unknown coefficients can be determined by

$$\mathbf{f}_a = \Lambda \mathbf{f}_b, \tag{19}$$

where $\Lambda = (-1)(\mathbf{Z}_a^T \mathbf{Z}_a)^{-1} \mathbf{Z}_a^T \mathbf{Z}_b$. In our algorithm, we take samples of f over the equiangular sampling grid Ω_M where $M > L$. We first pre-compute $(Z)_{\ell',\ell}^{m',m}$ using (14) and form matrix $\mathbf{Z}$ to determine Λ. We apply the space limiting operator defined in (7) to get space-limited function $\hat{f}(\hat{\boldsymbol{x}})$. In order to find the vector $\mathbf{f}$ in (17), we take the SHT of space-limited function $\hat{f}(\hat{\boldsymbol{x}})$, first modify it using (11) and then modify it again by updating $\mathbf{f}_a$ after computing $\mathbf{f}_a$ using (19). We take the inverse SHT of $\mathbf{f}$ and update the space-limited function. We use the same procedure iteratively for K number of iterations and summarize the evaluation of the unknown coefficients in the form of procedure 1 given below[2].

4. ANALYSIS

In this section, we conduct two numerical experiments to illustrate the accuracy of the proposed iterative extrapolation method. We compare the proposed algorithm with the the iterative conjugate gradient method [10]. In both the experiments, we consider the accessible region as

$$\mathcal{R} = \left\{ \hat{\boldsymbol{x}}(\theta,\phi) \in \mathbb{S}^2 | 0 \le \theta \le \pi - \theta_c \right\}, \tag{20}$$

[2]We represent the inverse SHT of $\mathbf{f}$ as $\hat{g}(\hat{\boldsymbol{x}})$.

Procedure 1 Iterative Extrapolation

Require: $(f)_\ell^m, \quad \forall\, 0 \le \ell < L,\ |m| \le \ell$

procedure ITERATIVE EXTRAPOLATION($f_{\mathcal{R}}(\hat{\boldsymbol{x}})$)
  $\hat{f}(\hat{\boldsymbol{x}}) = f_{\mathcal{R}}(\hat{\boldsymbol{x}})$
  compute $\mathbf{Z}$ using (14) and evaluate Λ
  for $k = 1, 2, \ldots, K$ **do**
    compute $\mathbf{f}$ from $\hat{f}(\hat{\boldsymbol{x}})$ using SHT
    update $\mathbf{f}$ using (11)
    compute $\mathbf{f}_a$ using (19)
    update $\mathbf{f}$ with $\mathbf{f}_a$
    compute $\hat{g}(\hat{\boldsymbol{x}})$ as inverse SHT of $\mathbf{f}$
    update $\hat{f}(\hat{\boldsymbol{x}}) \leftarrow \hat{f}(\hat{\boldsymbol{x}}) + (D_{\mathcal{R}^c}\hat{g})(\hat{\boldsymbol{x}})$
  end for
  evaluate $(f)_\ell^m$ by taking SHT of $\hat{f}(\hat{\boldsymbol{x}})$
  return $(f)_\ell^m$
end procedure

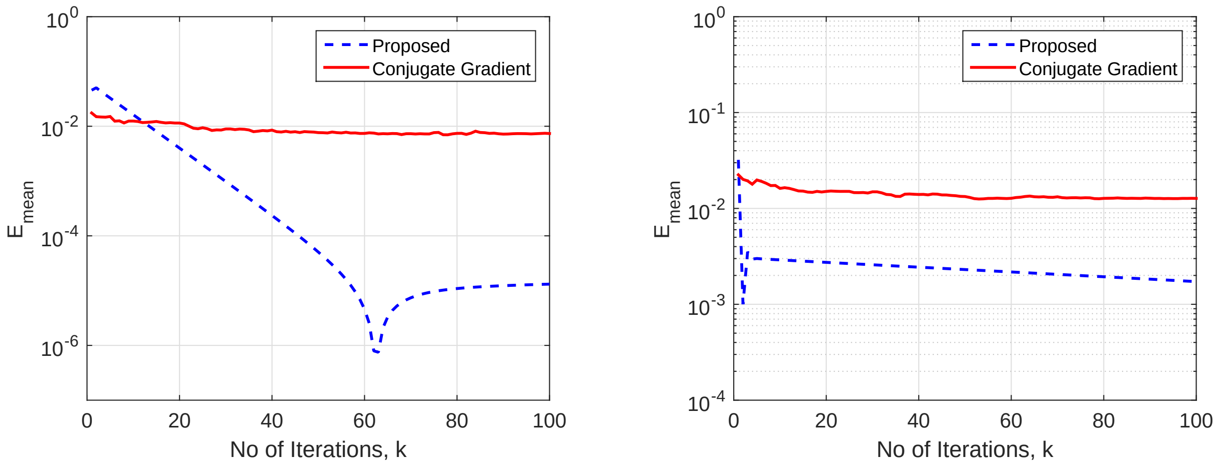
where θ_c represents the excluded polar cap region, that is, it represents the region where the measurements are unreliable or unaccessible. In both the experiments, we compute the mean extrapolation error defined as

$$E_{\text{mean}} \triangleq \frac{1}{L^2} \sum_{\ell,m}^{L-1} |(f)_\ell^m - (\hat{f})_\ell^m|, \tag{21}$$

where we take the mean of the absolute difference between actual $(f)_\ell^m$ and the extrapolated values $(\hat{f})_\ell^m$ in the harmonic domain.

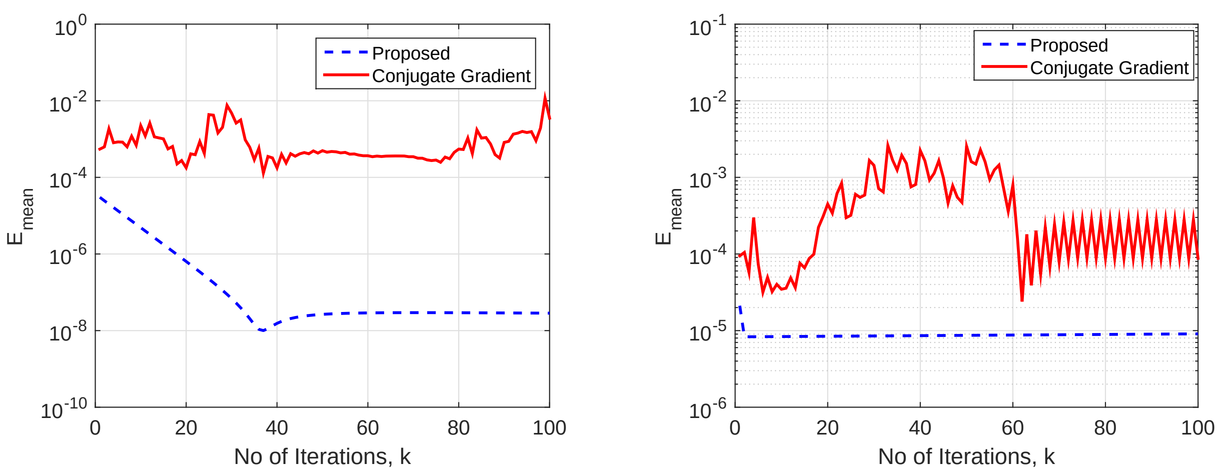
Experiment 1: In the first experiment, we consider random complex valued band-limited test signal with band-limit $L = 30$. We generate such test signal first randomly selecting $(f)_\ell^m$ with real and imaginary parts uniformly distributed in the interval [-1,1] and then synthesizing signal over Ω_M using (3). For different values of $\theta_c = \pi/8, \pi/6$, and $M = 60$, we apply the proposed algortihm and iterative conjugate gradient method to extrapolate the signal in region $\mathcal{R}^c$ and compute the error as defined in (21). We run both the algorithms for $K = 100$ number of iterations and plot the mean error in Fig. 1, where it is evident that the proposed algorithm enables more accurate extrapolation.

Experiment 2: In the second experiment, we apply the proposed method to extrapolate the head related transfer function (HRTF) on the sphere. We use spherical head model [15] to obtain synthetic HRTF data for the following parameters:head radius, $a = 0.09$ m, distance from head, $r = 1$ m, audible frequency range, $f_r = [5, 20]$ kHz, and speed of sound, $c = 340$ ms^{-1}. The effective HRTF band-limit is estimated by $L(\lambda) = \lceil \frac{e\pi a f_r}{c} \rceil + 1$, where λ is a wave number and is directly proportional to the frequency f_r. HRTF measurements are not reliable in South polar cap region due to the reflections from the ground and hence the samples cannot be taken on the South pole. For a given frequency, f_r, we obtain the HRTF measurements, $h(\hat{\boldsymbol{x}})$ over the sampling grid Ω_M and compute spherical harmonic coefficients, $(h)_\ell^m$ using (4). We then apply the proposed algorithm to extrapolate the signal in the region beyond $\theta_c = \pi/6$ and compute the mean error as defined in (21). The results of extrapolation of HRTF measurements of different frequencies $f_r = 5$ kHz,10 kHz, effective band-limits, $L = 15, 27$ and $M = 40, 70$ for $K = 100$ using the proposed and iterative con-



(a) (b)

Fig. 1: Mean extrapolation error E_{mean} given in (21), for Experiment 1, for a random signal, band-limited at $L = 30$ and sampled over Ω_M with $M = 60$ for (a) $\theta_c = \pi/8$ and (b) $\theta_c = \pi/6$.



(a) (b)

Fig. 2: Mean extrapolation error E_{mean} given in (21), for Experiment 2, for HRTF measurements over $\mathcal{R}$ with $\theta_c = \pi/6$ taken at (a) frequency $f_a = 5kHz$, effective band-limit $L = 15$ and $M = 40$ and (b) frequency f_r=10kHz, effective band-limit $L = 27$ and $M = 70$.

jugate gradient method are plotted in Fig. 2. Again the numerical analysis reveals that the proposed method gives more accurate result than the well-known iterative conjugate gradient method.

5. CONCLUSIONS

In this work , we develop an iterative algorithm for extrapolation of band-limited signals on the sphere from limited or incomplete measurements. Existing schemes focus on the use of the band-limited property of the signal, that is, the signal extrapolation is carried out iteratively by forcing the harmonic coefficients outside the band-limit of the signal to zero at each iteration. In the proposed algorithm, we do not only force the harmonic coefficients to zero but also use these to improve the extrapolation of the signal over the inaccessible region at each iteration. We conduct numerical experiments in order to check the accuracy of the proposed algorithm and use Iterative Conjugate Gradient method as benchmark for comparison. We also take HRTF measurements using synthetic head model and extrapolate the signal on the South pole. The numerical analysis show that the proposed algorithm enables more accurate extrapolation than the existing methods.

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