GENERALIZED UNCERTAINTY PRINCIPLES FOR THE TWO-SIDED QUATERNION LINEAR CANONICAL TRANSFORM

Yan-Na, Zhang [1,2] *Bing-Zhao, Li* [1,2,⋆]

1. School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 102488, P.R. China
2. Beijing Key Laboratory on MCAACI, Beijing Institute of Technology, Beijing 100081, P.R. China
E-mail:li_bingzhao@bit.edu.cn

ABSTRACT

An uncertainty principle (UP), which offers information about a function and its Fourier transform (FT) in the time-frequency plane, is particularly powerful in the field of signal processing. In this paper, based on the fundamental relationship between the quaternion linear canonical transform (QLCT) and quaternion Fourier transform (QFT), we propose two different UPs related to the two-sided QLCT. Different from existing results in the spatial and frequency domains, new derived consequences can be regarded as a general form of the UP of the QLCT, which present lower bounds for the product of spreads of a quaternion-valued function in two different QLCT domains.

Index Terms— Uncertainty principle, Quaternion linear canonical transform, Quaternion Fourier transform

1. INTRODUCTION

An uncertainty principle (UP), which provides a lower bound on the spreads of two specific transform domains, is of importance in various scientific fields such as mathematics, signal processing and information theory [1]. In quantum mechanics, one UP demonstrates that the impossibility of simultaneous precise measurements of a particle's momentum and its position. In signal processing, it states that a signal cannot be simultaneously sharply located in both time and frequency domains. On the mathematical side, the classical Heisenberg UP in the time-frequency plane is given by [2]

$$\int_{\mathbb{R}} (t-t_0)^2 \left|f(t)\right|^2 dt \cdot \int_{\mathbb{R}} (w-w_0)^2 \left|F(w)\right|^2 dw \geq \frac{1}{4}\left(\int_{\mathbb{R}} \left|f(t)\right|^2 dt\right)^2 \tag{1}$$

where

$$t_0 = \frac{1}{E}\int_{\mathbb{R}} t\left|f(t)\right|^2 dt \tag{2}$$

$$w_0 = \frac{1}{E}\int_{\mathbb{R}} w\left|F(w)\right|^2 dw \tag{3}$$

Herein, $E = \int_{\mathbb{R}} \left|f(t)\right|^2 dt = \int_{\mathbb{R}} \left|F(w)\right|^2 dw$ denotes the energy of $f(t)$. $F(w)$ denotes the Fourier transform (FT) of the signal $f(t)$. t_0 and w_0 stand for the mean time and mean frequency, respectively. With loss of generality, let $t_0 = w_0 = 0$ since UPs do not depend on the location of means. The UP in Eq.(1) describes that the product of spreads of a given signal $f(t)$ in the time and frequency domains is limited by a lower bound. In terms of the importance of the UP in mathematics, physics, optics and signal processing, there are many efforts to extend it into various types of functions and integral transforms in one dimensional form [2–7].

The linear canonical transform (LCT) [8] is a three free parameter class of linear integral transformation, which includes many important integral transforms as its special cases such as the FT, the fractional Fourier transform (FRFT), the Fresnel transform (FST), the Lorentz transform (LT), and other transforms. It is an effective tool for chirp signals and used widely in various fields of optics and signal processing [8, 9]. The UP in Eq.(1) has been generalized for a complex signal $f(t)$ with unit energy in two different LCT domains [7]

$$\int_{\mathbb{R}} u^2 \left|F_{A_1}(t)\right|^2 dt \cdot \int_{\mathbb{R}} v^2 \left|F_{A_2}(u)\right|^2 dw \geq \frac{(a_1 b_2 - a_2 b_1)^2}{4} \tag{4}$$

where F_{A_i} denotes the LCT of the signal $f(t)$ with the parameter matrix $A_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \in \mathbb{R}^{2\times 2}, i = 1, 2$ limited to $det(A_i) = 1$.

Recently, it has become popular to extend integral transforms from real and complex numbers to quaternion algebra to study higher dimension, for instance, the quaternion Fourier transform (QFT) [10], the fractional quaternion Fourier

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transform (FRQFT) [11], the quaternion wavelet transform (QWT) [12], the quaternion linear canonical transform (QLCT) [13, 14] and others [15]. Many properties of the classical FT have been extended for quaternion types including shift, modulation, differentiation, energy conservation, convolution and correlation, UP and so on. The conventional UP in Eq.(1) has been generalized for the right-sided QFT firstly by Bahri [10] in the name of component-wise UP, see **Lemma** 1. The work of Bahri motivates a large amount of attempts on generalizing various UPs from complex FT into the QFT setting [16–18]. Since the LCT has also been investigated with the context of quaternion algebra, and played a vital role in the representation of hyper-complex signal [13], it is natural to ask what the UPs in the QLCT setting obey? To the best of our knowledge, most of existing works are focused on lower bounds in the spatial and LCT frequency domains [13, 19–21]. However, the UP in Eq.(4) with respect to arbitrary two different QLCT domains have not derived yet. It is therefore interesting and worthwhile to investigate these kinds of UPs and obtain more valuable results associated with the QLCT.

The rest of this paper is organized as follows. Section 2 provides a brief introduction to some general definitions of quaternion algebra and the QLCT. New UPs in two different QLCT domains are given in Section 3. Finally, Section 4 concludes this paper.

2. PRELIMINARIES

2.1. Quaternion algebra

The quaternion algebra [10] was first invented by W. R. Hamilton in 1843 and is denoted by $\mathbb{H}$ in his honor. Every element of $\mathbb{H}$ can be written in the following form

$$\mathbb{H} = \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} | q_0, q_1, q_2, q_3 \in \mathbb{R}\} \tag{5}$$

where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ are imaginary units and obey Hamilton's multiplication rules

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \mathbf{ki} = -\mathbf{ik} = \mathbf{j} \tag{6}$$

For a quaternion number $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{H}$, q_0 is called as the scalar part of q and denoted by $Sc(q)$. $q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is named as the vector (or pure) part of q and conventionally notated by $\mathbf{q}$. The conjugate $\bar{q}$ of a quaternion q is given by

$$\bar{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k} \tag{7}$$

which is an anti-involution, i.e.,

$$\overline{qp} = \bar{p}\bar{q} \tag{8}$$

According to Eq.(7), the norm or modulus of $q \in \mathbb{H}$ is defined by

$$|q| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \tag{9}$$

and the inverse of $q \in \mathbb{H} \setminus \{0\}$ is

$$q^{-1} = \frac{\bar{q}}{|q|^2} \tag{10}$$

A quaternion-valued function $f : \mathbb{R}^2 \to \mathbb{H}$ can be expressed as

$$\begin{aligned} f(\mathbf{x}) = f_0(\mathbf{x}) + f_1(\mathbf{x})\mathbf{i} + f_2(\mathbf{x})\mathbf{j} + f_3(\mathbf{x})\mathbf{k}, \\ f_0(\mathbf{x}), f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}) \in \mathbb{R} \end{aligned} \tag{11}$$

2.2. The quaternion linear canonical transform

Definition 1 *Let $A_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \in \mathbb{R}^{2\times 2}$ be a matrix parameter satisfying $det(A_i) = 1$, for $i = 1, 2$. The two-sided QLCT of $f \in L^1(\mathbb{R}^2, \mathbb{H})$ is defined by [13, 14]*

$$\mathcal{L}^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{u}) = \int_{\mathbb{R}^2} K^{\boldsymbol{i}}_{A_1}(x_1, u_1) f(\boldsymbol{x}) K^{\boldsymbol{j}}_{A_2}(x_2, u_2) d\boldsymbol{x} \tag{12}$$

where the kernel functions of the QLCT above are given by

$$K^{\boldsymbol{i}}_{A_1}(x_1, u_1) = \begin{cases} \frac{1}{\sqrt{2\pi b_1}} e^{\boldsymbol{i}\left(\frac{a_1}{2b_1}x_1^2 - \frac{x_1 u_1}{b_1} + \frac{d_1}{2b_1}u_1^2 - \frac{\pi}{4}\right)}, & b_1 \neq 0 \\ \sqrt{d_1} e^{\boldsymbol{i}\frac{c_1 d_1}{2}u_1^2}, & b_1 = 0 \end{cases}$$

and

$$K^{\boldsymbol{j}}_{A_2}(x_2, u_2) = \begin{cases} \frac{1}{\sqrt{2\pi b_2}} e^{\boldsymbol{i}\left(\frac{a_2}{2b_2}x_2^2 - \frac{x_2 u_2}{b_2} + \frac{d_2}{2b_2}u_2^2 - \frac{\pi}{4}\right)}, & b_2 \neq 0 \\ \sqrt{d_2} e^{\boldsymbol{i}\frac{c_2 d_2}{2}u_2^2}, & b_2 = 0 \end{cases}$$

Here, $\mathbf{x} = x_1\mathbf{e_1} + x_2\mathbf{e_2}$ and $\mathbf{u} = u_1\mathbf{e_1} + u_2\mathbf{e_2}$. $\mathbf{e_1}$ and $\mathbf{e_2}$ are the unit 2D vectors and orthogonal to each other. It is noted that for $b_i = 0, i = 1, 2$ the QLCT of a signal is essentially a chirp multiplication and it is of no particular interest for our objective in this work. Hence, without loss of generality, we set $b_i \neq 0$ in the following section unless stated otherwise. Under some suitable conditions, the QLCT above is invertible and the inversion is given in **Definition** 2.

Definition 2 *Suppose $f \in L^1(\mathbb{R}^2, \mathbb{H})$, then the inversion of the QLCT of f is given by [13, 14]*

$$\begin{aligned} f(\mathbf{x}) &= \{\mathcal{L}^{\mathbb{H}}_{A_1,A_2}\}^{-1}\{f\}(\mathbf{x}) \\ &= \int_{\mathbb{R}^2} K^{-\mathbf{i}}_{A_1}(x_1, u_1) \mathcal{L}^{\mathbb{H}}_{A_1,A_2}\{f\}(\mathbf{u}) K^{-\mathbf{j}}_{A_2}(x_2, u_2) d\mathbf{u} \end{aligned} \tag{13}$$

Remark 1 *When $A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, Eq.*(12) *and Eq.*(13) *reduce to the QFT and inverse QFT of $f(\mathbf{x})$.*

Many other important properties of the QLCT have been derived [13, 14] and an application of the QLCT to study the generalized swept-frequency filters was presented in [22]. To emphasize, in this paper, we mainly investigate UP on the two-sided QLCT. Without explanation, $\mathcal{F}$ denotes the two-sided QFT operator and $\mathcal{L}^{\mathbb{H}}_{A_1,A_2}$ is the two-sided QLCT operator.

Lemma 1 *The conventional UP in Eq.*(1) *has been generalized for the right-sided QFT firstly by Bahri [10] in the name of component-wise UP, which takes the form*

$$\int_{\mathbb{R}^2} x_k^2 \left|f(\mathbf{x})\right|^2 d\mathbf{x} \cdot \int_{\mathbb{R}^2} u_k^2 \left|\mathcal{F}_r\left\{f\right\}(\mathbf{u})\right|^2 d\mathbf{u} \geq \frac{1}{4}\left(\int_{\mathbb{R}^2} \left|f(\mathbf{x})\right|^2 d\mathbf{x}\right)^2, k=1,2 \tag{14}$$

where $\mathbf{x}, \mathbf{u} \in \mathbb{R}^2, \mathbf{x} = x_1\mathbf{e_1} + x_2\mathbf{e_2}, \mathbf{u} = u_1\mathbf{e_1} + u_2\mathbf{e_2}$*.* $\mathcal{F}_r$ *is the right-sided QFT operator.*

The result in **Lemma** 1 provides a lower bound on the product of effective widths of a quaternion-valued signal in the spatial and frequency domains, which is also valid for the left-sided QFT and two-sided QFT.

3. MAIN RESULTS

In this section, some novel results of UPs are provided.

Theorem 1 *Let* $f \in L^1\left(\mathbb{R}^2, \mathbb{H}\right) \cap L^2\left(\mathbb{R}^2, \mathbb{H}\right)$ *be a quaternion-valued signal with unit energy, then we derive*

$$\int_{\mathbb{R}^2} u_1^2 \left|\mathcal{L}^{\mathbb{H}}_{A_1,A_2}\left\{f\right\}(\mathbf{u})\right|^2 d\mathbf{u} \cdot \int_{\mathbb{R}^2} v_1^2 \left|\mathcal{L}^{\mathbb{H}}_{A_3,A_4}\left\{f\right\}(\mathbf{v})\right|^2 d\mathbf{v} \geq \frac{\left(a_3b_1 - a_1b_3\right)^2}{4} \tag{15}$$

and

$$\int_{\mathbb{R}^2} u_2^2 \left|\mathcal{L}^{\mathbb{H}}_{A_1,A_2}\left\{f\right\}(\mathbf{u})\right|^2 d\mathbf{u} \cdot \int_{\mathbb{R}^2} v_2^2 \left|\mathcal{L}^{\mathbb{H}}_{A_3,A_4}\left\{f\right\}(\mathbf{v})\right|^2 d\mathbf{v} \geq \frac{\left(a_4b_2 - a_2b_4\right)^2}{4} \tag{16}$$

Proof 1 *Firstly, without loss of generality, we let* $F = \int_{\mathbb{R}^2} \left|f(\mathbf{x})\right|^2 d\mathbf{x} = 1$ *and* $b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + b_6^2 \neq 0$*. Set*

$$\mathcal{F}\{g\}(\mathbf{u}) = e^{-\mathbf{i}\frac{d_5}{2b_5}u_1^2}\mathcal{L}^{\mathbb{H}}_{A_1,A_2}\left\{f\right\}(\mathbf{u})e^{-\mathbf{j}\frac{d_6}{2b_6}u_2^2} \tag{17}$$

and

$$g(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^2}}\int_{\mathbb{R}^2} e^{\mathbf{i}x_1u_1}\mathcal{F}\left\{g\right\}(\mathbf{u})e^{\mathbf{j}x_2u_2}d\mathbf{u} \tag{18}$$

According to ***Lemma*** *1, we derive*

$$\int_{\mathbb{R}^2} x_k^2 \left|g(\mathbf{x})\right|^2 d\mathbf{x} \cdot \int_{\mathbb{R}^2} u_k^2 \left|\mathcal{F}\left\{g\right\}(\mathbf{u})\right|^2 d\mathbf{u} \geq \frac{1}{4}\left(\int_{\mathbb{R}^2} \left|g(\mathbf{x})\right|^2 d\mathbf{x}\right)^2, k=1,2 \tag{19}$$

Note the fact that

$$\begin{aligned}\left|\mathcal{F}\{g\}(\mathbf{u})\right| &= \left|e^{-\mathbf{i}\frac{d_5}{2b_5}u_1^2}\mathcal{L}^{\mathbb{H}}_{A_1,A_2}\left\{f\right\}(\mathbf{u})e^{-\mathbf{j}\frac{d_6}{2b_6}u_2^2}\right| \\ &= \left|\mathcal{L}^{\mathbb{H}}_{A_1,A_2}\left\{f\right\}(\mathbf{u})\right|\end{aligned} \tag{20}$$

We get

$$\int_{\mathbb{R}^2} u_k^2 \left|\mathcal{F}\left\{g\right\}(\mathbf{u})\right|^2 d\mathbf{u} = \int_{\mathbb{R}^2} u_k^2 \left|\mathcal{L}^{\mathbb{H}}_{A_1,A_2}\left\{f\right\}(\mathbf{u})\right|^2 d\mathbf{u} \tag{21}$$

From the definition of the QFT, we obtain

$$\begin{aligned}& g\left(\frac{x_1}{b_5}, \frac{x_2}{b_6}\right) \\ =& \frac{1}{\sqrt{(2\pi)^2}}\int_{\mathbb{R}^2} e^{\mathbf{i}\frac{x_1}{b_5}u_1}\mathcal{F}\left\{g\right\}(\mathbf{u})e^{\mathbf{j}\frac{x_2}{b_6}u_2}d\mathbf{u} \\ =& \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\mathbf{i}\frac{x_1}{b_5}u_1}e^{-\mathbf{i}\frac{d_5}{2b_5}u_1^2}\mathcal{L}^{\mathbb{H}}_{A_1,A_2}\left\{f\right\}(\mathbf{u})e^{-\mathbf{j}\frac{d_6}{2b_6}u_2^2}e^{\mathbf{j}\frac{x_2}{b_6}u_2}d\mathbf{u} \\ =& \sqrt{b_5b_6}e^{\mathbf{i}\frac{a_5}{2b_5}x_1^2 - \mathbf{i}\frac{\pi}{4}} \\ & \cdot \int_{\mathbb{R}^2} K^{-\mathbf{i}}_{A_5}(x_1,u_1)\mathcal{L}^{\mathbb{H}}_{A_1,A_2}\left\{f\right\}(\mathbf{u})K^{-\mathbf{j}}_{A_6}(x_2,u_2)d\mathbf{u} \cdot e^{\mathbf{j}\frac{a_6}{2b_6}x_2^2 - \mathbf{j}\frac{\pi}{4}} \\ =& \sqrt{b_5b_6}e^{\mathbf{i}\frac{a_5}{2b_5}x_1^2 - \mathbf{i}\frac{\pi}{4}} \\ & \cdot \mathcal{L}^{\mathbb{H}}_{A_5^{-1},A_6^{-1}}\left\{\mathcal{L}^{\mathbb{H}}_{A_1,A_2}\left\{f\right\}(\mathbf{u})\right\}(\mathbf{x}) \cdot e^{\mathbf{j}\frac{a_6}{2b_6}x_2^2 - \mathbf{j}\frac{\pi}{4}} \\ =& \sqrt{b_5b_6}e^{\mathbf{i}\frac{a_5}{2b_5}x_1^2 - \mathbf{i}\frac{\pi}{4}} \cdot \mathcal{L}^{\mathbb{H}}_{A_5^{-1}A_1,A_6^{-1}A_2}\left\{f\right\}(\mathbf{x}) \cdot e^{\mathbf{j}\frac{a_6}{2b_6}x_2^2 - \mathbf{j}\frac{\pi}{4}}\end{aligned}$$

Here, the additive property of the QLCT is used in the last equality. Furthermore,

$$\left|g\left(\frac{x_1}{b_5}, \frac{x_2}{b_6}\right)\right|^2 = b_5b_6\left|\mathcal{L}^{\mathbb{H}}_{A_5^{-1}A_1,A_6^{-1}A_2}\left\{f\right\}(\mathbf{x})\right|^2 \tag{22}$$

Set

$$A_3 = A_5^{-1}A_1 = \begin{bmatrix} d_5 & -b_5 \\ -c_5 & a_5 \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} \tag{23}$$

$$A_4 = A_6^{-1}A_2 = \begin{bmatrix} d_6 & -b_6 \\ -c_6 & a_6 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_4 & b_4 \\ c_4 & d_4 \end{bmatrix} \tag{24}$$

then we have $b_5 = a_3b_1 - a_1b_3$ *and* $b_6 = a_4b_2 - a_2b_4$*. In addition, we have*

$$\begin{aligned}&\int_{\mathbb{R}^2} x_1^2 \left|g(\mathbf{x})\right|^2 d\mathbf{x} \\ &= \int_{\mathbb{R}^2}\left(\frac{x_1}{b_5}\right)^2\left|g\left(\frac{x_1}{b_5}, \frac{x_2}{b_6}\right)\right|^2 \cdot \frac{1}{b_5b_6}d\mathbf{x} \\ &= \int_{\mathbb{R}^2}\left(\frac{x_1}{b_5}\right)^2 \cdot \left|\mathcal{L}^{\mathbb{H}}_{A_3,A_4}\left\{f\right\}(\mathbf{x})\right|^2 d\mathbf{x}\end{aligned} \tag{25}$$

That is to say,

$$\begin{aligned}&\int_{\mathbb{R}^2} x_1^2 |g(\mathbf{x})|^2 d\mathbf{x} \cdot \int_{\mathbb{R}^2} u_1^2 |\mathcal{F}\{g\}(\mathbf{u})|^2 d\mathbf{u} \\ &= \int_{\mathbb{R}^2} \left(\frac{x_1}{b_5}\right)^2 \left|\mathcal{L}^{\mathbb{H}}_{A_3,A_4}\{f\}(\mathbf{x})\right|^2 d\mathbf{x} \\ &\quad \cdot \int_{\mathbb{R}^2} u_1^2 \left|\mathcal{L}^{\mathbb{H}}_{A_1,A_2}\{f\}(\mathbf{u})\right|^2 d\mathbf{u} \geq \frac{1}{4}\end{aligned} \tag{26}$$

After a simple calculation, we derive

$$\begin{aligned}&\int_{\mathbb{R}^2} x_1^2 \left|\mathcal{L}^{\mathbb{H}}_{A_3,A_4}\{f\}(\mathbf{x})\right|^2 d\mathbf{x} \cdot \int_{\mathbb{R}^2} u_1^2 \left|\mathcal{L}^{\mathbb{H}}_{A_1,A_2}\{f\}(\mathbf{u})\right|^2 d\mathbf{u} \\ &\qquad \geq \frac{(a_3b_1 - a_1b_3)^2}{4}\end{aligned} \tag{27}$$

In a similar way, it is easy to derive

$$\begin{aligned}&\int_{\mathbb{R}^2} x_2^2 \left|\mathcal{L}^{\mathbb{H}}_{A_3,A_4}\{f\}(\mathbf{x})\right|^2 d\mathbf{x} \cdot \int_{\mathbb{R}^2} u_2^2 \left|\mathcal{L}^{\mathbb{H}}_{A_1,A_2}\{f\}(\mathbf{u})\right|^2 d\mathbf{u} \\ &\qquad \geq \frac{(a_4b_2 - a_2b_4)^2}{4}\end{aligned} \tag{28}$$

By taking $\mathbf{x} = \mathbf{v}$, the theorem is complete.

Remark 2 *Particularly, when $A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $A_3 = A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, Eq.*(15) *and Eq.*(16) *reduce to Eq.*(14)*; when $A_k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix}$, $k = 1, 2$ and $A_3 = A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, Eq.*(15) *and Eq.*(16) *reduce to the corresponding result of the QLCT in spatial and frequency domain [19].*

Theorem 2 *Let $f \in L^1(\mathbb{R}^2, \mathbb{H}) \cap L^2(\mathbb{R}^2, \mathbb{H})$ be a quaternion-valued signal, then we derive a directional UP related to the QLCT,*

$$\begin{aligned}&\int_{\mathbb{R}^2} |\mathbf{u}|^2 \left|\mathcal{L}^{\mathbb{H}}_{A_1,A_2}\{f\}(\mathbf{u})\right|^2 d\mathbf{u} \cdot \int_{\mathbb{R}^2} |\mathbf{v}|^2 \left|\mathcal{L}^{\mathbb{H}}_{A_3,A_4}\{f\}(\mathbf{v})\right|^2 d\mathbf{v} \\ &\qquad \geq \frac{(|a_3b_1 - a_1b_3| + |a_4b_2 - a_2b_4|)^2}{4}\end{aligned} \tag{29}$$

Proof 2 *For the sake of convenience, here we notate*

$$\Delta u_k^2 = \int_{\mathbb{R}^2} u_k^2 \left|\mathcal{L}^{\mathbb{H}}_{A_1,A_2}\{f\}(\mathbf{u})\right|^2 d\mathbf{u}, k = 1, 2 \tag{30}$$

$$\Delta v_k^2 = \int_{\mathbb{R}^2} v_k^2 \left|\mathcal{L}^{\mathbb{H}}_{A_3,A_4}\{f\}(\mathbf{u})\right|^2 d\mathbf{v}, k = 1, 2 \tag{31}$$

Then, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2} |\mathbf{u}|^2 \left|\mathcal{L}^{\mathbb{H}}_{A_1,A_2}\{f\}(\mathbf{u})\right|^2 d\mathbf{u} \cdot \int_{\mathbb{R}^2} |\mathbf{v}|^2 \left|\mathcal{L}^{\mathbb{H}}_{A_3,A_4}\{f\}(\mathbf{v})\right|^2 d\mathbf{v} \\ &= \left(\Delta u_1^2 + \Delta u_2^2\right) \cdot \left(\Delta v_1^2 + \Delta v_2^2\right) \\ &= \Delta u_1^2 \Delta v_1^2 + \Delta u_1^2 \Delta v_2^2 + \Delta u_2^2 \Delta v_1^2 + \Delta u_2^2 \Delta v_2^2 \\ &\quad \geq \Delta u_1^2 \Delta v_1^2 + 2\sqrt{\Delta u_1^2 \Delta v_2^2 \cdot \Delta u_2^2 \Delta v_1^2} + \Delta u_2^2 \Delta v_2^2 \\ &\quad \geq \frac{(a_3b_1 - a_1b_3)^2}{4} + 2\sqrt{\frac{(a_3b_1 - a_1b_3)^2}{4} \cdot \frac{(a_4b_2 - a_2b_4)^2}{4}} \\ &\quad + \frac{(a_4b_2 - a_2b_4)^2}{4} \\ &= \frac{(|a_3b_1 - a_1b_3| + |a_4b_2 - a_2b_4|)^2}{4}\end{aligned}$$

*The first inequality is obtained by $a^2 + b^2 \geq 2ab$. The second inequality is derived using **Theorem** 1. This completes the proof.*

Remark 3 *The QFT directional UP is only one special case of **Theorem** 2 when $A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $A_3 = A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (see [23],**Theorem** 13).*

Corollary 1 *When $A_3 = A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, a directional UP of the QLCT in spatial and frequency is given as following*

$$\begin{aligned}&\int_{\mathbb{R}^2} |\mathbf{x}|^2 |f(\mathbf{x})|^2 d\mathbf{x} \cdot \int_{\mathbb{R}^2} |\mathbf{u}|^2 \left|\mathcal{L}^{\mathbb{H}}_{A_1,A_2}\{f\}(\mathbf{u})\right|^2 d\mathbf{u} \\ &\qquad \geq \frac{(|b_1| + |b_2|)^2}{4}\end{aligned} \tag{32}$$

4. CONCLUSIONS

In this paper, based on the relationship between the QFT and QLCT, two novel UPs of the two-sided QLCT are presented. One is a general form of the component-wise UP, and the other is an extended form of the directional UP. Both of new derived results describe lower bounds of spreads of a quaternion-valued signal in arbitrary two different QLCT domains, which include those of UPs in the spatial and frequency domain as its special case. They offer a method to estimate the effective bandwidth in the QLCT settings. Besides, the discrete UPs relate closely to the problem of signal recovery; novel derived UPs in the QLCT domains can further contribute to solving the problem of quaternion-valued signal recovery in the practical applications.

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