

BANDLIMITED SPATIOTEMPORAL FIELD SAMPLING WITH LOCATION AND TIME UNWARE MOBILE SENSORS

Sudeep Salgia and Animesh Kumar

Department of Electrical Engineering
Indian Institute of Technology Bombay, Mumbai, India - 400076
Email: sudeepsalgia4@gmail.com, animesh@ee.iitb.ac.in

ABSTRACT

Sampling of smooth spatiotemporally varying fields is a well-studied topic in the literature. Classical approach assumes that the field is observed at known sampling locations and known timestamps ensuring field reconstruction. In a first, in this work the sampling and reconstruction of a spatiotemporal bandlimited field is addressed, where the samples are obtained by a *location-unaware, time-unaware* mobile sensor. The spatial and temporal order of samples is assumed to be known. It is assumed that the field samples are affected by measurement-noise. The spatial field's evolution is modeled by a linear constant coefficient partial differential equation. A regression style estimate is developed for reconstruction of the spatial field. The intersample spacings and the intersample timestamp differences are assumed to be from independent unknown renewal processes. If n is the average number of samples of the field obtained by the mobile sensor, then it is shown that the mean-squared error decreases as $O(1/n)$.

Index Terms— nonuniform sampling, signal reconstruction, signal sampling, regression analysis

1. INTRODUCTION

Sampling of smooth spatiotemporally varying fields is a well-studied topic in the literature [1, 2]. Classical approach assumes that the field is observed at known sampling locations and known timestamps ensuring field reconstruction [1, 2]. The field is assumed to evolve according a physical law, typically governed by a partial differential equation (PDE), such as the diffusion equation [3]. To the best of our knowledge, mobile sensing of spatial fields governed by a linear PDE with constant coefficients has not been studied.

Recently, mobile sensing has been proposed for sampling spatial fields [4, 5, 6]. For cost reduction, it is desirable to work with a location and time unaware, inexpensive mobile sensor. Accordingly, this work assumes that a mobile sensor is available for sampling a spatial field of interest in a finite

region, which is location-unaware and time-unaware. This results in a challenging field reconstruction problem: is it possible to sample and reconstruct a spatiotemporal field governed by a PDE, where the samples are obtained by a *location-unaware, time-unaware* mobile sensor? This work answers the question in affirmative. Our results also hold when additive measurement-noise is present in the samples.

The field is assumed to be bandlimited. The spatiotemporal field's evolution is given by a linear PDE with constant coefficients. Bandlimitedness implies a finite number of Fourier coefficients characterize the spatial field. The key aspects utilized in solution include oversampling, denoising, and utilization of spatial/temporal order of samples. The latter is assumed to be known.

Our solution works by developing a regression style estimate for the reconstruction of Fourier coefficients of the spatiotemporal field at time $t = 0$. With PDE known, the coefficients at all times can be predicted from the initial Fourier series coefficients. The intersample spacings and the intersample timestamp differences are assumed to be from independent unknown renewal processes. If n is the average number of samples of the field obtained by the mobile sensor, then our *main result shows that the mean-squared error in the spatiotemporal field estimation decreases as $O(1/n)$* . The sampling setup is illustrated in Fig. 1.

For mathematical tractability, the spatiotemporal field is assumed to be one dimensional in space evolving according to a known PDE. The smoothness of the field is modeled by its spatial bandlimitedness. The measurement noise is independent of the intersample spacing and intersample timestamp processes. It is also assumed to have zero mean and a finite variance. Oversampling is the key technique used to denoise and mitigate location-unawareness in this work.

Prior art: The literature consists of two parts: (i) papers which approach source reconstruction; and (ii) papers which address this problem as a sampling/reconstruction problem. Spatial sparsity of sources has been used to reconstruct sources [7] using maximum likelihood estimators. Lu and Vetterli have been sparsity aware super resolution techniques to obtain sources [8]. A method of localizing sources

A PART OF THE WORK WAS SUPPORTED BY BHARTI CENTRE FOR COMMUNICATION IN IIT BOMBAY.

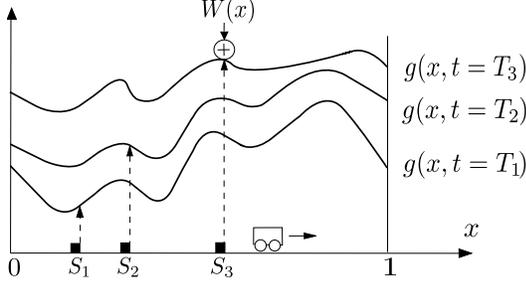


Fig. 1. The mobile sampling scenario under study is illustrated. A mobile sensor collects the spatial field's samples affected by measurement noise at unknown locations and unknown times denoted by $(S_1, T_1), (S_2, T_2), \dots$. The goal is to estimate $g(x, t)$ from the samples $g(S_1, T_1) + W_1, \dots, g(S_m, T_m) + W_m$.

has been proposed by Ranieri et al. [3]. Sampling with a mobile sensor has been a topic of interest recently [4, 5, 6]. However, mobile sensing typically assumes sampling locations to be known. When spatial field is fixed with time, sampling with a location-unaware mobile sensor or location-unaware sensor network has been addressed recently [9, 10]. Location-unaware sampling has also been reported for surface retrieval [11]. This work is novel since both the sampling locations and the timestamps of the samples are unknown. It is also assumed that the field is evolving while measurements are being made by the mobile sensor.

Notation: The spatiotemporally varying field will be denoted as $g(x, t)$ and its variants. Average sampling density will be denoted by n . All vectors will be denoted in bold. The expectation operator will be denoted by $\mathbb{E}[\cdot]$. The expectation is over all the random variables within the arguments. The set of natural numbers, integers, reals and complex numbers will be denoted by $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ and \mathbb{C} respectively. Also, $j = \sqrt{-1}$.

2. FIELD, SAMPLING AND NOISE MODEL, DISTORTION CRITERIA

2.1. Field Model

The spatial field is considered to be spatially smooth over a finite support, one dimensional in space, and its temporal evolution is given by a PDE. The PDE is assumed to be linear with constant coefficients, is known, and is given by

$$\sum_{i=0}^m p_i \frac{\partial^i}{\partial t^i} g(x, t) = \sum_{i=0}^{m'} q_i \frac{\partial^i}{\partial x^i} g(x, t) \quad (1)$$

where, $\frac{\partial^0}{\partial y^0} f(y) = f(y)$. For notational convenience, let $(\frac{\partial}{\partial z})^l = (\frac{\partial^l}{\partial z^l})$. With $p(z) = \sum_{i=0}^m p_i z^i$ and $q(z) =$

$\sum_{i=0}^{m'} q_i z^i$, the above PDE can be written as

$$p\left(\frac{\partial}{\partial t}\right)g(x, t) = q\left(\frac{\partial}{\partial x}\right)g(x, t). \quad (2)$$

The smoothness of the field will be modeled by bandlimitedness for all $t \geq 0$. Without loss of generality, the field's finite support is assumed to be $[0, 1]$. Its representation is,

$$g(x, t) = \sum_{k=-b}^b a_k(t) e^{j2\pi kx}; \quad a_k(t) = \int_{-\infty}^{\infty} g(x, t) e^{-j2\pi kx} dx. \quad (3)$$

The field is assumed to be bounded, i.e., $|g(x, t)| \leq 1$.

2.2. Distortion criteria

The distortion will be measured by the mean-squared error between the field and its estimate at $t = 0$. Let the field estimate be $\hat{G}(x, t)$ and its non-zero Fourier coefficients be $\hat{A}_k(t), |k| \leq b$, then the distortion is defined as

$$\begin{aligned} \mathbb{D}[\hat{G}, g] &:= \mathbb{E} \left[\int_0^1 |\hat{G}(x, t) - g(x, t)|^2 \right] \Big|_{t=0} \\ &= \sum_{k=-b}^b \mathbb{E} \left[|\hat{A}_k(0) - a_k(0)|^2 \right] \end{aligned} \quad (4)$$

2.3. Sampling model

It is assumed that there is a mobile sensor which moves from $x = 0$ to $x = 1$ while recording samples of the field $g(x, t)$. As the sensor moves, the field also evolves. This is the key distinction from models used in the literature. Field samples are collected on points generated by an *unknown* renewal process [9]. Let X_1, X_2, \dots be the intersample distances and N_1, N_2, \dots be the corresponding intersample time intervals. These variables are assumed to be the realizations of two *independent*, possibly different, and unknown renewal processes. Let the intersample distributions for X and N be $f(x)$ and $g(x)$, respectively. The sampling locations S_n are given by $S_n = \sum_{i=1}^n X_i$. The sampling is done over $[0, 1]$, and M is the random number of samples that lie in this interval. Note that the stopping condition $S_M \leq 1$ and $S_{M+1} > 1$ results in M ; and, it means M is a well defined random variable [12].

For analytical tractability, the support of the distributions of X and N are considered to be finite and inversely proportional to the sampling density. That is,

$$0 < X \leq \frac{\lambda}{n}, 0 < N \leq \frac{\mu}{n} \text{ and } \mathbb{E}[X] = \mathbb{E}[N] = \frac{1}{n} \quad (5)$$

where $1 < \lambda, \mu \ll n$ are fixed parameters that characterize the support of $f(x)$ and $g(x)$. Applying Wald's identity [12], on S_{M+1} and using equation (5), we can write

$$\mathbb{E}[M] = n\mathbb{E}[S_{M+1}] - 1 \quad (6)$$

By definition, $S_{M+1} > 1$ and $S_M \leq 1$. Since $S_{M+1} = S_M + X_{M+1}$, therefore, $S_{M+1} \leq 1 + X_{M+1} \leq 1 + \frac{\lambda}{n}$. Use these inequalities with equation (6), to obtain,

$$n - 1 < \mathbb{E}[M] \leq n + \lambda - 1 \quad (7)$$

Since $X_i \leq \frac{\lambda}{n}$ and $S_{M+1} > 1$, so

$$(M + 1)\frac{\lambda}{n} > 1 \text{ or } M > \frac{n}{\lambda} - 1 \quad (8)$$

The results in (6) and (8) imply that M scales linearly with n .

It is assumed that the sensing begins at $t = 0$. The timestamps are assumed to be $T_n = \sum_{i=1}^n N_i$. Let T_f be the timestamp *at the end* of the sensing experiment, i.e., when the sensor reaches $x = 1$. The timestamp T_f is assumed to be known. Note that $T_M \leq T_f$ and $T_{M+1} > T_f$ will be satisfied since M samples are obtained by the mobile sensor. The value of T_f may vary but is known at the end of the sensing.

2.4. Measurement noise model

It will be assumed that the obtained samples have been corrupted by measurement-noise $W(x, t)$. The noise samples for different pairs (x, t) will be independent, zero mean, finite variance, and identically distributed. The noise distribution is not required for mean-squared calculations done in this work.

3. FIELD ESTIMATION FROM SAMPLES

This section will highlight the estimation of the field from samples whose locations and timestamps come from two *unknown* independent renewal processes. A brief discussion of the field under its PDE is presented first. Since Fourier series is linear in coefficients, from (3) and (1), and linear independence of Fourier basis it follows that

$$\sum_{i=0}^m p_i \frac{\partial^i a_k(t)}{\partial t^i} - q(j2\pi k)a_k(t) = 0, \quad k = -b, \dots, b. \quad (9)$$

Each $a_k(t)$ evolves by an ordinary differential equation (ODE) with constant coefficients. The solution of ODE with constant coefficients is well known (via the Laplace transform [13]). For each k , let $r_i(k)$ be the m roots of $p(r) - q(j2\pi k) = 0$. Then, the Fourier coefficients in (9) are

$$a_k(t) = \sum_{i=1}^m a_{ki}(0) \exp(r_i(k)t), \quad (10)$$

where $a_{ki}(0)$ govern the initial conditions of the field before PDE based evolution begins. For a physically feasible field, $\Re[r_i(k)] \leq 0$ for all $i \leq m, |k| \leq b$, that is all roots have a non positive real part. This criteria can be checked using Routh Hurwitz algorithm [14]. While repeated roots are tractable, for notational simplicity, it is assumed that all the roots $r_1(k), \dots, r_m(k)$ that determine $a_k(t)$ are distinct.

From (10) and (3) the field samples at (S_n, T_n) are

$$g(S_n, T_n) = \sum_{k=-b}^b \sum_{i=1}^m a_{ki}(0) \exp(r_i(k)T_n + j2\pi k S_n). \quad (11)$$

Let $e_{k,i}(x, t) = \exp(r_i(k)t + j2\pi kx)$. Define

$$\begin{aligned} \mathbf{e}_k(x, t) &:= [e_{k,1}(x, t), \dots, e_{k,m}(x, t)] \\ \mathbf{a}_k &= [a_{k1}(0), a_{k2}(0), a_{k3}(0), \dots, a_{km}(0)] \\ \mathbf{a} &= [\mathbf{a}_{-b}, \dots, \mathbf{a}_{-1}, \mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_b]^T \\ \mathbf{e}(x, t) &= [\mathbf{e}_{-b}(x, t), \dots, \mathbf{e}_b(x, t)]^H. \end{aligned} \quad (12)$$

Observe that \mathbf{a} and $\mathbf{e}(x, t)$ are column vectors, while $\mathbf{e}_k(x, t)$ and \mathbf{a}_k are row vectors. Since $\Re[r_i(k)] \leq 0$, so $|e_{k,i}(x, t)| \leq 1$. This results in $\|\mathbf{e}_k(x, t)\|_2^2 \leq m$ and $\|\mathbf{e}(x, t)\|_2^2 \leq m(2b + 1)$. With notation from (12), (11) can be rewritten as

$$g(S_n, T_n) = \mathbf{e}^H(S_n, T_n) \mathbf{a}. \quad (13)$$

Due to measurement-noise, the observed field values will be $g(S_i, T_i) + W(S_i, T_i), 1 \leq i \leq M$. Define two vectors

$$\mathbf{g} = [g_1, g_2, \dots, g_M]^T \text{ and } \mathbf{w} = [w_1, w_2, \dots, w_M]^T \quad (14)$$

where, $g_i = g(S_i, T_i), w_i = W(S_i, T_i)$ for $1 \leq i \leq M$.

Let \mathbf{g}_s be the measurement-noise affected samples obtained at $(S_1, T_1), \dots, (S_M, T_M)$. Then, $\mathbf{g}_s = \mathbf{g} + \mathbf{w}$. Combining equation (13) and (14), we get

$$\mathbf{g} = Y \mathbf{a}; \text{ where } Y = [\mathbf{e}(S_1, T_1) \dots \mathbf{e}(S_M, T_M)]^H \quad (15)$$

Our estimation method follows next. The locations S_1, \dots, S_M and timestamps T_1, \dots, T_M are unknown, so equi-spaced approximations for them are defined as follows:

$$s_i = \frac{i}{M} \text{ and } t_i = \frac{iT_f}{M} \text{ for } 1 \leq i \leq M. \quad (16)$$

Corresponding to $(s_1, t_1), \dots, (s_M, t_M)$ define

$$\mathbf{g}_0 = [g_{u,1}, \dots, g_{u,M}]^T; Y_0 = [\mathbf{e}(s_1, t_1) \dots \mathbf{e}(s_M, t_M)]^H$$

where $g_{u,i} = g(s_i, t_i), i = 1, 2, \dots, M$. (17)

The main idea behind the reconstruction is that the sample locations and timestamps are “near” to equispaced sampling setup [9] due to renewal process structure on intersample spacings/timestamps. This motivates to define Y_0 as above and estimate the Fourier coefficients by assuming that samples have been obtained by multiplying the Fourier coefficient vector by Y_0 instead of the (unknown matrix) Y . The best least-squared estimate of the Fourier coefficients, $\hat{\mathbf{a}}$, would be

$$\hat{\mathbf{a}} = \arg \min_{\mathbf{b}} \|\mathbf{g}_s - Y_0 \mathbf{b}\|_2^2 \quad (18)$$

For sampling density exceeding the number of Fourier coefficients $n > m(2b + 1)$, this can be solved as a linear least squares estimation (regression). The solutions are [15]

$$\begin{aligned}\hat{\mathbf{a}} &= (Y_0^H Y_0)^{-1} Y_0^H \mathbf{g}_s \\ \mathbf{a} &= (Y_0^H Y_0)^{-1} Y_0^H \mathbf{g}_0\end{aligned}\quad (19)$$

where the second equation gives the exact Fourier coefficients of the field. The distortion in (4) is upper-bounded by

$$\mathbb{D}[\hat{G}, g] = \sum_{k=-b}^b \mathbb{E} \left[\left| \sum_{i=1}^m (\hat{A}_{ki}(0) - a_{ki}(0)) \right|^2 \right] \quad (20)$$

$$\leq \sum_{k=-b}^b \mathbb{E} \left[m \sum_{i=1}^m |\hat{A}_{ki}(0) - a_{ki}(0)|^2 \right] \quad (21)$$

$$= m(2b + 1) \mathbb{E} \left[\|\hat{\mathbf{a}} - \mathbf{a}\|_2^2 \right] \quad (22)$$

We now state our main result in the following theorem.

Theorem 3.1. *Let $\hat{\mathbf{a}}$ and \mathbf{a} be defined in equation (19). Then*

$$\mathbb{E} \left[\|\hat{\mathbf{a}} - \mathbf{a}\|_2^2 \right] \leq \frac{C'}{n}$$

where n is the average sampling density and C' is a positive constant independent of n . The constant C' depends on the bandwidth b , the renewal processes parameters λ and μ , the PDE coefficients, and noise variance σ^2 . The field reconstruction distortion can be bounded as $\frac{m(2b+1)C'}{n}$.

Proof. For details see [16]. The proof utilizes the properties of the matrix Y_0 . Let $A = (Y_0^H Y_0)^{-1} Y_0^H$. Then

$$\mathbb{E} \left[\|\hat{\mathbf{a}} - \mathbf{a}\|_2^2 \right] \leq 2\mathbb{E} \left[\lambda_{\max}^A \|\mathbf{g} - \mathbf{g}_0\|_2^2 \right] + 2\mathbb{E} \left[\|\mathbf{A}\mathbf{w}\|_2^2 \right] \quad (23)$$

where Cauchy-Schwarz inequality is used and λ_{\max}^A is the largest eigenvalue of $A^H A$. Now, it can be shown that

$$\lambda_{\max}^A \leq \text{tr}((Y_0^H Y_0)^{-1}) \quad (24)$$

where $\text{tr}(A^H A)$ means the trace of the $A^H A$. Since noise is zero mean, independent, and finite variance, so the second term in (23) satisfies

$$\mathbb{E} \left[\|\mathbf{A}\mathbf{w}\|_2^2 \right] \leq \sigma^2 \mathbb{E} \left[\text{tr}((Y_0^H Y_0)^{-1}) \right] \quad (25)$$

From [16], $\text{tr}((Y_0^H Y_0)^{-1}) \leq (C_t/M)$ for some constant C_t . So

$$\mathbb{E} \left[\text{tr}((Y_0^H Y_0)^{-1}) \right] \leq C_t \mathbb{E} \left[\frac{1}{M} \right] \leq \frac{C_t \lambda}{n - \lambda}. \quad (26)$$

from (8). In [16], it has been proved that

$$\mathbb{E} \left[\frac{1}{M} \|\mathbf{g} - \mathbf{g}_0\|_2^2 \right] \leq \frac{C_s + C_T}{n} \quad (27)$$

for some constants, C_s and C_T , independent of n . The inequalities in (25), (26), and (27) in (23) shows the result. \square

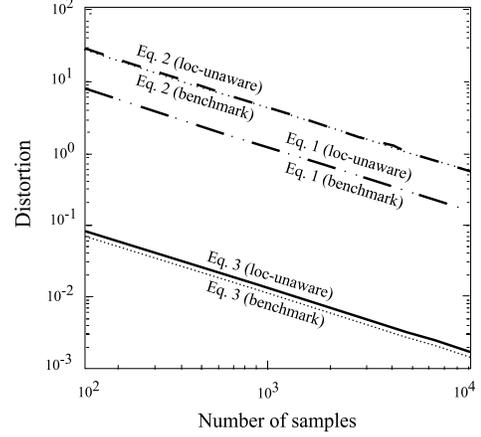


Fig. 2. The plots show that the distortion scales as $O(1/n)$ (see Theorem 3.1) for the three PDEs in Section 4.

4. SIMULATIONS

A sample field $g_s(x, 0)$ with $b = 3$ was generated. Its $a_k(0)$ Fourier coefficients for $k \geq 0$ were generated using independent Uniform $[-1, 1]$ random variables for the real and imaginary parts. Conjugate symmetry was used to obtain $a_k(0)$ for negative k . The field was finally scaled to have $|g(x)| \leq 1$. The following PDEs with initial condition $g_s(x, 0)$ were used for simulations: (i) $p_1(z) = z^2 + 3z$, $q_1(z) = 0.01(z^2 - 0.0125z^4)$; (ii) $p_2(z) = z^2 + 3z$, $q_2(z) = 0.01z^2$; and, (iii) $p_3(z) = z$, $q_3(z) = 0.01z^2$ (diffusion equation). The intersample distances were generated from Uniform $[\frac{0.2}{n}, \frac{1.8}{n}]$ distribution, while the timestamps were generated from Uniform $[\frac{0.6}{n}, \frac{1.4}{n}]$. Measurement noise was assumed to be independent Gaussian with variance $\sigma^2 = 0.125$. The mean-squared error for our estimates are illustrated in Fig. 2 for above three PDEs. The mean-squared error has slope of -1 in log-log plot confirming $O(1/n)$ decay. To benchmark our regression, distortion is compared when regression is performed with field samples equi-spaced in location and time. Surprisingly, there is feeble difference between the benchmark and the location-unaware regression.

5. CONCLUSIONS

The sampling of spatially bandlimited field evolving according to the constant coefficient linear PDE using a mobile sensor was studied. The field was estimated using the noisy samples obtained at unknown locations and time instants obtained from two independent and unknown renewal processes and it was shown that the mean squared error between the estimated field and the true field decreased as $O(1/n)$, where n was the average sampling density. The main idea that was leveraged was the fact that the sample locations approach the equi-spaced uniform sample locations as the sampling density increases, and oversampling reduced the error.

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