# EFFICIENT SAMPLING ON HEALPIX GRID

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# ABSTRACT

We propose an iso-latitude sampling scheme for the representation of band-limited signals on the sphere. The proposed scheme is designed as a variant of the widely used Hierarchical Equal Area iso-Latitude Pixelization (HEALPix) scheme on the sphere. We use HEALPix grid of resolution  $N_{\rm side}$  to represent a signal of bandlimit (spherical harmonic degree) L. To reduce the number of samples, the proposed algorithm takes only L iso-latitude rings out of  $4N_{\rm side} - 1$  rings of the HEALPix. This selection is carried out by ensuring that the spherical harmonic transform (SHT) of the signal is computed accurately from the samples. The number of samples required by the proposed sampling scheme is smaller than that required by HEALPix by at least a factor of 3/2. For the proposed sampling scheme, we also formulate the spherical harmonic transform and conduct numerical experiments to evaluate the number of samples required by the proposed sampling scheme and the accuracy of the associated SHT.

*Index Terms*— Sampling, HEALPix, unit 2-sphere, spherical harmonic transform, band-limited signals

# 1. INTRODUCTION

There are many disciplines of science and engineering in which signals exhibit angular dependence and hence, are inherently defined on the sphere. Areas where spherical signal processing techniques have been extensively used include wireless communication [1], computer graphics [2], medical imaging [3], acoustics [4], quantum chemistry [5], cosmology [6] and geodesy [7], to name a few. In many of these applications, signal is often analysed in the spatial or harmonic domain. The transformation from the spatial to harmonic domain is enabled by spherical harmonic transform (SHT).

As we can only process spatially discrete signals, many sampling schemes have been proposed for the computation of SHT [8– 12]. Equiangular sampling schemes in [9, 10] enable exact computation of SHT but suffer from massive oversampling near the poles which renders them sub-optimal for certain applications. Equal area and iso-latitude tessellation schemes are most desirable because not only do they avoid oversampling at the poles, they enable separation of variables which supports faster computation of SHT. One of the widely adopted among such schemes is the Hierarchical Equal Area iso-Latitude Pixelization (HEALPix) [12]. It supports hierarchical tree structure for the database of samples, facilitating various topological methods of analysis and allowing for fast computation of transforms through fast look-up of neighboring data elements. SHT associated with the HEALPix utilizes all the samples on the grid to evaluate the transform as an approximate quadrature. Although, HEALPix supports accurate computation of SHT, it is a high-resolution sampling grid and therefore, it is very desirable to design a sampling scheme that maintains the same order of accuracy in the computation of SHT but takes fewer number of samples.

In this context, we address the following research questions in this work:

- 1. Can we reduce the number of samples required by HEALPix for the computation of SHT?
- 2. Does the proposed reduction in number of samples compromise the accuracy of SHT?

In addressing these questions, we organize the rest of the paper as follows. In Section 2, we review the mathematical background for signal and harmonic analysis on the sphere and present an overview of the HEALPix sampling scheme. In Section 3, we formulate the SHT, outline the sampling requirements and devise an algorithm to design the sampling scheme. Before we present the concluding remarks in Section 5, we evaluate the reduction in number of samples achieved by the proposed sampling scheme and carry out accuracy analysis of the formulated SHT in Section 4.

# 2. MATHEMATICAL PRELIMINARIES

In this section, we present the necessary mathematical background for spatial and spectral representations of signals on the sphere and review the HEALPix sampling scheme.

# 2.1. Signals on 2-Sphere

We consider complex-valued signals  $f(\theta, \phi)$  on the 2-sphere (or sphere), denoted by  $\mathbb{S}^2$ . Here  $\theta \in [0, \pi]$  is the co-latitude angle measured from the positive z-axis and  $\phi \in [0, 2\pi]$  is the longitude angle measured from the positive x-axis in the x-y plane. The inner product between two such signals f, g is given by

$$\langle f,g \rangle \triangleq \int_{\mathbb{S}^2} f(\theta,\phi) \overline{g(\theta,\phi)} \sin \theta d\theta d\phi,$$
 (1)

where  $(\cdot)$  denotes the complex conjugate operation,  $\sin \theta d\theta d\phi$  is the differential area element on the sphere and integration is carried out over the whole sphere, i.e.,  $\int_{\mathbb{S}^2} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi}$ . With the inner product defined in (1), the set of signals on the sphere form a Hilbert space  $L^2(\mathbb{S}^2)$ . Energy and norm of the signal f are given by  $\langle f, f \rangle$  and  $||f|| \triangleq \langle f, f \rangle^{1/2}$  respectively.

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## 2.2. Spherical Harmonics

The Hilbert space  $L^2(\mathbb{S}^2)$  is separable and contains a complete set of orthonormal basis functions referred to as spherical harmonic functions or spherical harmonics for short, defined as [13]

$$Y_{\ell}^{m}(\theta,\phi) \triangleq \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_{\ell}^{m}(\cos\theta) e^{im\phi}, \qquad (2)$$

for integer degree  $\ell \ge 0$  and order<sup>1</sup>  $|m| \le \ell$ . Here  $P_{\ell}^{m}(\cos \theta)$  is the associated Legendre polynomial of degree  $\ell$  and order m [13]. Since spherical harmonics form a complete set of orthonormal basis functions on the sphere, any signal  $f \in L^{2}(\mathbb{S}^{2})$  can be expressed as

$$f(\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (f)_{\ell}^{m} Y_{\ell}^{m}(\theta,\phi), \qquad (3)$$

where

$$(f)_{\ell}^{m} = \langle f, Y_{\ell}^{m} \rangle = \int_{\mathbb{S}^{2}} f(\theta, \phi) \overline{Y_{\ell}^{m}(\theta, \phi)} \sin(\theta) d\theta d\phi \quad (4)$$

is the spherical harmonic (spectral) coefficient of degree  $\ell$  and order m and constitutes the spectral representation of the signal. The transformation (spatial to spectral) of the signal in (4) is referred to as the spherical harmonic transform (SHT) and the one (spectral to spatial) given in (3), is referred to as the inverse SHT (ISHT). A signal is considered band-limited to degree L if  $(f)_{\ell}^m = 0$  for  $\ell, m \geq L$ . Set of all such band-limited signals on the sphere forms an  $L^2$ -dimensional subspace of  $L^2(\mathbb{S}^2)$  and is denoted by  $\mathcal{H}_L$ . For any signal  $f \in \mathcal{H}_L$ , the sum over degree in (3) is truncated at L - 1.

#### 2.3. Hierarchical Equal Area iso-Latitude Pixelization

Hierarchical Equal Area iso-Latitude Pixelization [12], HEALPix for short, takes three iso-latitude rings of samples with four samples in each ring, dividing the sphere into 12 equal area regions at the base-resolution level. Sampling grid density is parameterized by  $N_{\rm side}$  which is defined as the number of divisions along the side of a base-resolution pixel needed to reach a desired high-resolution tessellation. An increase in resolution level by one divides each of the equal area regions on the sphere into four sub-regions. Total number of samples on the HEALPix grid is  $12N_{side}^2$  and are placed into three zones: Equatorial (-2/3 < z < 2/3), North polar  $(z \ge 2/3)$ and South polar ( $z \leq -2/3$ ), where  $z = \cos(\theta)$ . Total number of iso-latitude rings on the sampling grid is  $4N_{\rm side}-1$  out of which  $2N_{\rm side} - 1$  are located in the equatorial zone and  $N_{\rm side}$  are located in each polar zone. All equatorial rings contain maximum number of samples per ring, equal to  $4N_{side}$ , whereas the polar zone rings contain varying number of samples.

SHT associated with the HEALPix is computed by the approximate quadrature rule<sup>2</sup> given by

$$(\widehat{f})_{\ell}^{m} = \frac{4\pi}{N_{\text{pix}}} \sum_{k=0}^{N_{pix}-1} f(\theta_{k}, \phi_{k}) \overline{Y_{\ell}^{m}(\theta_{k}, \phi_{k})}.$$
(5)

Since the quadrature approximation in (5) is the zeroth-order estimator, the Jacobi iterative method is applied on it to improve its accuracy.

# 3. EFFICIENT SPHERICAL HARMONIC TRANSFORM FOR HEALPIX

Now we address the first question posed in Section 1 and propose a method for the computation of spherical harmonic transform of a band-limited signal  $f \in \mathcal{H}_L$  using only a subset of samples on the HEALPix grid. We propose to take samples on L iso-latitude rings with locations indexed in the vector  $\theta \equiv [\theta_0, \theta_1, \dots, \theta_{L-1}]^T$ . Before we determine the location of these iso-latitude rings and the number of samples along each ring, we present the formulation of SHT.

#### 3.1. Spherical Harmonic Transform - Formulation

For a signal  $f \in \mathcal{H}_L$ , its spherical harmonic coefficients of order |m| < L can be defined in terms of the Fourier transform of the signal along  $\phi$  in an iso-latitude ring placed at  $\theta = \theta_k$  as

$$G_m(\theta_k) \triangleq \int_0^{2\pi} f(\theta_k, \phi) e^{-im\phi} d\phi$$
$$= 2\pi \sum_{\ell=|m|}^{L-1} (f)_\ell^m \tilde{P}_\ell^m(\theta_k), \tag{6}$$

where  $\tilde{P}_{\ell}^{m}(\theta_{k}) \triangleq Y_{\ell}^{m}(\theta_{k}, 0)$  is the scaled associated Legendre polynomial. By defining a column vector  $\mathbf{g}_{m}$  as a vector of Fourier transform of the signal at L - |m| different iso-latitude rings, given by

$$\mathbf{g}_{m} \triangleq \left[ G_{m}(\theta_{|m|}), G_{m}(\theta_{|m|+1}), \dots, G_{m}(\theta_{L-1}) \right], \tag{7}$$

and a column vector  $\mathbf{f}_m$  containing spherical harmonic coefficients of order m as

$$\mathbf{f}_{m} \equiv \left[ f_{|m|}^{m}, f_{|m|+1}^{m}, \dots, f_{|L-1|}^{m} \right], \tag{8}$$

(9)

we can compactly express L - |m| equations of the form given in (6) as

 $\mathbf{g}_m = 2\pi \mathbf{P}_m \mathbf{f}_m, \quad |m| < L,$ 

where

$$\mathbf{P}_{m} \triangleq \begin{bmatrix} \tilde{P}_{|m|}^{m}(\theta_{|m|}) & \tilde{P}_{|m|+1}^{m}(\theta_{|m|}) & \cdots & \tilde{P}_{L-1}^{m}(\theta_{|m|}) \\ \tilde{P}_{|m|}^{m}(\theta_{|m|+1}) & \tilde{P}_{|m|+1}^{m}(\theta_{|m|+1}) & \cdots & \tilde{P}_{L-1}^{m}(\theta_{|m|+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{P}_{|m|}^{m}(\theta_{L-1}) & \tilde{P}_{|m|+1}^{m}(\theta_{L-1}) & \cdots & \tilde{P}_{L-1}^{m}(\theta_{L-1}) \end{bmatrix}$$
(10)

It becomes clear from (10) that in order for  $\mathbf{P}_m$  to be wellconditioned, Fourier transform in (6) must be evaluated along  $\phi$ for at least L different iso-latitude rings. By computing  $G_m(\theta_k)$  at different iso-latitude rings placed at  $\theta_k$ ,  $k = |m|, |m|+1, \ldots, L-1$ and inverting  $\mathbf{P}_m$  in (9), we can compute the spherical harmonic coefficients of order m and degrees  $|m| \leq \ell \leq L - 1$ .

## 3.2. Spherical Harmonic Transform - Computation

Spectral coefficients of order m contained in vector  $\mathbf{f}_m$  can be recovered from (9) provided  $\mathbf{g}_m$  is computed correctly and  $\mathbf{P}_m$  is well-conditioned to be invertible. Consequently, the accuracy of the formulated transform is dictated by the computation of  $G_m(\theta_k)$ and condition number of  $\mathbf{P}_m$ . Accurate computation of  $G_m(\theta_k)$ 

 $<sup>||\</sup>cdot|$  denotes the absolute value

<sup>&</sup>lt;sup>2</sup>http://healpix.sourceforge.net/documentation.php

depends on the number of samples along  $\phi$  in the chosen iso-latitude ring and the condition number of  $\mathbf{P}_m$  depends on the locations  $\theta_{|m|}, \theta_{|m|+1}, \ldots, \theta_{L-1}$  of the iso-latitude rings.

#### 3.2.1. Avoiding Aliasing in the Computation of $G_m(\theta_k)$

Using (6) and changing the order of summation in (3), a signal f band-limited to L and evaluated at samples in an arbitrary isolatitude ring placed at  $\theta_k$  can be written as

$$f(\theta_k, \phi) = \frac{1}{2\pi} \sum_{m=-(L-1)}^{L-1} G_m(\theta_k) e^{im\phi}.$$
 (11)

We observe that this signal has contribution from 2L - 1 complex exponentials  $e^{im\phi}$ , |m| < L. We therefore, require at least 2L - 1samples in the ring placed at  $\theta_k$  to avoid the effects of aliasing on  $G_m(\theta_k)$ . This is true regardless of the choice of the ring. However, we note that if we know the spectral coefficients of order |m| < 1 $p \leq L - 1$ , we can subtract their contribution from the samples of the signal in the iso-latitude ring placed at  $\theta_{|m|-1}$ , if this ring does not have 2L - 1 samples. This ring is then required to have only at least 2|m| - 1 samples for  $G_{|m|-1}(\theta_{|m|-1})$  to be free of any aliasing errors. We further elaborate on this concept. Equation (9) can be used to solve for  $(f)_{L-1}^{L-1}$  by computing  $G_{L-1}(\theta_{L-1})$  at a ring placed at  $\theta_{L-1}$  and having at least 2L-1 samples. If the next ring at  $\theta_{L-2}$  has at least 2L-1 samples, we compute  $G_{L-2}(\theta_{L-2})$  without aliasing. However, if the number of samples is less than 2L - 1 but at least 2L - 3, then we have to subtract the contribution of  $(f)_{L-1}^{L-1}$ and  $(f)_{L-1}^{-(L-1)}$  from the samples of f in the ring placed at  $\theta_{L-2}$  and update it as

$$f(\theta_{L-2},\phi) \leftarrow f(\theta_{L-2},\phi) - \tilde{f}_{L-1}(\theta_{L-2},\phi), \qquad (12)$$

where

$$\tilde{f}_{m}(\theta_{k},\phi) = \sum_{\ell=|m|}^{L-1} \{(f)_{\ell}^{m} \tilde{P}_{\ell}^{m}(\theta_{k}) e^{im\phi} + (f)_{\ell}^{-m} \tilde{P}_{\ell}^{-m}(\theta_{k}) e^{-im\phi} \}$$
$$= \frac{1}{2\pi} (G_{m}(\theta_{k}) e^{im\phi} + G_{-m}(\theta_{k}) e^{-im\phi})$$
(13)

is the contribution of spectral coefficients of order  $\pm m$ , for all degrees  $|m| \leq \ell \leq L - 1$ . Hence, we require the iso-latitude ring at  $\theta_{L-1}$  to have at least 2L - 1 samples, the iso-latitude ring at  $\theta_{L-2}$  to have at least 2L - 3 samples and so on.

# 3.3. Sampling Scheme – Requirements

Following the philosophy presented in the previous section and using the formulation given in (9), we note that SHT of the signal band-limited to L can be accurately computed by taking its samples over iso-latitude sampling scheme of L rings located at  $\theta_k$ ,  $k = 0, 1, \ldots, L - 1$ , provided the sampling scheme fulfills the following requirements:

- (**R1**) The iso-latitude ring located at  $\theta_k$  has at least 2k + 1 samples along longitude.
- (R2) Ring locations,  $\theta_k$ , k = 0, 1, ..., L 1 are chosen such that the matrix  $\mathbf{P}_m$  given in (10) is well-conditioned for each<sup>3</sup> m = 0, 1, ..., L 1.

SHT can be computed accurately if sampling scheme design takes into account these requirements as R1 and R2 ensure the accurate computation of  $\mathbf{g}_m$  and accurate inversion of (10) for each |m| < Lrespectively.

#### 3.4. Sampling Scheme – Design

We devise an algorithm to design the sampling scheme comprised of subset of HEALPix samples. Before we present the algorithm that selects the iso-latitude rings of samples from the HEALPix grid taking into account the sampling requirements, we establish a relation between the HEALPix resolution parameter  $N_{\rm side}$  and band-limit L. Since the number of samples required in the first ring is 2L-1 and all the rings in equatorial zone on the HEALPix grid contain maximum number of samples per ring, i.e.,  $4N_{\rm side}$ , first ring must be chosen from the equatorial zone. This puts an upper bound on the band-limit of the signal, i.e.,  $L \leq 2N_{\rm side}$ . Hence, for a given sampling grid with resolution parameter  $N_{\rm side}$ , we can compute the SHT for a maximum band-limit of  $2N_{\rm side}$ .

To select the iso-latitude rings of samples from the HEALPix grid with resolution parameter  $N_{\text{side}}$ , we propose the following isolatitude ring selection algorithm taking into account R1 and R2. We use  $\tilde{\theta}_h$  and  $n_h$  to denote the location of an iso-latitude ring and the number of samples along  $\phi$  in it with  $h = 1, 2, \ldots, H$ , where  $H = 4N_{\text{side}} - 1$  is the total number of rings on the HEALPix grid.

Procedure	1	Ring	Selection	Algorithm
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**Require:**  $\theta_k, k = 0, 1, \dots, L - 1$ 1: **procedure** RING SELECTION( $\tilde{\theta}_h, N_{\text{side}}$ ) 2:  $\Theta = {\tilde{\theta}_h}_{h=1}^H$ 

3:  $\theta_{L-1} = \pi/2$  (first ring)

4: **for**  $m = L - 2, L - 3, \dots, 0$  **do** 

- 5:  $\Theta_m = \{ \tilde{\theta}_h \in \Theta \mid n_h \ge 2m+1 \}$
- 6: Choose  $\theta_m \in \Theta_m$  which minimizes the condition number of  $\mathbf{P}_m$
- 7: end for

8: return 
$$\theta_k$$
,  $k = 0, 1, ..., L - 1$ .  
9: end procedure

The proposed algorithm identifies the rings from the HEALPix grid in such a way that each  $\mathbf{P}_m$  matrix is well-conditioned and the ring located at  $\theta_k$  has at least 2k + 1 samples along  $\phi$ , thus serving both sampling design requirements and ensuring the accurate computation of SHT.

#### 3.5. Multipass SHT

Like HEALPix, we also employ an iterative method to further improve the accuracy of the transform. After computing the spectral coefficients in the first pass, we reconstruct the signal in spatial domain using (3). Spectral coefficients of the difference between original and reconstructed spatial signals are computed and added to the previously computed coefficients, obtaining the spectral coefficients for the second pass. This process is continued until the quantity  $\sqrt{\mathbf{e}^H \mathbf{e}}$  either exceeds its value obtained in the previous pass or drops below a preset threshold of  $10^{-16}$ , where  $\mathbf{e}$  is the difference between the original and reconstructed spatial signal column vectors and  $(.)^H$  denotes the complex conjugate transpose.

<sup>&</sup>lt;sup>3</sup>We only need to ensure that  $\mathbf{P}_m$  is well-conditioned for non-negative orders m as  $\mathbf{P}_{-m} = (-1)^m \mathbf{P}_m$  which follows from the conjugate symmetry of spherical harmonics given by  $Y_{\ell}^{-m}(\theta, \phi) = (-1)^m \overline{Y_{\ell}^m(\theta, \phi)}$ .



Fig. 1: Number of samples used by HEALPix and proposed sampling scheme along with the theoretical bound established in Lemma 1, for band-limits in the range  $2 \le L \le 512$ .



Fig. 2: Visual comparison of sampling density between HEALPix and proposed sampling scheme for bandlimit L = 32.

# 4. EVALUATION

In this section, we compare the accuracy of the formulated SHT with SHT associated with the HEALPix and evaluate the reduction in number of samples achieved by the proposed sampling scheme. The formulated SHT is efficient compared to the one associated with the HEALPix in the sense that it uses lesser number of samples to accurately compute the spectral coefficients.

# 4.1. Reduction in Number of Samples

Since our ring selection algorithm chooses the iso-latitude rings from the HEALPix grid by minimizing the condition number of the matrix  $\mathbf{P}_m$ , we cannot analytically determine the exact decrease in the number of samples achieved by the proposed sampling scheme. However, we can work out the minimum guaranteed decrease in the number of samples which is presented in the following Lemma.

**Lemma 1** (Lower-bound on the Reduction in Number of Samples). The proposed sampling scheme requires at least 3/2 times less number of samples than HEALPix for the accurate computation of SHT of a signal band-limited to  $L \leq 2N_{side}$ .

*Proof.* Since  $L_{\text{max}} = 2N_{\text{side}}$  denotes the maximum band-limit for a given grid resolution parameter  $N_{\text{side}}$  and the number of samples on the HEALPix grid is  $N_{\text{pix}} = 12N_{\text{side}}^2$ , we have  $L_{\text{max}} = \sqrt{N_{\text{pix}}/3}$ . As the proposed sampling scheme requires  $L(\leq L_{\text{max}})$  iso-latitude rings for the accurate computation of SHT, total number of samples in the proposed sampling scheme, denoted by N, is given by  $N \leq L(4N_{\text{side}}) \leq L_{\text{max}}(4N_{\text{side}})$  or  $N_{\text{pix}}/N \geq 3/2$ .



Fig. 3:  $E_{\text{max}}$  and  $E_{\text{mean}}$  between spectral coefficients of the original and reconstructed signals, for band-limits in the range  $2 \le L \le 512$ .

In Fig. 1, we plot the number of samples used by SHT associated with the HEALPix, number of samples required by SHT formulated for the proposed sampling scheme and the theoretical bound established in Lemma 1 for band-limits in the range  $2 \le L \le 512$ . It can be seen that at moderately large band-limits, the proposed sampling scheme, in comparison with HEALPix, requires about half the number of samples. Fig. 2 provides a visual comparison of the number of samples used by HEALPix and proposed sampling scheme to compute SHT accurately for bandlimit L = 32.

# 4.2. Accuracy Analysis

SHT formulated for the proposed sampling scheme is evaluated on the test signal generated using test spectral coefficients,  $(f_T)_{\ell}^m$ , uniformly distributed between -1 and 1 in real and imaginary parts. We denote by  $(f_R)_{\ell}^m$  the reconstructed spectral coefficients. For both the sampling schemes, we evaluate the maximum and mean errors defined as

$$E_{\max} \triangleq \frac{1}{\|f_T\|} \max |(f_T)_{\ell}^m - (f_R)_{\ell}^m|,$$
 (14)

$$E_{\text{mean}} \triangleq \frac{1}{\|f_T\|} \sum_{\ell=0}^{L-1} \sum_{m=-\ell}^{\ell} |(f_T)_{\ell}^m - (f_R)_{\ell}^m|.$$
(15)

These errors are averaged over ten different realizations of the test signal for band-limits in the range  $2 \le L \le 512$  and plotted in Fig. 3. It is evident that the proposed sampling scheme, although requires lesser number of samples, enables accurate computation of SHT with errors on the order of numerical precision.

#### 5. CONCLUSION

We have presented an iso-latitude sampling scheme as a variant of the widely used Hierarchical Equal Area iso-Latitude Pixelization (HEALPix) scheme for the representation and reconstruction of band-limited signals on the sphere. The proposed sampling scheme requires at least 3/2 times less number of samples than HEALPix to accurately compute the spherical harmonic transform (SHT) of a signal band-limited to spherical harmonic degree  $L \leq 2N_{\rm side}$ , where  $N_{\rm side}$  is the HEALPix resolution parameter. We have also conducted numerical experiments, demonstrating the accurate computation of SHT with errors on the order of numerical precision.

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