KERNEL-INDUCED SAMPLING THEOREM FOR TRANSLATION-INVARIANT REPRODUCING KERNEL HILBERT SPACES WITH UNIFORM SAMPLING

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ABSTRACT

The kernel-induced sampling theorem enables us to determine whether the sampling theorem holds or not for a reproducing kernel Hilbert space and a given set of sampling points. However, it is not easy to specifically calculate the necessary and sufficient condition formula except in some special cases, since it includes the inverse of an infinite dimensional Gramian matrix. In this paper, we discuss the kernel-induced sampling theorem restricted to a translation-invariant reproducing kernel Hilbert space with uniform sampling; and introduce an alternative necessary and sufficient condition formula, in which the inverse of the Gramian matrix is explicitly treated, by incorporating the theory of Laurent operators.

Index Terms— reproducing kernel Hilbert space, kernelinduced sampling theorem, translation-invariant space, uniform sampling, Laurent operator

1. INTRODUCTION

The sampling theorem [1] is one of crucial foundations in the field of digital signal processing since it guarantees that obtained digital signals contain all information of the corresponding original analogue signals. So far, so many generalizations and extensions of the sampling theorem have been introduced (see [2, 3] and their references cited in). In particular, the relationship between the sampling theory and the theory of reproducing kernel Hilbert spaces [4, 5, 6] is noteworthy since the theory of reproducing kernel Hilbert spaces gives a unified mathematical framework for sampling and reconstruction processes in the sampling theory [7, 8]. Motivated by these works, we introduced a necessary and sufficient condition formula of the sampling theorem for general reproducing kernel Hilbert spaces in [9, 10], called the 'Kernel-Induced Sampling Theorem'. By using this formula, we can determine whether the sampling theorem holds or not for the given kernel, corresponding to a considered reproducing kernel Hilbert space, and a given set of sampling points. In fact, we gave alternative proofs of the Shannon's sampling theorem in [9] and the sampling theorem for the space of bandpass functions in [10]; and we proved that a certain Sobolev space did not have a sampling theorem in [9]. However, it is not so easy to specifically calculate the formula in more general cases since the formula includes the inverse of an infinite dimensional Gramian matrix. Due to this difficulty, it is hard to explicitly confirm whether the necessary and sufficient condition formula holds or not, and it is even difficult to confirm it numerically.

As can be seen in the Shannon's sampling theorem, the sampling theorem for translation-invariant spaces with uniform sampling has a special importance in the field of digital signal processing. Under such conditions, the Gramian matrix is reduced to a Laurent operator [11]. In this paper, we introduce an alternative necessary and sufficient condition formula of the kernel-induced sampling theorem specialized for translation-invariant reproducing kernel Hilbert spaces with uniform sampling on the basis of the theory of Laurent operators by which the difficulty of the inverse of the infinite dimensional Gramian matrix is resolved.

2. REPRODUCING KERNEL HILBERT SPACES

In this section, we give a brief overview of the theory of reproducing kernel Hilbert spaces [4, 5, 6].

Definition 1 [4] Let \mathbf{R}^d be a d-dimensional real vector space and let \mathcal{H} be a class of functions defined on \mathbf{R}^d , forming a Hilbert space of real-valued functions. The function $K(\boldsymbol{x}, \boldsymbol{y})$, $(\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^d)$ is called a reproducing kernel of \mathcal{H} , if (1) $\forall \boldsymbol{x} \in \mathbf{R}^d$, $K(\cdot, \boldsymbol{x}) \in \mathcal{H}$, and (2) $\forall \boldsymbol{x} \in \mathbf{R}^d$, $\forall f \in \mathcal{H}$,

$$f(\boldsymbol{x}) = \langle f(\cdot), K(\cdot, \boldsymbol{x}) \rangle_{\mathcal{H}},\tag{1}$$

hold, where $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ denotes the inner product of the Hilbert space \mathcal{H} .

The Hilbert space \mathcal{H} that has a reproducing kernel is called a reproducing kernel Hilbert space (RKHS). Eq.(1) is called the reproducing property of a kernel. Note that a reproducing kernel is positive definite and symmetric, and has the unique corresponding RKHS [4]. Hereafter, we use the symbol \mathcal{H}_K for the RKHS corresponding to a kernel K.

The following theorem is one of the most important properties of an RKHS.

Theorem 1 [9] The set $\{K(\cdot, \boldsymbol{y}) \mid \boldsymbol{y} \in \mathbf{R}^d\}$ is complete in \mathcal{H}_K .

According to this theorem, any function $f \in \mathcal{H}_K$ can be represented by

$$f(\cdot) = \sum_{\boldsymbol{y} \in \mathbf{R}^d} c_{\boldsymbol{y}} K(\cdot, \boldsymbol{y}), \qquad (2)$$

with certain coefficients $cy \in \mathbf{R}$.

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3. KERNEL-INDUCED SAMPLING THEOREM

In this section, we give a brief overview of the kernel-induced sampling theorem introduced in [9].

We consider the RKHS \mathcal{H}_K corresponding to a certain kernel K and set of sampling points $X = \{x_k \in \mathbb{R}^d \mid k \in \mathbb{N}\}$, where \mathbb{N} stands for the set of natural numbers. The goal of the kernel-induced sampling theorem is to obtain a necessary and sufficient condition for K and X to perfectly reconstruct any function $f \in \mathcal{H}_K$ as a linear combination of $K(\cdot, x_k)$, $(k \in \mathbb{N})$. In other words, we need a necessary and sufficient condition for K and X that leads $\mathcal{H}_K = S$, where

$$S = \operatorname{span}\{K(\cdot, \boldsymbol{x}_k) \mid \boldsymbol{x}_k \in X, \ k \in \mathbf{N}\}$$
(3)

with $\overline{}$ denoting the closure of a given set. Let P_S be the orthogonal projector onto S in \mathcal{H}_K , whose closed-form [9] is given by

$$P_S = \sum_{j,k \in \mathbf{N}} G_{jk}^{-1}(K(\cdot, \boldsymbol{x}_j) \otimes K(\cdot, \boldsymbol{x}_k)), \qquad (4)$$

where $G = (K(\boldsymbol{x}_j, \boldsymbol{x}_k))$ denotes the Gramian matrix of K with Xand \otimes denotes the Schatten product [12, 9]. Then, $\mathcal{H}_K = S$ if and only if $f(\cdot) - P_S f(\cdot) = 0$ for any $f \in \mathcal{H}_K$. Since any function $f \in \mathcal{H}_K$ can be represented by Eq.(2), we can say that $\mathcal{H}_K = S$ if and only if

$$K(\cdot, \boldsymbol{y}) - P_S K(\cdot, \boldsymbol{y}) = 0 \tag{5}$$

holds for any $\boldsymbol{y} \in \mathbf{R}^d$. Evaluating the squared norm of the both sides of Eq.(5) by using the reproducing property Eq.(1) and Eq.(4) and applying the Pythagorean theorem lead the following theorem (see [9] for more details).

Theorem 2 [9]
$$\mathcal{H}_K = S$$
 if and only if

$$K(\boldsymbol{y}, \boldsymbol{y}) - \sum_{j,k \in \mathbf{N}} K(\boldsymbol{y}, \boldsymbol{x}_j) G_{jk}^{-1} K(\boldsymbol{y}, \boldsymbol{x}_k) = 0 \quad (6)$$

holds for any $y \in \mathbf{R}^d$.

By checking whether Eq.(6) holds or not, we can determine whether the sampling theorem holds for \mathcal{H}_K with X or not. However in general, it is hard to explicitly calculate G^{-1} in Eq.(6), and it is even hard to calculate numerically.

4. KERNEL-INDUCED SAMPLING THEOREM FOR TRANSLATION-INVARIANT RKHS WITH UNIFORM SAMPLING

In this section, we discuss the kernel-induced sampling theorem for a one dimensional translation-invariant RKHS with uniform sampling.

A translation-invariant RKHS [13] is defined as the RKHS \mathcal{H}_{K} such that

$$f(\cdot) \in \mathcal{H}_K \Rightarrow f(\cdot - \mathbf{r}) \in \mathcal{H}_K$$
 (7)

for any $\boldsymbol{r} \in \mathbf{R}^d$. When a kernel $K(\boldsymbol{x}, \boldsymbol{y})$ only depends on $\boldsymbol{x}-\boldsymbol{y}$, that is, $K(\boldsymbol{x}, \boldsymbol{y}) = K_T(\boldsymbol{x}-\boldsymbol{y})$ holds with some one variable function K_T , the corresponding \mathcal{H}_K is a translation-invariant RKHS since when $K(\cdot, \boldsymbol{x}) \in \mathcal{H}_K$ for a certain $\boldsymbol{x} \in \mathbf{R}^d$ (which is guaranteed by Definition 1), we have

$$K(\cdot - \boldsymbol{r}, \boldsymbol{x}) = K_T(\cdot - \boldsymbol{r} - \boldsymbol{x}) = K(\cdot, (\boldsymbol{x} + \boldsymbol{r})) \in \mathcal{H}_K$$

for any $r \in \mathbf{R}^d$, which immediately implies that Eq.(7) holds from Eq.(2). Hereafter, we assume that a kernel *K* is corresponding to a translation-invariant RKHS defined on **R**.

Let us consider the set of one dimensional equidistant sampling points $X = \{sk \mid k \in \mathbf{Z}\}$ with a certain positive real number s^{-1} , where \mathbf{Z} stands for the set of integers. Then, the Gramian matrix of K with X is reduced to

$$G = (K(sj, sk)) = (K_T(s(j - k)))$$

$$= \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & K_T(0) & K_T(-s) & K_T(-2s) & \ddots \\ \ddots & K_T(s) & K_T(0) & K_T(-s) & \ddots \\ \ddots & K_T(2s) & K_T(s) & K_T(0) & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, (8)$$

where $K_T(0)$ is the (0,0) position of the doubly infinite matrix G. It is trivial that G is symmetric, since a kernel is symmetric. It is known that the matrix given by Eq.(8) (not limited to be symmetric in general) is called a Laurent operator [11].

Here, we briefly describe the theory of Laurent operators according to the descriptions in [11] (see Chapter III in [11] for more details).

Definition 2 [11] A Laurent operator A is a bounded linear operator on $\ell_2(\mathbf{Z})$ with the property that the matrix of A with respect to the standard orthonormal basis $\{e_j\}_{j \in \mathbf{Z}}$ of $\ell_2(\mathbf{Z})$ is of the form

$$A = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & a_0 & a_{-1} & a_{-2} & \ddots \\ \ddots & a_1 & \boxed{a_0} & a_{-1} & \ddots \\ \ddots & a_2 & a_1 & a_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} .$$
(9)

Let $L_2([-\pi, \pi])$ be the Hilbert space of square integrable functions defined on $[-\pi, \pi]$, provided the inner product²

$$\langle f,g\rangle_{L_2([-\pi,\pi])} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)}dt, \qquad (10)$$

where $\overline{\cdot}$ stands for the complex conjugate of a give complex value (or a complex valued function).

Let $\alpha(t)$ be a bounded complex valued Lebesgue measurable function defined on $[-\pi, \pi]$, and let M be the corresponding operator of multiplication by $\alpha(t)$ on $L_2([-\pi, \pi])$, that is,

$$(Mf)(t) = \alpha(t)f(t), \quad f \in L_2([-\pi,\pi]).$$
 (11)

Then, the operator M is a bounded linear operator on $L_2([-\pi, \pi])$ and the matrix of M with respect to the orthonormal basis³ $\phi_n(t) = e^{int}$, $(n \in \mathbb{Z})$ is given by Eq.(9), where

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \alpha(t) e^{-int} dt, \quad n \in \mathbf{Z}.$$
 (12)

²In [11], division by 2π is missing. However, it is needed for the contents in [11] to be consistent.

¹Here, s represents the interval of sampling points.

 $^{{}^{3}}$ In [11], $e^{int}/\sqrt{2\pi}$ is used for the orthonormal basis function with the standard L_{2} inner product. However, it is also inconsistent with the other contents in [11].

It follow that the operator A in Eq.(9) is represented by $A = \mathcal{F}M\mathcal{F}^{-1}$, where \mathcal{F} is the Fourier transform on $L_2([-\pi,\pi])$, defined as

$$\mathcal{F}f(t) = (f_n)_{n \in \mathbf{Z}}, \ f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$$
 (13)

whose corresponding inverse Fourier transform is given by

$$\mathcal{F}^{-1}(f_n)_{n\in\mathbf{Z}} = \sum_{n\in\mathbf{Z}} f_n e^{int}.$$
 (14)

When $(a_k)_{k \in \mathbb{Z}}$ is given a priori,

$$\alpha(t) = \mathcal{F}^{-1}(a_k)_{k \in \mathbf{Z}} = \sum_{k \in \mathbf{Z}} a_k e^{ikt}, \qquad (15)$$

holds, which is confirmed by substituting Eq.(15) into Eq.(12). Accordingly, $A^{-1} = \mathcal{F}M^{-1}\mathcal{F}^{-1}$ holds, where the operator M^{-1} is specified by

$$(M^{-1}f)(t) = \frac{1}{\alpha(t)}f(t), \quad f \in L_2([-\pi,\pi]).$$
 (16)

Note that \mathcal{F}^{-1} is the adjoint operator of \mathcal{F} , denoted by \mathcal{F}^* , since for any $f(t) = \sum_{n \in \mathbb{Z}} f_n e^{int} \in L_2([-\pi, \pi])$ and any $g = (g_n)_{n \in \mathbb{Z}} \in \ell_2(\mathbb{Z})$, we have

$$\begin{split} \langle f(t), \mathcal{F}^{-1} \boldsymbol{g} \rangle_{L_2([-\pi,\pi])} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n \in \mathbf{Z}} f_n e^{int} \right) \overline{\left(\sum_{n \in \mathbf{Z}} g_n e^{int} \right)} dt \\ &= \frac{1}{2\pi} \sum_{n \in \mathbf{Z}} f_n \overline{g_n} \int_{-\pi}^{\pi} dt = \sum_{n \in \mathbf{Z}} f_n \overline{g_n} = \langle \mathcal{F} f(t), \boldsymbol{g} \rangle_{\ell_2(\mathbf{Z})}. \end{split}$$

On the basis of the above discussions, we introduce an alternative formula of the kernel-induced sampling theorem specialized for a one dimensional translation-invariant RKHS with the set of equidistant sampling points $X = \{sk \mid k \in \mathbb{Z}\}$ with s > 0.

Let $\boldsymbol{v}_y = (K(y, sk))_{k \in \mathbf{Z}} \in \ell_2(\mathbf{Z})$ and let

$$v_y(t) = \mathcal{F}^{-1} \boldsymbol{v}_y = \sum_{k \in \mathbf{Z}} K(y, sk) e^{ikt} \in L_2([-\pi, \pi]).$$
 (17)

Also let $G = \mathcal{F}M\mathcal{F}^{-1}$, where M is an operator of multiplication by

$$u(t) = \sum_{k \in \mathbf{Z}} K(0, sk) e^{ikt} \in L_2([-\pi, \pi]),$$
(18)

which is obtained from the components of the matrix G as the same with Eq.(15). Then, the second term of the left hand side in Eq.(6) is reduced to

$$\sum_{j,k\in\mathbf{Z}} K(y,sj)G_{jk}^{-1}K(y,sk)$$

$$= \mathbf{v}_{y}^{T}G^{-1}\mathbf{v}_{y} = \langle G^{-1}\mathbf{v}_{y},\mathbf{v}_{y}\rangle_{\ell_{2}(\mathbf{Z})}$$

$$= \langle \mathcal{F}M^{-1}\mathcal{F}^{-1}\mathbf{v}_{y},\mathbf{v}_{y}\rangle_{\ell_{2}(\mathbf{Z})}$$

$$= \langle M^{-1}\mathcal{F}^{-1}\mathbf{v}_{y},\mathcal{F}^{*}\mathbf{v}_{y}\rangle_{L_{2}([-\pi,\pi])}$$

$$= \langle M^{-1}\mathcal{F}^{-1}\mathbf{v}_{y},\mathcal{F}^{-1}\mathbf{v}_{y}\rangle_{L_{2}([-\pi,\pi])}$$

$$= \left\langle \frac{1}{u(t)}v_{y}(t),v_{y}(t)\right\rangle_{L_{2}([-\pi,\pi])}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{v_{y}(t)\overline{v_{y}(t)}}{u(t)}dt.$$
(19)

Therefore, we have the following theorem.

Theorem 3 Let K be the kernel corresponding to a one dimensional translation-invariant RKHS \mathcal{H}_K defined on \mathbf{R} and let

$$S = \overline{\operatorname{span}\{K(\cdot, sk) \mid k \in \mathbf{Z}\}}, \ s > 0.$$
(20)

Then $\mathcal{H}_K = S$ if and only if

$$K(y,y) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{v_y(t)v_y(t)}{u(t)} dt = 0$$
(21)

holds for any $y \in \mathbf{R}$, where $v_y(t)$ and u(t) are given by Eqs.(17) and (18).

5. EXAMPLES

We give some examples of Theorem 3. The former two examples are the same results obtained in [9], and the others lead the knowledge which is clarified for the first time by our Theorem 3.

5.1. Shannon's Sampling Theorem

Let

$$K_1(x,y) = \frac{\sin \pi (x-y)}{\pi (x-y)}, \quad x,y \in \mathbf{R}$$
 (22)

be the sinc kernel corresponding to the RKHS consisting of all π band-limited functions, and let $X = \{k \mid k \in \mathbb{Z}\}$ be the set of sampling points.

It is trivial that $K_1(y, y) = 1$ for any $y \in \mathbf{R}$. Since

$$u(t) = \sum_{k \in \mathbf{Z}} \frac{\sin \pi (0-k)}{\pi (0-k)} e^{ikt} = 1$$
$$v_y(t) = \sum_{k \in \mathbf{Z}} \frac{\sin \pi (y-k)}{\pi (y-k)} e^{ikt}.$$

we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{v_y(t)\overline{v_y(t)}}{u(t)} dt$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k \in \mathbf{Z}} \left(\frac{\sin \pi (y-k)}{\pi (y-k)}\right)^2 dt$$
$$= \sum_{k \in \mathbf{Z}} \left(\frac{\sin \pi (y-k)}{\pi (y-k)}\right)^2 = 1$$

for any $y \in \mathbf{R}$. Thus, Eq.(21) is satisfied, which is almost the same proof with that obtained in [9] (see Section VI-A in [9]).

5.2. Sobolev Space $W^{1,2}$

The Sobolev space $W^{1,2}$ is the RKHS whose norm is defined by

$$||f||_{W^{1,2}} = \left(\int_{-\infty}^{\infty} |f(t)|^2 + |f'(t)|^2 dt\right)^{1/2}$$
(23)

and the corresponding kernel is given by

$$K_2(x,y) = \frac{1}{2}e^{-|x-y|}, \quad x,y \in \mathbf{R}$$
 (24)

as shown in [6]. It is trivial that $K_2(y, y) = 1/2$ for any $y \in \mathbf{R}$. Let $X = \{sk \mid k \in \mathbf{Z}\}$ with s > 0 be the set of sampling points. In [9], it is proved that the sampling theorem does not hold for this case. Thus, it is enough to consider $y = s/2 \notin X$ instead of any $y \in \mathbf{R}$

in order to follow the result obtained in [9]. Note that the following calculations were conducted by Mathematica [14]. Since

$$u(x) = \frac{1}{2} \sum_{k \in \mathbf{Z}} e^{-s|k|} e^{ikt} = \frac{1}{2\operatorname{csch}(s)(\cosh(s) - \cos(t))},$$

$$v_{s/2}(t) = \frac{1}{2} \sum_{k \in \mathbf{Z}} e^{-s|1/2 - k|} e^{ikt} = \frac{(1 + e^{it})\sinh(s/2)}{2(\cosh(s) - \cos(t))},$$

we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{v_{s/2} y(t) \overline{v_{s/2}(t)}}{u(t)} dt$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\tanh\left(s/2\right) \cos^{2}\left(t/2\right)}{\cosh(s) - \cos(t)} dt = \frac{1}{e^{s} + 1}, \quad (25)$$

which can not be equal to 1/2 with a positive s. Thus, the sampling theorem for $W^{1,2}$ with X does not hold with any s > 0.

Note that this result agrees to the result obtained in [9] (see Eq.(53) in Section VI-C in [9]). In fact, Eq.(25) is trivially obtained by substituting y = 1/2 to Eq.(53) in [9]. It is noteworthy that our calculation does not use the explicit formula of G^{-1} .

5.3. Sobolev Space $W^{2,2}$

The Sobolev space $W^{2,2}$ is the RKHS whose norm is defined by

$$||f||_{W^{2,2}} = \left(\int_{-\infty}^{\infty} |f(t)|^2 + |f'(t)|^2 + |f''(t)|^2 dt\right)^{1/2}$$
(26)

and the corresponding kernel is given by

$$K_3(x,y) = \frac{1}{4}e^{-|x-y|}(1+|x-y|), \quad x,y \in \mathbf{R}$$
(27)

as shown in [6]. It is trivial that $K_3(y, y) = 1/4$ for any $y \in \mathbf{R}$. Let $X = \{sk \mid k \in \mathbf{Z}\}$ with s > 0 be the set of sampling points. As the same with the previous example, we consider y = s/2, by which we intend to show that the sampling theorem does not hold for $W^{2,2}$ with X for any s > 0. We have

$$u(x) = \frac{1}{4} \sum_{k \in \mathbf{Z}} e^{-s|k|} (1+s|k|) e^{ikt}$$

$$= \frac{\cosh(s)(s\cos(t) + \sinh(s)) - \sinh(s)\cos(t) - s}{4(\cos(t) - \cosh(s))^2}$$

$$v_{s/2}(t) = \frac{1}{4} \sum_{k \in \mathbf{Z}} e^{-s|1/2-k|} (1+s|1/2-k|) e^{ikt}$$

$$= \frac{e^{2s + \frac{5it}{2}}\cos(t/2)}{(-1+e^{s+it})^2 (e^s - e^{it})^2}$$

$$\times [s\cosh(s/2) (\cosh(s) + \cos(t) - 2) + 2\sinh(s/2) (\cosh(s) - \cos(t))].$$

Table 1 shows numerical results of the left hand side of Eq.(21) with respect to some specific s's ⁴.

According to these results, it is suggested that the left hand side of Eq.(21) is positive for any s > 0, which implies that the sampling theorem does not hold for $W^{2,2}$ with a positive s, although

Table 1. Numerical results of the left hand side of Eq.(21), denoted by LHS, with some specific *s*.

s	LHS
10^{1}	$2.491832090521 \times 10^{-1}$
10^{0}	$9.620518266448 \times 10^{-3}$
10^{-1}	$1.091301103237 \times 10^{-5}$
10^{-2}	$1.092733542065 \times 10^{-8}$
10^{-3}	$1.092747881898 \times 10^{-11}$
10^{-4}	$1.092748025298 \times 10^{-14}$
10^{-5}	$1.092748026732 \times 10^{-17}$
10^{-6}	$1.092748026746 \times 10^{-20}$

theoretical asymptotic behavior is not revealed. However, it is noteworthy that these results are obtained for the first time by our formula Eq.(21), while we have no way to obtain the same results by the original formula Eq.(6).

5.4. Gaussian RKHS

Let

$$K_4(x,y) = e^{-\sigma(x-y)^2}, \quad x,y \in \mathbf{R}$$
 (28)

be the popular Gaussian kernel with the shape parameter $\sigma > 0$, which is frequently used in many applications, such as machine learning problems. However, the existence (or non-existence) of the sampling theorem for the corresponding RKHS have not been discussed so far.

Is is trivial that $K_4(y, y) = 1$ for any $y \in \mathbf{R}$. Let $X = \{k \mid k \in \mathbf{Z}\}$ be the set of sampling points⁵. We have

$$u(t) = \sum_{k \in \mathbf{Z}} e^{-\sigma k^2} e^{ikt}$$

= $\sqrt{\frac{\pi}{\sigma}} e^{-\frac{t^2}{4\sigma}} \vartheta_3 \left(-\frac{i\pi t}{2\sigma}, e^{-\frac{\pi^2}{\sigma}}\right),$
 $v_y(t) = \sum_{k \in \mathbf{Z}} e^{-\sigma(y-k)^2} e^{ikt}$
= $\sqrt{\frac{\pi}{\sigma}} e^{-\frac{t^2}{4\sigma} + ity} \vartheta_3 \left(-\frac{i\pi t}{2\sigma} - \pi y, e^{-\frac{\pi^2}{\sigma}}\right),$

where

$$\vartheta_3(u,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos(2nu),$$
 (29)

denotes the one of the four elliptic theta functions⁶. Unfortunately, it seems hopeless to specifically calculate the integral in Eq.(19) since the elliptic theta function is a transcendental function. However, it is meaningful that our theorem 3 can point out the relationship between the sampling theorem of the Gaussian RKHS and the elliptic theta function.

6. CONCLUSION

In this paper, we discussed the kernel-induced sampling theorem for a translation-invariant reproducing kernel Hilbert space with uniform sampling; and introduced an alternative and convenient necessary and sufficient condition formula specialized for these cases.

⁴Mathematica failed to calculate the integral Eq.(19) with the interval parameter s remaining in $v_{s/2}(t)$. Thus, we calculate the exact integral value of Eq.(19) by substituting some specific number to s in advance, and then numerized.

⁵The shape parameter can be regarded as the square root of the interval.

 $^{^{6}}$ The elliptic theta functions has several definitions. The definition Eq.(29) follows the definition in Mathematica.

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