DESIGN OF OPTIMAL ENTROPY-CONSTRAINED UNRESTRICTED POLAR QUANTIZER FOR BIVARIATE CIRCULARLY SYMMETRIC SOURCES

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ABSTRACT

This paper proposes an algorithm for the design of entropy-constrained unrestricted polar quantizer (ECUPQ) for bivariate circularly symmetric sources. The algorithm is globally optimal for the class of ECUPQs with magnitude quantizer thresholds confined to a finite set. The optimization problem is formulated as the minimization of a weighted sum of the distortion and entropy and the proposed solution is based on modeling the problem as a minimum-weight path problem in a certain weighted directed acyclic graph. The proposed algorithm enables solving the overall problem in $O(K^2 \log |\hat{\mathcal{P}}|)$ time, where K is the size of the set of possible magnitude thresholds and $\hat{\mathcal{P}}$ is the set of the number of phase levels for the uniform phase quantizers.

***Index Terms*—** Unrestricted polar quantization, globally optimal solution, entropy-constrained quantizer, minimum-weight path problem.

1. INTRODUCTION

A polar quantizer quantizes the magnitude and the phase of a two dimensional source vector represented in polar coordinates. The phase quantizer is uniform while the magnitude quantizer may be nonuniform. Polar quantization of bivariate sources with circularly symmetric densities, has been extensively investigated either for the general case or for the specific Gaussian case [2]– [11].

Early work on polar quantization uses independent quantizers for the two components. Such a scheme is referred in the literature as a *strictly* polar quantizer (SPQ) [4] or a *conventional* polar quantizer [7]. In later work the *unrestricted* polar quantizer is introduced (UPQ) [5], where the phase quantizer depends on the magnitude level, and UPQ is shown to outperform SPQ.

Most of the work on the analysis and design of polar quantizers relies on the high resolution assumption. In particular, the asymptotic analysis of the uniform polar quantizers, i.e., where the quantizer of the magnitude is also uniform, was performed in [6, 8, 11] for the conventional case and in [9] for the unrestricted case. On the other hand, the design of optimal practical polar quantizers, i.e., without the high rate assumption, was considered in [3, 4] for the level-constrained SPQ and in [5] for the level-constrained UPQ.

Further, to increase the efficiency of the polar quantizer, entropy coding may be applied to the quantizer's outputs. This was done, for instance, in [5]. However, for optimal performance the polar quantizer has to be optimized under a constraint on the entropy. Such a quantizer is called entropy-constrained quantizer. Work [10] is the only work addressing the design of entropy-constrained polar quantizers, up to our knowledge. The authors of [10] derive the optimal entropy-constrained UPQ (ECUPQ) and the optimal entropy-constrained SPQ (ECSPQ) using high resolution assumptions. However, the asymptotic expression cannot be applied to rates smaller than $\log_2(2\pi e) \approx 4.1$ bits/pair.

This paper formulates the problem of optimal ECUPQ design as the minimization of the Lagrangian for a given multiplier λ, which is the same formulation as in [10]. Thus, the cost function is actually a weighted sum of the quantizer distortion and entropy. This formulation readily simplifies the problem of rate allocation between the magnitude quantizer and phase quantizers. The proposed algorithm models the design problem as a minimum-weight path (MWP) problem in a certain weighted directed acyclic graph (WDAG), where each edge represents a possible bin of the magnitude quantizer. In order to expedite the computation of all weights we develop a fast strategy for finding the optimal number of phase levels for all possible magnitude bins. The overall running time of the solution algorithm is $O(K^2 \log |\hat{\mathcal{P}}|)$, where K is the size of the set from which the magnitude thresholds are selected, while $\hat{\mathcal{P}}$ is the set of the number of phase levels corresponding to a magnitude bin.

We point out that the design approach based on modeling the problem as an MWP problem in some WDAG, with or without a constraint on the number of edges, has been used in the past for the design of other scalar quantizer systems. For instance, it was employed for the design of fixed-rate quantizers [12], entropy-constrained and Wyner-Ziv quantizers [13, 14], multi-resolution and multiple description quantizers [13–17], joint source-channel quantizer with random index assignment [18], as well as quantizers for sequential source coding [19]. However, the problem we address here is significantly different than the problems considered in the above mentioned work, and as a consequence, the graph to model the problem is different. Precisely, the graph in our work is different in structure (i.e., in terms of vertexes and edges) than the graphs used for the multiple description problem. Furthermore, although our graph may have similar nodes and edges as the graphs employed for some of the other problems mentioned above, the edge weights are different, because the meaning assigned to an edge is different.

The main contribution of our work include: a) We propose the first globally optimal ECUPQ design algorithm for finite rates. The optimality claim holds for the class of UPQs with the thresholds for the magnitude quantizer restricted to a finite set; b) Practical results for a bivariate memoryless Gaussian source show that at small rates our algorithm considerably outperforms the best entropy-coded and entropy-constrained UPQ schemes known to date; c) Our proposed algorithm is the first algorithm for practical polar quantizer design, which handles efficiently the problem of rate allocation between the magnitude and phase quantizers.

The rest of the paper is organized as follows. The next section

A longer version [1] of this work will be published in *IEEE Trans. Communications.*

introduces the notations and formulates the optimization problem. Section 3 shows how the problem can be modeled as the MWP in some WDAG. Section 4 finalizes the design algorithm and presents the pseudocode. The experimental results are presented in Section 5, and Section 6 concludes the paper.

2. PROBLEM FORMULATION

Consider a bivariate random variable with the following circularly symmetric density, as a function of the polar coordinates r and θ,

$$p(r,\theta)=\frac{1}{2\pi}g(r),\ 0\le r<\infty,\ 0\le\theta<2\pi.$$

Note that $g(r)$ is the marginal probability density function (pdf) of the magnitude variable, while the phase variable is uniformly distributed over the interval $[0,2\pi)$. Additionally, notice that the magnitude and phase variables are independent. An example of such a variable is a two-dimensional memoryless Gaussian vector (X_1,X_2), i.e., where X_1 and X_2 are independent and have identical marginal pdfs.

Let M denote the number of magnitude levels of the UPQ and let $\mathbf{r}\triangleq(r_1,r_2,\cdots,r_{M-1})$ denote the vector of thresholds of the magnitude quantizer, i.e., $r_0=0<r_1<r_2<\cdots<r_{M-1}<r_M=\infty$. For $1\le m\le M$, let C_m denote the m-th cell (or bin) of the magnitude quantizer, i.e., $C_m=\{r|r_{m-1}\le r<r_m\}$. Further, let $\mathbf{P}\triangleq(P_1,P_2,\cdots,P_M)$, where P_m denotes the number of phase regions of the phase quantizer corresponding to C_m, $1\le m\le M$. Each phase quantizer is uniform, consequently, each quantization bin of the UPQ can be represented as

$$\mathcal{R}(m,s)=\left\{re^{j\theta}|r_{m-1}\le r<r_m,(s-1)\frac{2\pi}{P_m}\le\theta<s\frac{2\pi}{P_m}\right\},$$

for $1\le m\le M$, and $1\le s\le P_m$. Clearly, the total number of quantization bins of the UPQ is $N=\sum_{m=1}^{M}P_m$.

The reconstruction for quantizer bin $\mathcal{R}(m,s)$ is $A_me^{j\theta_{m,s}}$, where A_m is the reconstruction value of the magnitude for the m-th magnitude level, and $\theta_{m,s}$ is the reconstruction value for the phase.

We will use the squared error as a distortion measure. Therefore, the expected distortion of the UPQ can be expressed as [2,3,5]

$$\begin{aligned}D&=\sum_{m=1}^{M}\sum_{s=1}^{P_m}\int_{r_{m-1}}^{r_m}\int_{(s-1)\frac{2\pi}{P_m}}^{s\frac{2\pi}{P_m}}\|re^{j\theta}-A_me^{j\theta_{m,s}}\|^2p(r,\theta)d\theta dr\\&=\sum_{m=1}^{M}\sum_{s=1}^{P_m}\int_{r_{m-1}}^{r_m}\int_{(s-1)\frac{2\pi}{P_m}}^{s\frac{2\pi}{P_m}}(r^2+A_m^2-2rA_m\cos(\theta-\theta_{m,s}))\frac{g(r)}{2\pi}d\theta dr.\end{aligned}$$

The best reconstruction values, which minimize the distortion, were determined in prior work [2,3,5] by solving $\partial D/\partial\theta_{m,s}=0$ and $\partial D/\partial A_m=0$, leading to

$$\theta_{m,s}=(2s-1)\pi/P_m, \tag{1}$$

$$A_m=sinc\left(\tfrac{1}{P_m}\right)\frac{\int_{r_{m-1}}^{r_m}rg(r)dr}{\int_{r_{m-1}}^{r_m}g(r)dr}, \tag{2}$$

where $sinc(\frac{1}{P_m})=\frac{\sin(\pi/P_m)}{\pi/P_m}$. By exploiting (1) and (2), the expected distortion can be simplified as

$$D=\int_0^\infty r^2g(r)dr-\sum_{m=1}^{M}A_m^2\int_{r_{m-1}}^{r_m}g(r)dr. \tag{3}$$

Clearly, since the reconstruction values of the UPQ are given by (1) and (2), it follows that the tuples $\mathbf{r}$ and $\mathbf{P}$ completely specify the UPQ.

Let I_a and I_θ denote the random variables representing the magnitude and phase quantization indexes, respectively. Then the entropy of the UPQ equals the joint entropy of (I_a,I_θ), denoted by $H(I_a,I_\theta)=H(I_a)+H(I_\theta|I_a)$, which can be expressed as follows

$$H(I_a,I_\theta)=\sum_{m=1}^{M}q(m)(-\log_2q(m)+\log_2P_m), \tag{4}$$

where $q(m)=\int_{r_{m-1}}^{r_m}g(r)dr$ is the probability of cell C_m.

We formulate the problem of ECUPQ design as the minimization of Lagrangian as follows

$$\begin{aligned}&\min_{M,\mathbf{r},\mathbf{P}}\ \ \mathcal{L}(\mathbf{r},\mathbf{P},\lambda),\\&\text{subject to } r_i\in\mathcal{A},1\le i\le M-1.\end{aligned} \tag{5}$$

for fixed Lagrangian multiplier $\lambda>0$, where $\mathcal{L}(\mathbf{r},\mathbf{P},\lambda)\triangleq D+\lambda H(I_a,I_\theta)$, and $\mathcal{A}=\{a_1,a_2,\cdots,a_K\}$ is a finite set from which the magnitude thresholds of the UPQ are selected.

Note that the problem (5) will be solved with the thresholds of the magnitude quantizer confined to the finite set $\mathcal{A}$. This set can be obtained by finely discretizing the interval $[0,B]$ for some B chosen such that the probability that the magnitude level is larger than B to be sufficiently small.

It is known [20,21] that the set of solutions to problem (5), when λ varies over $(0,\infty)$, is the set of UPQs such that the corresponding pair $(H(I_a,I_\theta),D)$ is on the lower boundary of the convex hull of the set of all possible pairs $(H(I_a,I_\theta),D)$. Thus, a UPQ which is a solution to problem (5) minimizes the distortion for the corresponding entropy, therefore it is an ECUPQ.

3. GRAPH MODEL

In this section we show how the minimization problem (5) can be modeled as an MWP problem in a certain WDAG. Notice that the first term in (3) is constant, therefore we can remove it from the cost function. Then, minimizing $\mathcal{L}(\mathbf{r},\mathbf{P},\lambda)$ becomes equivalent to minimizing $C(\mathbf{r},\mathbf{P})$, where

$$C(\mathbf{r},\mathbf{P})\triangleq-\sum_{m=1}^{M}A_m^2\int_{r_{m-1}}^{r_m}g(r)dr+\lambda H(I_a,I_\theta).$$

Further, substituting (2) and (4) into the above equation leads to

$$C(\mathbf{r},\mathbf{P})=\sum_{m=1}^{M}\int_{r_{m-1}}^{r_m}g(r)dr\left(-sinc^2\left(\frac{1}{P_m}\right)x_m^2+\lambda\log_2\frac{P_m}{\int_{r_{m-1}}^{r_m}g(r)dr}\right), \tag{6}$$

where $x_m=\frac{\int_{r_{m-1}}^{r_m}rg(r)dr}{\int_{r_{m-1}}^{r_m}g(r)dr}$.

Now it can be seen that if the vector of thresholds $\mathbf{r}$ is fixed, then P_m can be optimized separately for each m. Specifically, the optimal value of P_m, $1\le m\le M$, is

$$P_m^*=\arg\min_{P_m}\left(-sinc^2\left(\frac{1}{P_m}\right)x_m^2+\lambda\log_2P_m\right),$$

since $\int_{r_{m-1}}^{r_m}g(r)dr$ and x_m are fixed, for fixed $\mathbf{r}$.

Consider now the following notations. For each $0\le\alpha<\beta\le\infty$, denote

$$q(\alpha,\beta)\triangleq\int_\alpha^\beta g(r)dr,$$

$$x(\alpha,\beta) \triangleq \frac{\int_\alpha^\beta rg(r)dr}{\int_\alpha^\beta g(r)dr},$$

$$P^*_{(\alpha,\beta)} \triangleq \min\arg\min_P \left(-sinc^2\left(\frac{1}{P}\right)x(\alpha,\beta)^2 + \lambda\log_2 P\right), \tag{7}$$

where the minimization is over all positive integers P. Note that, if there are more values P minimizing the cost in (7), we select the smallest one as $P^*_{(\alpha,\beta)}$.

Further, by replacing P_m in (6) by $P^*_{(r_{m-1},r_m)}$, we obtain a new cost function which only depends on $\mathbf{r}$, denoted by $\bar{C}(\mathbf{r})$, where

$$\bar{C}(\mathbf{r}) \triangleq \sum_{m=1}^M q(r_{m-1},r_m)\left(\lambda\log_2\frac{P^*_{(r_{m-1},r_m)}}{q(r_{m-1},r_m)} - sinc^2\left(\frac{1}{P^*_{(r_{m-1},r_m)}}\right)x(r_{m-1},r_m)^2\right). \tag{8}$$

As a consequence, problem (5) is equivalent to the following

$$\min_{M,\mathbf{r}} \ \bar{C}(\mathbf{r}) \quad \text{subject to } r_i \in \mathcal{A}, 1 \le i \le M-1. \tag{9}$$

The next step is based on the observation that the cost $\bar{C}(\mathbf{r})$ can be expressed as a summation of costs of the individual intervals (r_{m-1}, r_m), fact which allows us to regard it as the weight of a path in a certain WDAG, as we show next.

Let us assume that the elements of $\mathcal{A}$ are labeled in increasing order, i.e., $0 < a_i < a_{i+1}$, for $1 \le i \le K-1$. Additionally, let us denote $a_0 = 0$ and $a_{K+1} = \infty$. Construct now the WDAG $G = (V, E, w)$, where $V = \{0, 1, 2, \cdots, K+1\}$ is the vertex set, and $E = \{(u,v) \in V^2 \mid 0 \le u < v \le K+1\}$ is the edge set. Further, the weight of each edge (u, v) is defined as follows,

$$w(u,v) \triangleq q(a_u,a_v)\left(-sinc^2\left(\frac{1}{\hat{P}(u,v)}\right)x(a_u,a_v)^2 + \lambda\log_2\frac{\hat{P}(u,v)}{q(a_u,a_v)}\right), \tag{10}$$

where we simplify the notation $P^*_{(a_u,a_v)}$ to $\hat{P}(u,v)$.

The source node in this graph is vertex 0 and the final node is $K+1$. A path in this graph from some node u to some node v is any sequence of connected edges starting at u and ending at v. Clearly, any path from the source to the final node can be represented as an $(s+1)$-tuple of vertexes $\mathbf{t} = (t_0, t_1, \cdots, t_s)$, satisfying $t_0 = 0$, $t_s = K+1$ and $t_{m-1} < t_m$, $1 \le m \le s$, for some $s \ge 1$. Note that s equals the number of edges on the path. Let us denote by $\mathcal{T}(s)$ the set of all paths from the source to the final node with exactly s edges, for each $s \ge 1$. The weight $W(\mathbf{t})$ of path $\mathbf{t}$ is defined as the sum of the weights of its edges, i.e.,

$$W(\mathbf{t}) \triangleq \sum_{i=1}^s w(t_{i-1}, t_i).$$

Let us associate now to each $(M-1)$-tuple of thresholds $\mathbf{r}$, with components from the set $\mathcal{A}$, where $M \ge 1$, the M-edge path $\mathbf{t} \in \mathcal{T}(M)$, such that $r_m = a_{t_m}$ for each $1 \le m \le M-1$. In other words the m-th edge on this path, which is (t_{m-1}, t_m), corresponds to the m-th magnitude cell $[r_{m-1}, r_m)$. Then we see that the weight of path $\mathbf{t}$ equals the cost $\bar{C}(\mathbf{r})$. Additionally, the above correspondence is one-to-one. Therefore, we conclude that problem (9) is equivalent to the MWP problem in the graph G, i.e., the problem of finding the minimum-weight path , from the source to the final node.

In the following section we present an efficient algorithm to evaluate the weight of each edge and finalize the solution to problem (5).

4. SOLUTION ALGORITHM

Let us assume we know some value P_{max} such that $\hat{P}(u,v) \le P_{max}$, for all $(u,v) \in E$, and further denote $\mathcal{P} \triangleq \{1, 2, 3, \cdots, P_{max}\}$. Additionally, we denote $f(y) = -sinc^2(\frac{1}{y})$ and $g(y) = \ln y$ for any $y > 0$, and consider the following minimization problem

$$\min_{P\in\mathcal{P}} (f(P) + \mu g(P)), \tag{11}$$

where $\mu > 0$. It can be easily verified from (7) that $\hat{P}(u,v)$ is a solution to problem (11) for $\mu = \frac{\lambda}{x(a_u,a_v)^2 \ln 2}$.

For $P \in \mathcal{P}$ let us denote by $S(P)$ the point in the plane of coordinates $(g(P), f(P))$. Additionally, let $\mathcal{U}$ denote the set of points $\{S(P)|P \in \mathcal{P}\}$. It is known [20, 21] that some value P^* minimizes the cost in (11) if and only if the point $S(P^*)$ is situated on the lower boundary of the convex hull of $\mathcal{U}$, and the line of slope $-\mu$ passing through $S(P^*)$ is a support line to $\mathcal{U}$.

Note that any point $S(P)$ lies on the lower boundary of the convex hull of $\mathcal{U}$ is called an extreme point. Moreover, the set of extreme points of $\mathcal{U}$ is the union of line segments connecting any two consecutive extreme points, since the set $\mathcal{U}$ is finite. Any such line segment is called a convex hull edge. Let $\hat{\mathcal{P}}$ denote the set of integers $P \in \mathcal{P}$ such that $S(P)$ is an extreme point of $\mathcal{U}$. For each $P \in \hat{\mathcal{P}}$, except for the first and the last ones, further denote by $left\_slope(P)$ (respectively, $right\_slope(P)$) the slope of the convex hull edge to the left (respectively, right) of $S(P)$, i.e., connecting $S(P)$ with the previous (respectively, next) extreme point. Finally, the condition that the line of slope $-\mu$ passing through some extreme point $S(P)$ is a support line to $\mathcal{U}$, is equivalent to the relation $left\_slope(P) \le -\mu \le right\_slope(P)$.

As a result, we obtain the following property of $\hat{P}(u,v)$.

Lemma. For each $(u,v) \in E$, the value $\hat{P}(u,v)$ equals the smallest $P \in \hat{\mathcal{P}}$ satisfying

$$left\_slope(P) \le -\frac{\lambda}{x(a_u,a_v)^2 \ln 2} \le right\_slope(P). \tag{12}$$

The above lemma implies that $\hat{P}(u,v)$ can be found using a binary search over the set $\hat{\mathcal{P}}$, and it is proved in [1] that $\hat{\mathcal{P}} = \mathcal{P} \setminus \{2\}$. Moreover, we can set $P_{max} = \hat{P}(K, K+1)$ [1], which can be found by inspecting all positive integers $P \in \hat{\mathcal{P}}$, in increasing order until relation (12) is satisfied.

Algorithm 1 on the following page describes the algorithm to solve problem (5) including the procedure for determining the values $\hat{P}(u,v)$. We point out that $\hat{W}(v)$ denotes the weight of the minimum-weight path from the source to node v, and $\varepsilon(v)$ records the node preceding v on this optimal path. At the end, the MWP can be tracked back by utilizing the values of $\varepsilon(v)$. For simplicity, we denote $E(P,u,v) \triangleq \left(-sinc^2\left(\frac{1}{P}\right)x(a_u,a_v)^2 + \lambda\log_2 P\right)$.

In order to enable the computation of each edge weight in constant time, the following cumulative probabilities and first moments need to be precomputed and stored during the preprocessing step,

$$Cum_i(u) = Cum_i(u-1) + \int_{a_{u-1}}^{a_u} r^i g(r)dr,$$

for $i = 0, 1$, and $1 \le u \le K+1$, where $a_0 = 0$, $a_{K+1} = \infty$ and $Cum_i(0) = 0$ by convention. Therefore, assuming that the evaluation of each integral $\int_{a_{u-1}}^{a_u} r^i g(r)dr$ takes constant time, the computation of all these cumulative values takes $O(K)$ time.

Algorithm 1 Solution algorithm for problem (5).

$\hat{W}(0) = 0$
for $v = 1$ **to** $K + 1$ **do**
  $\hat{P}(v-1, v) := \min \arg\min_{P \in \hat{\mathcal{P}}} E(P, v-1, v)$
  $\hat{W}(v) := \hat{W}(v-1) + w(v-1, v)$
  $\varepsilon(v) := v - 1$
  for $u = v - 2$ **down to** 0 **do**
    $\hat{P}(u, v) := \min \arg\min_{P \in \hat{\mathcal{P}}} E(P, u, v)$
    if $\hat{W}(u) + w(u, v) < \hat{W}(v)$ **do**
      $\hat{W}(v) := \hat{W}(u) + w(u, v)$
      $\varepsilon(v) := u$
    end if
  end for
end for
Restoring the MWP by back-tracking the values of $\varepsilon(v)$.

Based on these values, when the weight of edge (u, v) is needed, the quantities $q(a_u, a_v)$ and $x(a_u, a_v)$ will be computed in $O(1)$ time.

Finally, by applying binary search for each graph edge leads to a time complexity $O(K^2 \log |\hat{\mathcal{P}}|)$ of solving the problem (5).

5. EXPERIMENTAL RESULTS

This section assesses the practical performance of the proposed ECUPQ design algorithm and compares it with the designs of [5] and [10]. The experiments are conducted for a two-dimensional random vector (X_1, X_2), where X_1 and X_2 are independent and identically distributed Gaussian variables with zero-mean and unit-variance, with the following joint pdf in polar coordinates

$$p(r, \theta) = \frac{r}{2\pi} \exp\left(-\frac{r^2}{2}\right),\ 0 \le r < \infty,\ 0 \le \theta < 2\pi,$$

where $r = \sqrt{x_1^2 + x_2^2}$, and $\theta = \tan^{-1}(x_2/x_1)$. It then follows that $g(r) = r\exp(-r^2/2)$.

The finite set of possible thresholds $\mathcal{A}$ is obtained by dividing the range $[0, 6]$ into subintervals of size 0.001 and picking the thresholds between intervals. In other words, $K = 6000$ and $a_i = 0.001i$, for $1 \le i \le K$. Moreover, we set $P_{max} = 600$ in the optimization of the number of phase regions. In order to design an ECUPQ achieving some target rate R_t we run the algorithm for various values of λ until the entropy of the UPQ becomes sufficiently close to R_t. We use D to denote the distortion of the proposed approach computed using (3). The distortion is converted in dB using $10\log_{10} D$. The rate R is computed as the entropy of the UPQ, i.e., $H(I_a, I_\theta)$.

The comparison against the entropy-coded UPQ of [5] is performed for rates in the range from 1 to 5 bit/pair based on the results reported in [5]. The comparison with the asymptotically optimal ECUPQ of [10] is performed for rates higher than 4.1. Moreover, we also list the gap between the ECUPQ distortion and the distortion-rate function $D_G(R)$ of the circularly symmetric Gaussian variable, i.e., $D_G(R) = 2 \times 2^{-R}$.

Table 1 illustrates the comparison with entropy-coded UPQ in [5]. Note that all the results related to the UPQs of [5] are taken from [5]. It can be seen that our algorithm always outperforms the design of [5] with gains always higher than 0.2 dB, and even larger than 0.6 dB when $R \ge 3$. Additionally, a peak improvement of 0.755 dB is achieved for $R = 4.512$ bit/pair. Note that the gap away from $D_G(R)$ takes values between 0.883 dB at rate $R = 1$ and 1.527 dB at $R = 4.990$ bit/pair.

Table 1: Performance comparison of the proposed ECUPQ with the entropy-coded UPQ of [5] and $D_G(R)$, for rates $R \le 5$ bit/pair.

Rate	$10\log_{10} D$	$10\log_{10} D^{[5]}$	$10\log_{10} \frac{D^{[5]}}{D}$	$10\log_{10} \frac{D}{D_G(R)}$
1.000	0.883	1.348	0.465	0.883
1.585	−0.550	−0.334	0.216	1.211
2.000	−1.681	−1.391	0.290	1.328
2.556	−3.295	−2.941	0.354	1.391
3.139	−4.987	−4.271	0.716	1.450
3.508	−6.079	−5.436	0.643	1.473
3.895	−7.224	−6.615	0.609	1.492
4.512	−9.058	−8.303	0.755	1.515
4.990	−10.486	−9.763	0.723	1.527

Table 2: Performance comparison of the proposed ECUPQ with ASY, PASY and $D_G(R)$, for rates $R \ge \log_2(2\pi e)$ bit/pair.

Rate	$10\log_{10} D$	$10\log_{10} D_{ASY}$	$10\log_{10} D_{PASY}$	$10\log_{10} \frac{D}{D_G(R)}$
4.100	−7.832	−7.800	−7.213	1.501
4.512	−9.058	−9.041	−8.481	1.515
4.990	−10.486	−10.480	−9.973	1.527
5.996	−13.500	−13.507	−13.144	1.539
6.995	−16.506	−16.514	−16.287	1.541
8.000	−19.532	−19.539	−19.408	1.540
9.000	−22.547	−22.549	−22.481	1.538
9.990	−25.528	−25.530	−25.493	1.536
10.992	−28.544	−28.546	−28.528	1.536
11.991	−31.550	−31.553	−31.542	1.536

Next we compare the performance of the proposed design scheme with the ECUPQ optimized in [10]. We will use the acronym ASY to refer to the asymptotical ECUPQ performance derived in [10], and the acronym PASY to refer to the practical ECUPQ based on the asymptotic point density functions given in [10]. Table 2 illustrates the performance of the proposed algorithm in comparison with ASY and PASY for several rates in the range 4.1 to 12 bit/pair, where $D_{ASY} = 2^{-(R - \log_2(2\pi e))}/6$, for rates $R \ge \log_2(2\pi e) \approx 4.094$.

We see that the proposed algorithm performs extremely close to ASY. Specifically, for the rates higher than 4.99 the absolute value of the performance difference is smaller than 0.01 dB, while for the rates lower than 4.99, our design is actually slightly better reaching improvements of up to 0.032 dB. Additionally, it can be observed that the proposed algorithm outperforms PASY for all rates examined. The performance improvement is between 0.5 and 0.619 dB for rates up to 4.99. The gap gradually decreases as the rate increases, but it still remains higher than 0.1 dB for rates up to 8.0. Finally, for $R \approx 12$ the gap falls below 0.01 dB. Moreover, the difference in performance versus the $D_G(R)$ is also presented in Table 2, where the gap ranges from 1.501 dB to 1.541 dB.

6. CONCLUSIONS

This paper focuses on the design of entropy-constrained unrestricted polar quantizer for bivariate circularly symmetric sources. We propose a design algorithm which is globally optimal when the thresholds of the magnitude quantizer are confined to a finite set. The algorithm consists of solving the minimum-weight path problem in a certain weighted directed acyclic graph, in conjunction with the procedure to find the optimal number of phase regions for each possible magnitude quantizer bin. The experimental results show significant improvements over the prior practical designs at low rates, and performance very close to the optimal asymptotical performance.

7. REFERENCES

[1] H. Wu and S. Dumitrescu, "Design of optimal entropy-constrained unrestricted polar quantizer for bivariate circularly symmetric sources", to appear in *IEEE Trans. Commun.*.

[2] G. H. Senge, "Quantization of image transforms with minimum distortion", *Technical Report No. ECE-77-8,* Dept. of Elec. and Comp. Eng., University of Wisconsin, Madison, WI, Jun. 1997.

[3] N. C. Gallagher, Jr., "Quantizing schemes for the discrete Fourier transform of a random time-series," *IEEE Trans. Inform. Theory*, vol. IT-24, no. 2, pp. 156-163, Mar. 1978.

[4] W. A. Pearlman, "Polar quantization of a complex Gaussian random variable", *IEEE Trans. Commun.,* vol. COM-27, no. 6, pp. 892-899, Jun. 1979.

[5] S. G. Wilson, "Magnitude/phase quantization of independent Gaussian variates", *IEEE Trans. Commun.,* vol. COM-28, no. 11, pp. 1924-1929, Nov. 1980.

[6] P. F. Swaszek, "Uniform spherical coordinate quantization of spherically symmetric sources," *IEEE Trans. Commun.*, vol. COM-33, no. 6, pp. 518-521, Jun. 1985.

[7] D. L. Neuhoff, "Polar quantization revisited," in *Proc. IEEE Int. Symp. Inform. Theory (ISIT 1997),* pp. 60, Ulm, Germany, Jun. 1997.

[8] P. W. Moo and D. L. Neuhoff, "Uniform Polar Quantization Revisited", in *Proc. IEEE Int. Symp. Inform. Theory (ISIT 1998),* pp. 100, Cambridge, MA, USA, Aug. 1998.

[9] Z. H. Peric and M. C. Stefanovic, "Asymptotic analysis of optimal uniform polar quantization," *AEU Int. J. Electron. Commun.,* vol. 56, no. 5, pp. 345-347, 2002.

[10] R. Vafin and W. B. Kleijn, "Entropy-constrained polar quantization and its application to audio coding", *IEEE Trans. Speech and Audio Process.,* vol. 13, no. 2, pp. 220-232, Mar. 2005.

[11] A. Z. Jovanovic, Z. H. Peric, J. R. Nikolic and M. R. Dincic, "Asymptotic analysis and design of restricted uniform polar quantizer for Gaussian sources", *Digital Signal Process.,* vol. 49, pp. 24-32, Feb. 2016.

[12] A. Aggarwal, B. Schieber, and T. Tokuyama, "Finding a minimum weight k-link path in graphs with the concave monge property and applications," *Discr. Computat. Geometry,* vol. 12, no. 1, pp. 263-280, Dec. 1994.

[13] D. Muresan and M. Effros, "Quantization as histogram segmentation: globally optimal scalar quantizer design in network systems", in *Proc. Data Compression Conf. (DCC 2002)*, pp. 302-311, Snowbird, UT, USA, Apr. 2002.

[14] D. Muresan and M. Effros, "Quantization as Histogram Segmentation: Optimal Scalar Quantizer Design in Network Systems," *IEEE Trans. Inform. Theory,* vol. 54, no. 1, pp. 344-366, Jan. 2008.

[15] S. Dumitrescu and X. Wu, "Optimal Multiresolution Quantization for Scalable Multimedia Coding," in *Proc. IEEE Inform. Theory Workshop (ITW 2002),* pp. 139-142, Bangalore, India, Oct. 2002.

[16] S. Dumitrescu and X. Wu, "Optimal two-description scalar quantizer design," *Algorithmica,* vol. 41, no. 4, pp. 269-287, Feb. 2005.

[17] S. Dumitrescu and X. Wu, "Lagrangian optimization of two-description scalar quantizers," *IEEE Trans. Inform. Theory,* vol. 53, no. 11, pp. 3990-4012, Nov. 2007.

[18] S. Dumitrescu, "On the Design of Optimal Noisy Channel Scalar Quantizer with Random Index Assignment," *IEEE Trans. Inform. Theory,* vol. 62, no. 2, pp. 724-735, Feb. 2016.

[19] H. Wu and S. Dumitrescu, "Design of optimal entropy-constrained scalar quantizer for sequential coding of correlated sources," in *Proc. IEEE Inform. Theory Workshop (ITW 2017)*, pp. 524-528, Kaohsiung, Taiwan, Nov. 2017.

[20] H. Everett III, "Generalized lagrange multiplier method for solving problems of optimum allocation of resources," *Operat. Res.,* vol. 11, no. 3, pp. 399-417, Jun. 1963.

[21] D. G. Luenberger, *Optimization by Vector Space Methods*. New York: Wiley, 1969.