ON THE COMPUTABILITY OF SYSTEM APPROXIMATIONS UNDER CAUSALITY CONSTRAINTS

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ABSTRACT

Approximating the transfer function of stable causal linear systems by a basis expansion is a common task in signal- and system theory. This paper characterizes a scale of signal spaces, containing stable causal transfer functions, with a very simple basis (the Fourier basis) but which is not computable. Thus it is not possible to determine the coefficients of this basis expansion on any digital computer such that the approximation converges to the desired function.

Since the Fourier basis is not computable, the second part of the paper investigates whether there exist better bases. To this end, the notion of a computational basis is introduced and it is shown that there exists no computational basis in these spaces. The paper characterizes also subspaces on which computational bases do exist.

***Index Terms*—** Basis expansion, causality, computability, sampling, stability

1. BASIC CONCEPTS AND PROBLEM FORMULATION

The approximation of linear time-invariant (LTI) systems by simpler systems plays a fundamental role in system- and signal theory [1–3]. The goal is always to represent a general LTI system by a system which has a fairly simple structure such that it can easily be analyzed, synthesized, and implemented. In doing so, basis representations of transfer functions are of central importance [4–6]. Before giving a more detailed problem formulation, we shortly introduce basic notations.

A *causal* LTI systems is an operator S mapping input sequences $\boldsymbol{x} = \{x[n]\}_{n\in\mathbb{Z}} \in \ell^2$ onto output sequences $\boldsymbol{y} = \{y[n]\}_{n\in\mathbb{Z}} \in \ell^2$ according to

$$y[n] = \big(\mathrm{S}\boldsymbol{x}\big)[n] = \textstyle\sum_{k=0}^{\infty} f[k]\, x[n-k]\;, \qquad n \in \mathbb{Z}\,. \tag{1}$$

Therein $\{f[k]\}_{k=0}^{\infty}$ is said to be the *impulse response* of S, and ℓ^2 stand for the usual signal space of sequences $\boldsymbol{x}$ over $\mathbb{Z}$ of finite energy $\|\boldsymbol{x}\|_2^2 = \sum_{n\in\mathbb{Z}} |x[n]|^2 < \infty$. Taking the *discrete Fourier transform (DFT)* of (1), which is given by

$$x(\mathrm{e}^{\mathrm{i}\omega}) = \big(\mathcal{F}\boldsymbol{x}\big)(\mathrm{e}^{\mathrm{i}\omega}) = \textstyle\sum_{n\in\mathbb{Z}} x[n]\, \mathrm{e}^{in\omega}\;, \qquad \omega \in [-\pi,\pi)\;,$$

the input-output relation of S can equivalently be written in the frequency domain as

$$y(\mathrm{e}^{\mathrm{i}\omega}) = \big(\mathrm{S}x\big)(\mathrm{e}^{\mathrm{i}\omega}) = f(\mathrm{e}^{\mathrm{i}\omega})\, x(\mathrm{e}^{\mathrm{i}\omega})\;, \qquad \omega \in [-\pi,\pi)$$

wherein f is the *transfer function* of S and Parseval's theorem implies that x and y belong to the usual space $L^2(\mathbb{T})$ of square integrable functions on the *unit circle* $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The system S is said to be *stable* if $\boldsymbol{y} = \mathrm{S}\boldsymbol{x} \in \ell^2$ whenever $\boldsymbol{x} \in \ell^2$, and it is well known that S is stable if and only if its transfer function f is bounded, i.e. if and only if $f \in L^\infty(\mathbb{T})$. Similarly, the *causality* of S is reflected by the fact that its impulse response $\boldsymbol{f} = \{f[k]\}_{k\in\mathbb{Z}}$ is zero on the negative half axis, i.e. $f[k] = 0$ for all $k < 0$. This condition is equivalent to the requirement that the $\mathcal{Z}$-transform

$$f(z) = \big(\mathcal{Z}\boldsymbol{f}\big)(z) = \textstyle\sum_{n=0}^{\infty} f[n]\, z^n \tag{2}$$

of $\boldsymbol{f}$ is an analytic function for all $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

So any causal, stable LTI system can be identified with a transfer function f in the Hardy space $H^\infty(\mathbb{D})$ of bounded analytic functions in $\mathbb{D}$ [7]. Conversely, if the transfer function f of S is known then its impulse response $\boldsymbol{f} = \{f[n]\}_{n\in\mathbb{Z}}$ can, in principle, be determined by the inverse DFT

$$f[n] = \tfrac{1}{2\pi} \textstyle\int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}\omega})\, \mathrm{e}^{-\mathrm{i}n\omega}\, \mathrm{d}\omega\;, \qquad n \in \mathbb{Z}\,. \tag{3}$$

where $\{f[n]\}_{n\in\mathbb{Z}}$ are also called the *Fourier coefficients* of f.

The design of stable, causal LTI systems is often based on optimization techniques which derive the transfer function f of the desired system based on some optimality criteria (optimal filtering, pre-whitening, etc.) [8–15]. The so obtained f has often a very complicated structure without any closed form analytic representation. As a result of the optimization, f is often only given by its values on a certain discrete sampling set $\mathcal{Z} = \{\mathrm{e}^{\mathrm{i}\omega_k} : k = 1,\dots,M\}$. Then a usual approach is to approximate the optimal transfer functions by simpler stable systems which are known analytically.

A very natural and common technique is to represent f in a basis [3–5]. Let $\mathcal{B}$ be an arbitrary Banach space (the signal space) and let $\boldsymbol{\varphi} = \{\varphi_n\}_{n=0}^{\infty}$ be a basis for $\mathcal{B}$. Then every $f \in \mathcal{B}$ can be written as

$$f = \textstyle\sum_{n=0}^{\infty} c_n(f)\, \varphi_n \tag{4}$$

with coefficients $\{c_n(f)\}_{n=0}^{\infty} \subset \mathbb{C}$. An approximation $\widetilde{f}$ of f is obtained by restricting the sum in (4) to its first N terms, i.e. by

$$\widetilde{f}_N = \mathrm{P}_N f = \textstyle\sum_{n=0}^{N} c_n(f)\, \varphi_n\;, \qquad N = 0,1,2,\dots \tag{5}$$

Since $\boldsymbol{\varphi}$ is a basis for $\mathcal{B}$, the approximation error $\big\|f - \widetilde{f}_N\big\|_{\mathcal{B}}$ gets arbitrarily small as the approximation degree N becomes sufficiently large. Therewith, the desired approximation of f is obtained [16].

To make this approximation procedure effective, one only has to determine numerically the coefficients $\{c_n(f)\}_{n=0}^{\infty}$ in (4) for every $f \in \mathcal{B}$. This paper investigates whether this is always possible, i.e. whether the approximation (5) is *computable*. We are going to show that in a whole scale of Banach spaces $\mathcal{B} \subset H^\infty(\mathbb{D})$, which posses a very simple basis (the Fourier basis), there exists no numerical procedure to determine the coefficients $c_n(f)$ in the expansion (5) and such that the approximation (5) converges in the norm of $\mathcal{B}$ (and uniformly) to the desired $f \in \mathcal{B}$. Thus, we are going to show that these bases are *not computable*.

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2. SCHAUDER BASES & CAUSAL APPROXIMATIONS

Schauder bases in Banach spaces Let $\mathcal{B}$ be a separable Banach space with norm $\|\cdot\|_{\mathcal{B}}$. One says that $\mathcal{B}$ possesses a *(Schauder) basis* $\{\varphi_n\}_{n=0}^{\infty}$ if to every $f \in \mathcal{B}$ there exists a unique sequence $\{c_n(f)\}_{n=0}^{\infty} \subset \mathbb{C}$ such that (4) holds and where the series converges in the norm of $\mathcal{B}$ [16–18]. If $\{\varphi_n\}_{n=0}^{\infty}$ is a Schauder basis then the corresponding coefficient functionals $c_n : \mathcal{B} \to \mathbb{C}$ are known to be linear and continuous. Moreover, the series (4) converges in $\mathcal{B}$ if and only if the partial sums $\mathrm{P}_N : \mathcal{B} \to \mathcal{B}$, given in (5), satisfy

$$\sup_{N\in\mathbb{N}} \|\mathrm{P}_N\|_{\mathcal{B}\to\mathcal{B}} < +\infty$$

where the operator norm of a mapping $\mathrm{P} : \mathcal{B}_{\mathrm{a}} \to \mathcal{B}_{\mathrm{b}}$ from Banach space $\mathcal{B}_{\mathrm{a}}$ to $\mathcal{B}_{\mathrm{b}}$ is defined as usual by

$$\|\mathrm{P}\|_{\mathcal{B}_{\mathrm{a}}\to\mathcal{B}_{\mathrm{b}}} = \sup_{f\in\mathcal{B}_{\mathrm{a}},\|f\|_{\mathcal{B}_{\mathrm{a}}}\leq 1} \|\mathrm{P}f\|_{\mathcal{B}_{\mathrm{b}}} . \tag{6}$$

The following lemma provides a useful characterization of bases in Banach spaces $\mathcal{B}$ in terms of series of bounded linear operators [16].

Lemma 1: *If $\{\varphi_n\}_{n=0}^{\infty}$ is a Schauder basis for a Banach space $\mathcal{B}$ then the corresponding operators* (5) *satisfy the following properties*

1) $\lim_{N\to\infty} \|\mathrm{P}_N f - f\|_{\mathcal{B}} = 0$ *for all* $f \in \mathcal{B}$
2) $\dim\big[\mathrm{P}_N(\mathcal{B})\big] = N + 1$
3) $\mathrm{P}_N\mathrm{P}_M = \mathrm{P}_{\min(N,M)}$.

Conversely, assume $\{\mathrm{P}_N\}_{N=0}^{\infty}$ is a sequence of linear operators which satisfies properties 1) – 3) then every sequence $\{\varphi_n\}_{n=0}^{\infty}$ with

$$\varphi_0 \in \mathrm{P}_0(\mathcal{B}) \quad \textit{and} \quad \varphi_n \in \mathrm{P}_n(\mathcal{B}) \cap \mathrm{kern}(\mathrm{P}_{n-1}) , \ n \in \mathbb{N}$$

is a Schauder basis for $\mathcal{B}$.

Signal spaces of causal functions The Banach space of all functions continuous on $\mathbb{T}$ with the norm $\|f\|_{\infty} = \max_{\zeta\in\mathbb{T}} |f(\zeta)|$ is denoted by $\mathcal{C}(\mathbb{T})$, and the *disk algebra* $\mathcal{A}(\mathbb{D}) \subset \mathcal{C}(\mathbb{T})$ is the set of all functions which are analytic in $\mathbb{D}$ and continuous in the closed unit disk $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ [7, 19]. It becomes a Banach space if it is equipped with the norm $\|f\|_{\infty} = \max_{z\in\overline{\mathbb{D}}} |f(z)|$.

Every $f \in \mathcal{A}(\mathbb{D})$ possesses a power series expansion (2) with Fourier coefficients $\{f[n]\}_{n=0}^{\infty}$ of f given by (3). It is important to note that for $f \in \mathcal{A}(\mathbb{D})$ the power series (2) converges only pointwise for every $z \in \overline{\mathbb{D}}$ but it does not converge in the norm of $\mathcal{A}(\mathbb{D})$ (i.e. not uniformly), in general. This implies that the set $\boldsymbol{\zeta} = \{\zeta_n(z) := z^n\}_{n=0}^{\infty}$ is *not a basis* for $\mathcal{A}(\mathbb{D})$. This shortcoming of $\boldsymbol{\zeta}$ motivates the definition of the subset $\mathcal{U}$ of all $f \in \mathcal{A}(\mathbb{D})$ which posses a uniformly converging power series [20]: Let $f \in \mathcal{A}(\mathbb{D})$ be arbitrary with Fourier coefficients (3). For every $N \in \mathbb{N}$ let

$$\big(\mathrm{P}_N f\big)(z) = \textstyle\sum_{n=0}^{N} f[n]\, \zeta_n(z) = \sum_{n=0}^{N} f[n]\, z^n \tag{7}$$

be its partial power series. Therewith, we define the norm

$$\|f\|_{\mathcal{U}} = \sup_{N\in\mathbb{N}} \|\mathrm{P}_N f\|_{\infty} \tag{8}$$

and the space $\mathcal{U} = \overline{\mathrm{span}\{f \in \mathcal{P}\}}^{\|\cdot\|_{\mathcal{U}}}$ as the closed linear span of all polynomials, with the closure taken with respect to the norm (8). The so defined $\mathcal{U}$, together with norm (8), is a separable Banach space and the Theorem of Banach–Steinhaus implies that

$$\mathcal{U} = \big\{f \in \mathcal{A}(\mathbb{D}) : \lim_{N\to\infty} \|f - \mathrm{P}_N f\|_{\infty} = 0\big\} . \tag{9}$$

By the definition of $\mathcal{U}$, it is clear that $\{\zeta_n\}_{n=0}^{\infty}$ is a basis for $\mathcal{U}$, that $\|\zeta_n\|_{\mathcal{U}} = 1$ for all n, and that $\|\mathrm{P}_N\|_{\mathcal{U}\to\mathcal{U}} = 1$ for all $N \in \mathbb{N}$.

Next, we introduce a collection of subspaces of $\mathcal{A}(\mathbb{D})$ [19]. To this end, we define for any $\alpha, \beta \geq 0$ the functional

$$\|f\|_{\alpha,\beta} = \big(\textstyle\sum_{n=1}^{\infty} n^{\alpha}\,(1 + \log n)^{\beta} \big|f[n]\big|^2\big)^{1/2}$$

on $\mathcal{A}(\mathbb{D})$, and therewith the scale of Banach spaces

$$\mathcal{B}_{\alpha,\beta} = \big\{f \in \mathcal{A}(\mathbb{D}) \ : \ \|f\|_{\alpha,\beta} < +\infty\big\} , \quad \alpha, \beta \geq 0 ,$$

equipped with the norm $\|f\|_{\mathcal{B}_{\alpha,\beta}} = \max\big(\|f\|_{\infty}, \|f\|_{\alpha,\beta}\big)$. The parameters α, β characterize the decay of the Fourier coefficients and it is clear that $\mathcal{B}_{\alpha',\beta} \subset \mathcal{B}_{\alpha,\beta} \subset \mathcal{B}_{0,\beta}$ for all $\alpha' > \alpha > 0$ and similarly $\mathcal{B}_{\alpha,\beta'} \subset \mathcal{B}_{\alpha,\beta} \subset \mathcal{B}_{\alpha,0}$ for all $\beta' > \beta > 0$.

Subsequently, we consider mainly the space $\mathcal{B}_{1,1}$ which will be denoted by $\mathcal{B}_1$, for simplicity of notation. Finally, we notice in the following lemma that $\mathcal{B}_1$ has the same basis as $\mathcal{U}$.

Lemma 2: *We have $\mathcal{B}_1 \subset \mathcal{U}$ and $\boldsymbol{\zeta} = \{\zeta_n(z) = z^n\}_{n=0}^{\infty}$ is a basis (the Fourier basis) of $\mathcal{B}_1$ with coefficient functionals*

$$c_n(f) = \tfrac{1}{2\pi}\textstyle\int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}\omega})\,\overline{\zeta_n(\mathrm{e}^{\mathrm{i}\omega})}\,\mathrm{d}\omega = f[n] . \tag{10}$$

Proof: To prove that $\boldsymbol{\zeta}$ is a basis for $\mathcal{B}_1$, one has to show that the operator norms $\|\mathrm{P}_N\|_{\mathcal{B}_1\to\mathcal{B}_1}$ are uniformly bounded. To see this, we note that there is a constant C_1 such that $\|\mathrm{P}_N f\|_{\infty} \leq C_1 \|f\|_1$. Indeed for any $f \in \mathcal{B}_1$ and all $z \in \mathbb{D}$, we have

$$\begin{aligned}|(\mathrm{P}_N f)(z)| &\leq \textstyle\sum_{n=0}^{N} |c_n(f)|\ |\zeta_n(z)| \leq \sum_{n=0}^{N} \frac{\sqrt{n\log n}|c_n(f)|}{\sqrt{n\log n}}\\ &\leq \Big(\textstyle\sum_{n=0}^{N} \frac{1}{n\log n}\Big)^{1/2} \Big(\sum_{n=0}^{N} n\log n\,|c_n(f)|^2\Big)^{1/2}\\ &\leq C_1\ \|f\|_1 \leq C_1\ \|f\|_{\mathcal{B}_1} ,\end{aligned}$$

using Cauchy–Schwarz inequality to obtain the second line. In combination with the obvious inequality $\|\mathrm{P}_N f\|_{1,1} \leq \|f\|_{1,1} \leq \|f\|_{\mathcal{B}_1}$, one obtains $\|\mathrm{P}_N f\|_{\mathcal{B}_1} \leq \max(1, C_1)\ \|f\|_{\mathcal{B}_1}$ showing that the operator norms $\|\mathrm{P}_N\|_{\mathcal{B}_1\to\mathcal{B}_1}$ are uniformly bounded. ■

Computable approximations Both spaces $\mathcal{U}$ and $\mathcal{B}_1$ possess the *Fourier basis* $\boldsymbol{\zeta} = \{\zeta_n\}_{n=0}^{\infty}$. If we are able to calculate the coefficients (10) for every f in $\mathcal{U}$ or $\mathcal{B}_1$ then we can determine the approximation (5) with $\varphi_n = \zeta_n$ for every $n \in \mathbb{N}$. Since $\boldsymbol{\zeta}$ is a basis, the approximation $\mathrm{P}_N f$ converges in the norm of $\mathcal{B}_1$ and $\mathcal{U}$ (and also uniformly) to f. So for $\mathcal{B} = \mathcal{U}$ or $\mathcal{B} = \mathcal{B}_1$, one has

$$\lim_{N\to\infty} \|\mathrm{P}_N f - f\|_{\mathcal{B}} = 0 \qquad \text{for all } f \in \mathcal{B} .$$

However, the integral in (10) can usually not calculated perfectly, because $f(\mathrm{e}^{\mathrm{i}\omega})$ is not known for all $\omega \in [-\pi, \pi)$ but only at finitely many sampling points on $\mathbb{T}$. Moreover, in order to evaluate (10) on a digital computer, it is only possible to incorporate finitely many samples of f, otherwise the computational time goes to infinity. Here, we discuss two common and natural ways to evaluate (10) from samples of f. 1) The calculation of (10) via numerical integration and 2) the existence of a so called computational bases. We are going to show that both methods fail on $\mathcal{U}$ and $\mathcal{B}_1$.

3. APPROXIMATION VIA NUMERICAL INTEGRATION

To determine the values of the coefficient functionals (10), one may apply numerical integration methods to evaluate numerical approximations $c_{N,n}$ of the coefficients (10).

Example 1: A simple way to approximate the integral in (10) is the application of the rectangular formula of the Riemann sum bases on $M = M(N)$ equidistant samples of f on $\mathbb{T}$, i.e.

$$c_n(f) \approx c_{N,n}(f) = \tfrac{1}{M(N)} \sum_{k=1}^{M(N)} f(z_{N,k})\, \zeta_n(z_{N,k}) \qquad (11)$$

wherein $\mathcal{Z}_N = \left\{z_{N,k} = \mathrm{e}^{\mathrm{i}\frac{2\pi}{M}(k-1)}\right\}_{k=1}^{M(N)} \subset \mathbb{T}$ is the sampling set.

Remark: More approximation methods are obtained by applying other quadrature formulas or by allowing for different (not necessarily equidistant) sampling sets.

Now one can use the approximated coefficients $c_{N,n}(f)$ instead of the unknown true values $c_n(f)$ and determine the approximations

$$\mathrm{Q}_N f = \sum_{n=0}^{N} c_{N,n}(f)\, \zeta_n \ , \qquad N \in \mathbb{N} \qquad (12)$$

of f. The question is whether $\mathrm{Q}_N f$ converges to f as $N \to \infty$ for any f in $\mathcal{U}$ or $\mathcal{B}_1$. Here we make no restrictions on the quadrature methods apart from two natural assumptions:

(i) To every $N \in \mathbb{N}$ and there exists an $M(N) \in \mathbb{N}$ and a set $\mathcal{Z}_N = \{z_{N,1}, \dots, z_{N,M(N)}\} \subset \mathbb{T}$ such that the functionals $c_{N,n}(f)$ depend only on the values of f on $\mathcal{Z}_N$.

(ii) The operators Q_N defined by (12) satisfies

$$\lim_{N\to\infty} \|\mathrm{Q}_N \zeta_n - \zeta_n\|_\infty = 0 \qquad \text{for all } n = 0, 1, 2, \dots \ .$$

Condition (i) requires that the functionals $c_{N,n}(f)$ are uniquely determined by the values of f on a finite set $\mathcal{Z}_N$. Again, this requirement is necessary to implement the integration method on a digital computer. Condition (ii) requires that the approximative functionals $c_{N,n}$ are such that at least the basis elements ζ_n are perfectly recovered by Q_N as $N \to \infty$. Clearly, this is a minimal requirement in order that $\mathrm{Q}_N f$ converges to f for all f in $\mathcal{U}$ or $\mathcal{B}_1$. Moreover, we notice that because of

$$\big(\mathrm{Q}_N \zeta_k\big)(z) - \zeta_k(z) = \sum_{n=0}^{N} c_{N,n}(\zeta_k) z^k - z^k \ ,$$

Condition (ii) implies that

$$\lim_{N\to\infty} c_{N,n}(\zeta_k) = c_n(\zeta_k) = \begin{cases} 0 & \text{if} \quad n \neq k \\ 1 & \text{if} \quad n = k \end{cases} \ . \qquad (13)$$

Thus the integration method which yields the functionals $c_{N,n}$ has the property that at least for the basis functions ζ_k the approximation $c_{N,n}(\zeta_k)$ converges to the correct Fourier coefficients $c_n(\zeta_k)$.

Note that the sampling set $\mathcal{Z}_N$ and the integration method can depend on the approximation degree $N \in \mathbb{N}$. For each N one may choose completely new sets $\mathcal{Z}_N$ and different quadrature methods.

Banach spaces without convergent methods We are going to show that there are Banach spaces ($\mathcal{U}$ and $\mathcal{B}_1$) which posses a nice basis (the Fourier basis) but for which there exists no way to replace the exact coefficient functionals c_n by some reasonable approximation $c_{N,n}$ which satisfy (i) and (ii) and such that the approximations (12) converge to the desired f for all f in this Banach space.

Theorem 3: *Let $\boldsymbol{\zeta} = \{\zeta_n\}_{n=0}^\infty$ be the Fourier basis of $\mathcal{U}$ and $\mathcal{B}_1$ and let $\{\mathrm{Q}_N\}_{n=1}^\infty$ be the sequence* (12) *of operators associated with $\boldsymbol{\zeta}$ and which satisfies conditions (i) and (ii), then*

$$\lim_{N\to\infty} \|\mathrm{Q}_N\|_{\mathcal{U}\to\mathcal{A}(\mathbb{D})} = \lim_{N\to\infty} \|\mathrm{Q}_N\|_{\mathcal{B}_1\to\mathcal{A}(\mathbb{D})} = +\infty \ ,$$

with the operator norms defined as in (6).

Theorem 3 shows that the norm of all operators (12) satisfying conditions (i) and (ii) becomes arbitrarily large as the approximation degree N goes to infinity. Applying the Theorem of Banach–Steinhaus [21, Chapt. 5] one can reformulate this result as follows.

Corollary 4: *Let $\mathcal{B}$ be equal to $\mathcal{U}$ or $\mathcal{B}_1$ and let $\{\mathrm{Q}_N\}_{N\in\mathbb{N}}$ be as in Theorem 3. Then there exists a residual subset $\mathcal{R} \subset \mathcal{B}$ such that*

$$\limsup_{N\to\infty} \|\mathrm{Q}_N f\|_\infty = +\infty \qquad \textit{for all } f \in \mathcal{R} \ .$$

So the set of all f in $\mathcal{U}$ and $\mathcal{B}_1$ for which the peak value of the approximation $\mathrm{Q}_N f$ can be controlled is a meager set. As an immediate consequence we have the following statement.

Corollary 5: *Let $\mathcal{B}$ be equal to $\mathcal{U}$ or $\mathcal{B}_1$, and let $\boldsymbol{\zeta}$ be the Fourier basis of $\mathcal{B}$. There exists no numerical integration method such that the sequence of approximation operators $\{\mathrm{Q}_N\}_{N\in\mathbb{N}}$ in* (12) *satisfies properties (i) and (ii) and such that*

$$\lim_{N\to\infty} \|\mathrm{Q}_N f - f\|_\infty = 0 \qquad \textit{for all } f \in \mathcal{B} \ .$$

Remark: Corollary 4 showed divergence in the uniform norm. Since $\|f\|_{\mathcal{U}} \geq \|f\|_\infty$ and $\|g\|_{\mathcal{B}_1} \geq \|g\|_\infty$ for all $f \in \mathcal{U}$ and $g \in \mathcal{B}_1$ these results imply the divergence in the Banach space norm of $\mathcal{U}$ and $\mathcal{B}_1$.

Remark: Since $\mathcal{B}_1 \subset \mathcal{B}_{\alpha,\beta}$ for all $0 \leq \alpha, \beta \leq 1$, it follows that Thm. 3 and Corollaries 4 and 5 hold for all $\mathcal{B}_{\alpha,\beta}$ with $0 \leq \alpha, \beta \leq 1$, and since $\mathcal{B}_1 \subset \mathcal{U}$ (cf. Lemma 2), these results hold also for $\mathcal{U}$.

Banach spaces with convergent methods Considering smaller and smaller subsets of $\mathcal{A}(\mathbb{D})$, one should finally find subspaces of $\mathcal{A}(\mathbb{D})$ such that approximations of the form (12) converge for all f in this subspace. Indeed, our next theorem presents such subspaces.

Theorem 6: *Let $\alpha \geq 1$ and $\beta > 1$ and let $\{\mathrm{Q}_N\}_{N\in\mathbb{N}}$ be the sequence defined in* (12) *with the coefficients $c_{N,n}(f)$ calculated by* (11) *with $M(N) \geq N$. Then*

$$\lim_{N\to\infty} \|\mathrm{Q}_N f - f\|_{\mathcal{B}_{\alpha,\beta}} \qquad \textit{for all } f \in \mathcal{B}_{\alpha,\beta} \ . \qquad (14)$$

Remark: Note that the norm convergence (14) implies the uniform convergence for all $f \in \mathcal{B}_{\alpha,\beta}$.

So we see that the negative result of Theorem 3 is sharp in the scale of the Banach spaces $\{\mathcal{B}_{\alpha,\beta}\}$: For all $\alpha, \beta \leq 1$ no convergent method exist whereas for any pair α, β with $\alpha \geq 1$ and $\beta > 1$ even very simply approximations as in Example 1 always converge.

4. COMPUTATIONAL BASES

It is known that there exist Banach spaces $\mathcal{B}$ which posses a basis $\{\varphi_n\}_{n=0}^\infty$ such that the corresponding coefficient functionals $\{c_n(f)\}$ depend only on finitely many samples of f.

Definition 1 (Computational basis): *Let $\mathcal{B}$ be a separable Banach space of continuous functions on $\mathbb{T}$ and let $\boldsymbol{\varphi} = \{\varphi_n\}_{n=0}^\infty$ be a basis for $\mathcal{B}$. We call $\boldsymbol{\varphi}$ a* computational basis *if the corresponding coefficient functionals $\{c_n(f)\}$ of $\boldsymbol{\varphi}$ have the following property:*
To every $n = 0, 1, 2, \dots$ there exists an $\mu(n) \in \mathbb{N}$ and distinct numbers $z_{1,n}, \dots, z_{\mu,n} \in \mathbb{T}$ such that $c_n(f)$ does only depend on the values $f(z_{k,n})$, $1 \leq k \leq \mu(n)$ for every $f \in \mathcal{B}$.

Remark: In other words, $\boldsymbol{\varphi}$ is a computational basis if and only if for all functions $f, g \in \mathcal{B}$ with $f(z_{k,n}) = g(z_{k,n})$ for all $k = 1, \dots, \mu(n)$ one has $c_n(f) = c_n(g)$ for all $n = 0, 1, 2, \dots$.

So a computational basis allows to determine the approximation (5) perfectly from only finitely many samples of f. Then $\mathrm{P}_N f \to f$, since $\{\varphi_n\}_{n=0}^{\infty}$ is a basis. For a large number of Banach spaces of continuous functions on $\mathbb{T}$, computational bases are known [18]. One concrete example is the *spline basis* for $C(\mathbb{T})$.

We want to investigate whether $\mathcal{U}$ and $\mathcal{B}_1$ posses a computational basis. Since $\mathcal{A}(\mathbb{D})$ is a closed dense subspace of $C(\mathbb{T})$ and since $\mathcal{U}, \mathcal{B}_1 \subset \mathcal{A}(\mathbb{D})$, it is clear that every f in $\mathcal{U}$ or $\mathcal{B}_1$ can arbitrarily well approximated in any computational basis of $\mathcal{C}(\mathbb{T})$. However, the corresponding approximate function *does not* belong to $\mathcal{U}$ or $\mathcal{B}_1$ but only to $\mathcal{C}(\mathbb{T})$. So the approximating function $\widetilde{f}_N$ may not be causal, thought we want to approximate a causal f. Thus a computational basis for $\mathcal{C}(\mathbb{T})$ does generally not give a causal system approximation. Therefore, we investigate next whether the causal subspaces $\mathcal{U}$ and $\mathcal{B}_1$ posses a computational basis. The following theorem gives a negative answer.

Theorem 7: *The spaces $\mathcal{U}$ and $\mathcal{B}_1$ possess no computational basis.*

Remark: Again, since $\mathcal{B}_1 \subset \mathcal{B}_{\alpha,\beta}$ for all $0 \le \alpha, \beta \le 1$, it easily follows that Theorem 7 holds for all spaces $\mathcal{B}_{\alpha,\beta}$ with $0 \le \alpha, \beta \le 1$.

Remark: We note without proof that in all $\mathcal{B}_{\alpha,\beta}$ with $\alpha \ge 1$ and $\beta > 1$ it is possible to find a computational basis $\{\varphi_n\}_{n=0}^{\infty}$ such that $\lim_{N\to\infty} \left\| \sum_{n=0}^{N} c_n(f)\, \varphi_n - f \right\|_\infty = 0$ for all $f \in \mathcal{B}_{\alpha,\beta}$.

5. PROOF SKETCHES OF THE MAIN RESULTS

We start with a detailed sketch of proof of Theorem 3.

Sketch of proof (Thm. 3): The poof is divided into several steps:
1) We assume the contrary, namely that

$$\liminf_{N\to\infty} \|\mathrm{Q}_N\|_{\mathcal{B}_1\to\mathcal{A}(\mathbb{D})} < \infty \,. \qquad (15)$$

By passing to a subsequence if necessary, we can assume that $C_1 := \sup_{N\in\mathbb{N}} \|\mathrm{Q}_N\|_{\mathcal{B}_1\to\mathcal{A}(\mathbb{D})} < +\infty$.
2) One considers the approximation error $\mathrm{R}_N f := f - \mathrm{Q}_N f$. Assumption (15) shows that for every $k = 0, 1, \ldots$ the sequence $\{(\mathrm{R}_N \zeta_k)(z)\}_{N\in\mathbb{N}}$ is uniformly bounded in $\mathbb{D}$ and so it converges for every $z \in \mathbb{D}$. Using a theorem of Vitali [22, § 5.2] and Cauchy's integral theorem, one shows that for all $r \in (0,1)$

$$\lim_{N\to\infty} \max_{|z|<r} \left| \left(\mathrm{R}_N \zeta_k\right)(z) \right| = \lim_{N\to\infty} \max_{|z|<r} \left|\zeta_k(z) - \mathrm{Q}_N \zeta_k(z)\right| = 0 \,.$$

3) Let $p(z) = \sum_{k=-M}^{M} a_k z^k$ be an arbitrary trigonometric polynomial with its causal part denoted by $p_+(z) = \sum_{k=0}^{M} a_k\, \zeta_k(z)$. Then Step 2) implies $\lim_{N\to\infty} \left(\mathrm{Q}_N\, p_+\right)(z) = p_+(z)$ for all $z \in \mathbb{D}$ and one can show that there exists a constant C_2 such that

$$\begin{aligned} \left| \left(\mathrm{Q}_N\, p_+\right)(z) \right| &\le \tfrac{1}{2\pi} \int_{-\pi}^{\pi} \left| \left(\mathrm{Q}_N\, p_\theta\right)(z \mathrm{e}^{\mathrm{i}\theta}) \right| \mathrm{d}\theta \\ &\le C_2\, \|\mathrm{Q}_N\|_{\mathcal{B}_1\to\mathcal{A}(\mathbb{D})}\, \|p\|_{\mathcal{C}(\mathbb{T})} \le C_2\, C_1\, \|p\|_{\mathcal{C}(\mathbb{T})} \,. \end{aligned}$$

Together with the previous equation we thus get

$$|p_+(z)| \le C_2\, C_1\, \|p\|_{\mathcal{C}(\mathbb{T})} \,, \qquad z \in \mathbb{D} \,. \qquad (16)$$

4) Finally, one considers the particular trigonometric polynomial

$$q(z) = \tfrac{1}{2\mathrm{i}} \sum_{k=1}^{K} \tfrac{1}{k} \left(z^k - z^{-k}\right)$$

for which there exists a constant C_3 such that $\|q\|_{\mathcal{C}(\mathbb{T})} \le C_3$ for all $K \in \mathbb{N}$ [23]. Then (16) yields $|q_+(z)| \le C_3\, C_2\, C_1$ for all $z \in \mathbb{D}$. Since the right hand side does not depend on z this inequality holds also for $|z| = 1$. However, for $z = 1$ we get

$$C_1\, C_2\, C_3 \ge |q_+(1)| = \left| \sum_{k=1}^{K} \tfrac{1}{k} \right| \ge \log(K+1) \,,$$

and for sufficiently large K this yields a contradiction. ∎

Next, we are going to prove Theorem 7. This proof is based on a lemma from [24]. It makes a statement on a class of linear operators characterized by two simple axioms, which are defined next.

Definition 2: *Let $\{\mathrm{T}_N\}_{N\in\mathbb{N}}$ be a sequence of bounded linear operators $\mathrm{T}_N : \mathcal{A}(\mathbb{D}) \to \mathcal{A}(\mathbb{D})$. We say that $\{\mathrm{T}_N\}_{N\in\mathbb{N}}$ satisfies Axiom*

(A) if to every $N \in \mathbb{N}$ there is an $M(N) \in \mathbb{N}$ and a finite set $\mathcal{Z}_N = \{z_{N,1}, \ldots, z_{N,M}\} \subset \mathbb{T}$ so that for all $f_1, f_2 \in \mathcal{A}(\mathbb{D})$

$$f_1(z_{N,k}) = f_2(z_{N,k}) \qquad \forall\, k = 1, \ldots, M(N)$$

$$\textit{implies} \quad \left(\mathrm{T}_N f_1\right)(z) = \left(\mathrm{T}_N f_2\right)(z) \quad \forall\, z \in \overline{\mathbb{D}} \,.$$

(B) if there exists a dense subset $\mathcal{M} \subset \mathcal{A}(\mathbb{D})$ such that

$$\lim_{N\to\infty} \|\mathrm{T}_N f - f\|_\infty = 0 \qquad \textit{for all } f \in \mathcal{M} \,.$$

For this class of operators the following result can be proven [24].

Lemma 8: *Let $\{\mathrm{T}_N\}_{N\in\mathbb{N}}$ be a sequence of bounded linear operators $\mathrm{T}_N : \mathcal{A}(\mathbb{D}) \to \mathcal{A}(\mathbb{D})$ satisfying Axioms (A) and (B), then there are functions $f_1 \in \mathcal{U}$ and $f_2 \in \mathcal{B}_1$ such that*

$$\limsup_{N\to\infty} \|\mathrm{T}_N f_i\|_\infty = +\infty \,, \qquad i = 1, 2 \,.$$

Based on this Lemma, Theorem 7 can easily verified.

Proof (Thm. 7): We prove the statement for $\mathcal{B}_1$. Let $\boldsymbol{\varphi} = \{\varphi_n\}_{n=0}^{\infty}$ be a computational basis for $\mathcal{B}_1$ with coefficient functionals $\{c_n\}_{n=0}^{N}$. For any $f \in \mathcal{B}_1$ and $N \in \mathbb{N}$ we consider the partial sum $\mathrm{P}_N f$ given in (5). It is obvious that $\{\mathrm{P}_N\}_{N\in\mathbb{N}}$ satisfies Axiom (A) with the sampling sets $\mathcal{Z}_N = \cup_{n=0}^{N} \{z_{k,n}\}_{k=1}^{\mu(n)}$. Since $\boldsymbol{\varphi}$ is a basis for $\mathcal{B}_1$ we have $\lim_{N\to\infty} \|\mathrm{P}_N f - f\|_{\mathcal{B}_1} = 0$ for every $f \in \mathcal{B}_1$, and there exists a constant C_4 such that

$$\|\mathrm{P}_N\|_{\mathcal{B}_1\to\mathcal{A}(\mathbb{D})} \le \|\mathrm{P}_N\|_{\mathcal{B}_1\to\mathcal{B}_1} \le C_4 \qquad \text{for all } N \in \mathbb{N} \,, \quad (17)$$

where the first inequality follows from $\|\mathrm{P}_N f\|_\infty \le \|\mathrm{P}_N f\|_{\mathcal{B}_1}$. So

$$\lim_{N\to\infty} \|\mathrm{P}_N f - f\|_\infty = 0 \qquad \text{for all } f \in \mathcal{B}_1 \,. \qquad (18)$$

Next we observe that $\mathcal{B}_1$ is dense in $\mathcal{A}(\mathbb{D})$. This follows because $\boldsymbol{\varphi}$ is a basis for $\mathcal{B}_1$ and because the Fejér-means of the power series (7) converges uniformly to f for any $f \in \mathcal{A}(\mathbb{D})$ [23]. So (18) shows that $\{\mathrm{P}_N\}_{N\in\mathbb{N}}$ satisfies (B) with $\mathcal{M} = \mathcal{B}_1$. Then Lemma 8 implies $\lim_{N\to\infty} \|\mathrm{P}_N\|_{\mathcal{B}_1\to\mathcal{A}(\mathbb{D})} = +\infty$. This contradicts (17) and so the assumption that $\boldsymbol{\varphi}$ is a computational basis for $\mathcal{B}_1$ was wrong. The proof for $\mathcal{U}$ is almost exactly the same. ∎

6. DISCUSSION AND OUTLOOK

Up to now there exist only results showing that *particular* filter design and approximation methods diverge on the disk algebra $\mathcal{A}(\mathbb{D})$ [25]. There was no observation showing that to *every* proposed design method there always exist some transfer functions for which divergence occurs. So a general theory to investigate divergence and convergence of practical methods is missing. This paper presented some results in this direction and gave a precise characterization of subspaces of $\mathcal{A}(\mathbb{D})$ on which approximations due to basis expansions are practically possible and on which they are impossible.

In [26] a result similar to Theorem 3 but for approximations of the Hilbert transform was proven, and in [27] this result was extended to a scale of Sobolev-like spaces similar to $\mathcal{B}_{\alpha,\beta}$. For all these results, it is crucial that the approximation operators (12) are *linear*. Then standard techniques from functional analysis can be applied. In a recent work [28], it was possible to drop this linearity condition for Hilbert transform approximation and it is an interesting open question whether Theorem 3 holds also for *non-linear* approximation operators Q_N having Properties (i) and (ii) from Sec. 3.

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