WHEN DOES PERIODICITY IN DISCRETE-TIME IMPLY THAT IN CONTINUOUS-TIME?

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ABSTRACT

If the sampled version $x(n) = x_c(nT)$ of a continuous-time signal $x_c(t)$ is periodic, it does not necessarily imply that $x_c(t)$ is periodic. This paper presents some conditions under which periodicity of $x_c(t)$ is indeed implied. The conditions for this implication are more relaxed than bandlimitedness. The results place in evidence a multriate method to estimate the period of $x_c(t)$ from the samples $x(n)$. The method works better than DFT based methods when the available data segment is short and multiple hidden periods are to be estimated.

Index Terms— Periodicity, multirate sampling, bandlimited signal, Ramanujan sums.

1. INTRODUCTION

Consider the continuous-time signal $x_c(t) = e^{j2\pi t/\tau}$ which is periodic with period τ. From basic courses on signals and systems we know (e.g., Problem 1.36 in [8]) that the uniformly sampled version $x(n) = x_c(nT)$ is periodic (i.e., $x(n) = x(n+P)$ for *integer* P) if and only if the sample spacing T is such that τ/T is rational. That is, $\tau/T = p/q$ where p and q are integers assumed coprime w.l.o.g. More generally, for *any* $x_c(t)$ with period τ, one can show that $x(n)$ has integer period P if and only if τ/T is rational.

In this paper we address a different question: suppose the uniformly sampled version of some signal $x_c(t)$ is found to be periodic with integer period P. Does it necessarily mean that $x_c(t)$ is periodic? The answer is of course *no* because we can easily create counter examples. Thus Fig. 1 shows a period-3 sequence $x_c(nT)$. If we define $x_c(t)$ by filling the space between samples in an arbitray way as shown, we get an example of a non-periodic $x_c(t)$ whose sampled version $x(n)$ is periodic.

This raises the following question: under what conditions does the periodicity of $x(n)$ imply that of $x_c(t)$? Indeed, in many practical applications involving pitch estimation, one often estimates the pitch of $x_c(t)$ from the periodicity of the samples $x(n)$. How is this justified? The answer is that we have to have some apriori knowledge about $x_c(t)$. For example if $x_c(t)$ is assumed bandlimited and the sampling rate $1/T$ exceeds Nyquist rate, then periodicity of $x(n)$ *does* imply that of $x_c(t)$. However bandlimitedness is not necessary; there is a much larger class (of which bandlimited signals are special cases), as we shall show in Sec. 2 and Sec. 3. Based on this we also present a novel method to estimate the period of $x_c(t)$ from the sampled version $x(n)$. To resolve certain ambiguities inherent in the method, a multirate approach is then proposed in Sec. 4. We demonstrate that this works better than DFT based methods when the available data segment is short, and closely spaced hidden periods are to be estimated.

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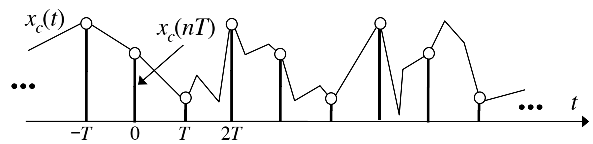

Fig. 1. A continuous-time signal $x_c(t)$ and its sampled version $x(n) = x_c(nT)$. Note that $x(n)$ is periodic with period 3, but $x_c(t)$ is not periodic.

Preliminaries. We say $x_c(t)$ is periodic if $x_c(t + R) = x_c(t)$ for all t, for some constant $R > 0$. This R is called a repetition interval. The smallest positive repetition interval is the period, denoted as τ. Similar definitions hold for discrete time signals $x(n)$ but the repetition interval and period are integers. Clearly $x_c(t) = x_c(t + k\tau)$ for any integer k so that $k\tau$ is always a repetition interval. Conversely any repetition interval R is an integer multiple of the period τ. *Proof:* We can always write $R = k\tau + r$ where $0 \leq r < \tau$ and k is an integer. Thus $x_c(t) = x_c(t + R) = x_c(t + k\tau + r) = x_c(t + r)$. Since $r < \tau$ this contradicts the fact that τ is the period, unless $r = 0$, i.e., $R = k\tau$. Similarly in discrete time, if P is the period of $x(n)$ and R a repetition interval, then $R = kP$ for integer k.

2. SIGNAL MODELS, SAMPLING, AND PERIODICITY

Even though periodicity of $x_c(nT)$ does not imply that of $x_c(t)$ in general, if we restrict $x_c(t)$ to certain classes, then periodicity of $x_c(nT)$ will indeed imply that of $x_c(t)$. Thus, consider a continuous-time signal which can be represented by the model

$$x_c(t) = \sum_{k=-\infty}^{\infty} c(k)\phi(t - kT) \tag{1}$$

Models with finite innovation rates [27] such as this arise in a number of contexts: (i) The special case where $\phi(t)$ is a sinc function will correspond to bandlimited signals. (ii) In digital communcation systems [9], a symbol stream $c(n)$, transmitted with a baseband pulse $\phi(t)$ with intersymbol spacing T (Fig. 2) gives rise to a transmitted waveform as in Eq. (1). (iii) The above signal model also arises in wavelet theory [2],

[26], [13], sampling theory [20], [19], and signal interpolation [16], [17], [25]. Observe that (1) implies in particular that $x(n)\stackrel{\Delta}{=}x_c(nT) = \sum_{k=-\infty}^{\infty} c(k)\phi(nT-kT)$. Defining

$$\Phi_d(z) = \sum_{n=-\infty}^{\infty} \phi_d(n)z^{-n} \qquad (2)$$

with impulse response $\phi_d(n) = \phi(nT)$ we see that $x(n) = \sum_{k=-\infty}^{\infty} c(k)\phi_d(n-k)$, which is a convolution. So, $x(n)$ can be regarded as the output of the digital filter $\Phi_d(z)$ in response to input $c(n)$. If the inverse filter $1/\Phi_d(z)$ is stable, we can therefore compute $c(n)$ as the output of this filter in response to the input $x(n)$. See Fig. 3.

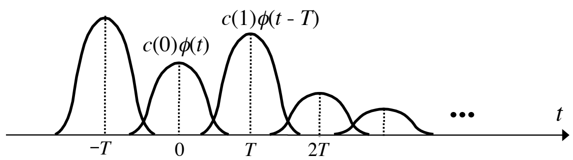

Fig. 2. The functions $c(k)\phi(t-kT)$ for consecutive values of k.

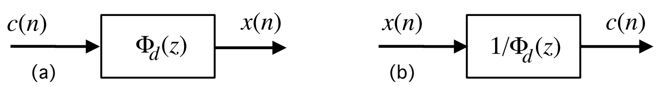

Fig. 3. (a) The sampled signal $x(n)$ regarded as the output of a digital filter, and (b) reconstruction of the coefficients $c(n)$ from the samples $x(n)$.

The beauty of the signal model (1) is that if $x(n)$ has an integer period P then $x_c(t)$ is guaranteed to be periodic:

Theorem 1. *Discrete periodicity implies continuous periodicity.* Let $x(n) = x_c(nT)$ where $x_c(t)$ has the form (1) and define the digital filter $\Phi_d(z) = \sum_{n=-\infty}^{\infty} \phi_d(n)z^{-n}$ where $\phi_d(n) = \phi(nT)$. Assume the inverse filter $1/\Phi_d(z)$ is stable. Then, if $x(n)$ is periodic-P, it implies that $x_c(t)$ is periodic with period

$$\tau = \frac{PT}{Q} \qquad (3)$$

for some integer Q coprime to P. $\diamondsuit$

Proof. Since $c(n)$ is the output of an LTI system in response to input $x(n)$ (Fig. 3(b)), it follows that if $x(n)$ has period P then $c(n) = c(n+P)$ as well. So

$$\begin{aligned} x_c(t) &= \sum_{k=-\infty}^{\infty} c(k+P)\phi(t-kT) \\ &= \sum_{k=-\infty}^{\infty} c(k)\phi(t-kT+PT) = x_c(t+PT) \end{aligned}$$

Thus $x_c(t)$ is periodic, and PT is a repetition interval. So the period can be written as $\tau = PT/Q$ for some integer $Q > 0$ (see *Preliminaries* in Sec. 1). It remains to show that P and Q are coprime. Assume the contrary so that $\tau = P_1T/Q_1$ for some integers $P_1 < P$ and $Q_1 < Q$. Then

$$\begin{aligned} x_c((n+P_1)T) &= \sum_{k=-\infty}^{\infty} c(k)\phi(nT-kT+P_1T) \\ &= \sum_{k=-\infty}^{\infty} c(k)\phi(nT-kT+Q_1\tau) \\ &= x_c(nT+Q_1\tau) = x_c(nT). \qquad (4) \end{aligned}$$

So $x(n) = x(n+P_1)$ which is a contradiction, since P is the smallest repetition interval. So P and Q are coprime. $\bigtriangledown \bigtriangledown \bigtriangledown$

3. DISCUSSION OF THE RESULT

Theorem 1 shows that the period τ of $x_c(t)$ is related to the integer period P of $x(n)$ by the formula $\tau = PT/Q$. The only difficulty is that we do not know Q, which can be any integer coprime to P. If we have the apriori information that $\tau > T$ (period is larger than the sample spacing which is typical), then $Q < P$. For example if $P = 4$ then $Q \in \{1, 3\}$, whereas if $P = 15$, then $Q \in \{1, 2, 4, 7, 8, 11, 13, 14\}$. We will see later how this Q-ambiguity can be resolved.

For the special case where $\phi(t)$ is the sinc function $\phi_{sinc}(t) = \sin(\pi t/T)/(\pi t/T)$, $x_c(t)$ is *bandlimited* to $|\omega| < \pi/T$. According to the sampling theorem [8] all such signals can be represented as in Eq. (1) where $c(k) = x_c(kT)$ (samples taken at the Nyquist rate). In this case $\phi_d(n) = \phi_{sinc}(nT) = \delta(n)$ so $\Phi_d(z) = 1$. So by Theorem 1, if $x_c(t)$ is bandlimited to $|\omega| < \pi/T$, and if $x_c(nT)$ has period P, then $x_c(t)$ has period $\tau = PT/Q$ for some integer Q coprime to P. Since $x_c(t)$ is bandlimtied to $|\omega| < \pi/T$, the fundamental frequency $2\pi/\tau < \pi/T$, which shows that $\tau > 2T$, so that $1 \leq Q < P/2$ in this case. Thus, τ can be identified with reduced Q-ambiguity. For example if $P = 10$ then $Q \in \{1, 3\}$ whereas if $P = 9$ then $Q \in \{1, 2, 4\}$.

Theorem 1 holds whether $x_c(t)$ is bandlimited or not. Imagine $\phi(t)$ (hence $x_c(t)$) is a high frequency narrowband signal as in Fig. 4(a). This can be sampled at a low rate proportional to the bandwidth 2σ to estimate its period. Another example is where a signal has missing fundamental, and contains only a few of high frequency harmonics (Fig. 4(b)). More generally $\phi(t)$ can even be a spline, and so forth.

In the bandlimited case, $X_c(j\Omega)$ is a line spectrum with finite number of lines. So if $x(n)$ is sampled above the Nyquist rate then $X(e^{j\omega})$ is the same line spectrum except for a scaling of the axis. If the line frequencies of $X(e^{j\omega})$ with nonzero amplitudes are at $k_i\beta$ and the gcd of the nonzero integers k_i is 1, then β is the fundamental, and $\eta\stackrel{\Delta}{=}2\pi/\beta$ is sometimes regarded as the "period" of $x(n)$ even if it is not an integer. In such cases, the period of $x_c(t)$ is indeed ηT [1]. But when $x(n)$ has an integer period P, the estimated period of $x_c(t)$ is not necessarily $\tau = PT$. This is because, periodicity of P does not necessarily imply there is line at $2\pi/P$, it only implies there are lines at $2\pi n_i/P$ where the nonzero n_i's have a gcd g coprime to P. However many authors [28], [6], [12], [3], [11] essentially assume, explicitly or implicitly, that $\tau = PT$, although in reality the estimate can only be claimed to be $\tau = PT/Q$ where Q is an unknown integer.

In problems where the pitch (e.g., of speech) is known to lie in certain reasonable ranges, the ambiguity Q gets resolved from this practical knowledge. Also, it is well-known that interpolation of difference function [3] or autocorerlation [28] is useful to refine integer period estimates. We next show how we can even resolve the Q-ambiguity through oversampling or interpolation using knowledge of the signal model alone.

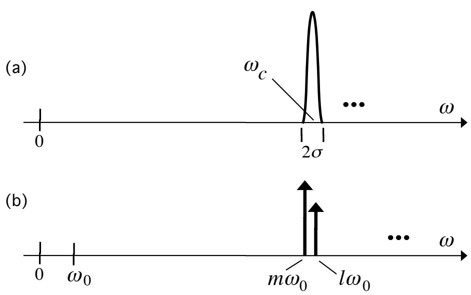

Fig. 4. (a) Fourier transform of a narrowband bandpass periodic signal, and (b) periodic signal with missing fundamental, and two high-frequency harmonics.

4. RESOLVING THE AMBIGUITY IN PERIODICITY

Assume $x_c(t)$ not only admits a model of the form (1) where $1/\Phi_d(z) = 1/\sum_n \phi(nT)z^{-n}$ is a stable digital filter, but furthermore it can *also* be modeled as

$$x_c(t) = \sum_{k=-\infty}^{\infty} d(k)f(t - k\frac{T}{L}) \qquad (5)$$

for some integer L. This would be the case, for example, when $x_c(t)$ is bandlimited to $|\omega| < \pi/T$ because it would then be bandlimited to $|\omega| < \pi L/T$ as well. In this case $\phi(t) = \sin(\pi t/T)/(\pi t/T)$ and $f(t) = \sin(\pi Lt/T)/(\pi Lt/T)$. Fig. 5 demonstrates $\phi(t - kT)$ and $f(t - k\frac{T}{L})$ for $L = 2$. Another situation is where $\phi(t)$ is the scaling function arising in multiresolution theory [5], [2]. In this case $\{\phi(t - kT)\}$ spans V_0 and $\{\phi(2(t - kT/2))\}$ spans V_1, and $V_0 \subset V_1$.

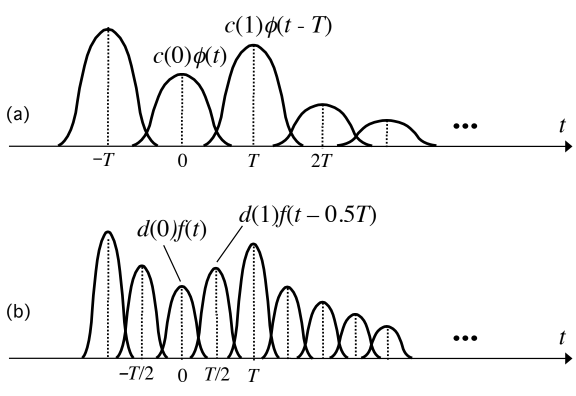

Fig. 5. (a) The functions $c(k)\phi(t - kT)$ for consecutive values of k, and (b) the functions $d(k)f(t - k\frac{T}{L})$ for consecutive values of k demonstrated for $L = 2$.

Define again the digital filter $F_d(z) = \sum_n f_d(n)z^{-n}$ with oversampled impulse response $f_d(n) = f(nT/L)$ and assume $1/F_d(z)$ is stable, so Theorem 1 is applicable to (5) as well. By applying Theorem 1 to model (1) we conclude that if $x(n) \triangleq x_c(nT)$ is periodic-P then $x_c(t)$ is periodic with period $\tau = PT/Q$ where P and Q are coprime and Q belongs to some ambiguity set $Q \in \{Q_1, Q_2, \cdots Q_K\}$. So the densely sampled version

$$x_L(n) \triangleq x_c(nT/L) \qquad (6)$$

is certainly periodic with a repetition interval PL because

$$\begin{aligned} x_L(n + PL) &= x_c((n + PL)\frac{T}{L}) \\ &= x_c(n\frac{T}{L} + PT) \\ &= x_c(n\frac{T}{L} + Q\tau) = x_c(n\frac{T}{L}) = x_L(n) \end{aligned}$$

Let P_L be the actual period of $x_L(n)$. Evidently P_L is a divisor of PL (see *Preliminaries* in Sec. 1). Applying Theorem 1 to the model (5) it follows that if $x_L(n)$ is periodic-P_L then $x_c(t)$ is periodic with period

$$\tau = \frac{P_L}{R_L}\frac{T}{L} \qquad (7)$$

where R_L is coprime to P_L. Thus

$$\tau = \frac{PT}{Q} = \frac{P_L}{R_L}\frac{T}{L} \qquad (8)$$

so that $P = QP_L/R_L L$. Choosing the oversampling rate as

$$L = \mathrm{lcm}\{Q_1, Q_2, \cdots Q_K\}, \qquad (9)$$

L/Q becomes an integer. So $P = P_L/R_L J$ where J is an integer. Since P itself is an integer, this is possible only if R_L is a divisor of P_L. But since R_L is coprime to P_L, this implies that $R_L = 1$. So (8) becomes

$$\tau = \frac{P_L T}{L} \qquad (10)$$

and the Q-ambiguity has disappeared! Notice finally that the oversampled version $x_L(n)$ can in principle be obtained from $x(n)$ itself (without having to oversample $x_c(t)$ directly) by computing $c(n)$ using the inverse filter $1/\Phi_d(z)$ (Fig. 3(b)), and then using Eq. (1). Summarizing we have proved:

Theorem 2. *Unambiguous estimation of periodicity using multirate sampling.* Assume that $x_c(t)$ can be modeled both as (1) and as (5). Define $\Phi_d(z) = \sum_n \phi_d(n)z^{-n}$, and $F_d(z) = \sum_n f_d(n)z^{-n}$ where $\phi_d(n) = \phi(nT)$ and $f_d(n) = f(nT/L)$ and assume $1/\Phi_d(z)$ and $1/F_d(z)$ are stable. Also define the sampled version $x(n) = x_c(nT)$ and oversampled version $x_L(n) = x_c(nT/L)$. If $x(n)$ is periodic-P then the following are true:

1. $x_c(t)$ is periodic-τ where $\tau = PT/Q$, where Q is some integer coprime to Q.
2. $x_L(n)$ is periodic with some period P_L.

3. Assume Q belongs to some known ambiguity set $Q \in \{Q_1, Q_2, \cdots, Q_K\}$. If the oversampling factor L is chosen as $L = \text{lcm}(Q_1, Q_2, \cdots, Q_K\}$ then the period of $x_c(t)$ can be uniquely determined to be equal to $\tau = P_L T/L$. $\diamondsuit$

As this involves examination of samples at two different rates, it is a *multirate method [18], [4] to estimate the period τ.* The following example demonstrates how this new method compares with traditional DFT based methods.

An Example. Assume $x_c(t)$ is a superposition of signals $x_{c,i}(t)$ with periods $\tau_1, \tau_2, \cdots, \tau_J$ where no τ_i is an integer multiple[1] of another period τ_j. We say that τ_i are *hidden periods.* If τ_i/T are rational the sampled version $x(n) = x_c(nT)$ has hidden integer periods P_i. We can identify τ_i from P_i using $\tau_i = P_i T/Q_i$ where $(Q_i, P_i) = 1$ (Theorem 1). Each Q_i belongs to an ambiguity set $\{Q_{i,1}, Q_{i,2}, \cdots, Q_{i,K_i}\}$. By Theorem 2 we can resolve these ambiguities by using an L-fold oversampled version by using $L =$ lcm of all the $Q_{i,j}$'s.

In our example $x_c(t)$ is a sum of three continuous-time signals with periods 4.5, 4.667 and 5.333 secs, the sample spacing is $T = 1$ sec, and we assume 100 consecutive samples $x(n) = x_c(nT)$ are available. Fig. 6(a) shows the 100-point DFT of $x(n)$. The figure also indicates the quantity $2\pi/\omega_i$ at locations of the peaks. The last two peaks correspond to harmonics of the first two peaks. From the first two peaks we estimate that there are two periods: 5.2623 and 4.5464. The former approximately represents the correct answer 5.333. But the periods 4.5 and 4.667 cannot be resolved by conventional DFT methods, as they merge into 4.5464.

Instead of using the DFT, we now use the methods of this paper. First, we obtain estimates of the hidden *integer* periods P_i in $x(n)$ using the dictionary approach [23], [15] based on Ramanujan-sums [10], [21], [22], [24]. This approach yields a so-called strength-vs-period plot [22], [15] as shown in Fig. 6(b). It reveals three integer periods: $P_1 = 9$, $P_2 = 14$ and $P_3 = 16$ (the other peaks are integer submultiples representing harmonics). By Theorem 1 the integer period P_i implies a continuous-time period $\tau_i = P_i T/Q_i$ where $Q_i < P_i$ and $(P_i, Q_i) = 1$. Thus the Q-ambiguity set for Q_1 is $\{1, 2, 4, 5, 7, 8\}$, and similarly for Q_2 and Q_3. To resolve these ambiguities we have to use an oversampling factor L which is the lcm of integers in all the three ambiguity sets. This turns out to be huge: $L = 360, 360$. To avoid such excessive oversampling, we use the apriori information that, in this example there is at least one harmonic for each hidden period (besides the fundamental). It can be shown that this meagre knowledge reduces the ambiguity sets to $Q_1 \in \{1, 2\}, Q_2 \in \{1, 3\}, Q_3 \in \{1, 3\}$. The lcm of these sets is $L = 6$. So we use an oversample factor $L = 6$. Let $x_6(n)$ be this oversampled version of $x_c(t)$. It has 600 samples now. Using the Ramanujan dictionary method [15] we now find the hidden integer periods in $x_6(n)$ to be 27, 28, and 32 (see Fig. 6(c)), from which the hidden periods in $x_c(t)$ are found (using Theorem 2) to be $\tau_1 = 27/6 = 4.5, \tau_2 = 28/6 = 4.667$, and $\tau_3 = 32/6 = 5.333$. The answers are exact. In particular the closely spaced periods 4.5, 4.667 are resolved perfectly in this noise-free case. On the other hand the 600 point DFT still does not reveal the closely spaced periods (see Fig. 6(d)). If the number of samples available is very large (in the thousands) then DFT performs well. But with short data records, the proposed new method works better. Comparison with well-known multipitch estimation methods [1] will be presented in future work.

[1] If $\tau_i = n\tau_j$ for some integer $n > 1$, then the period of $x_{c,i}(t) + x_{c,j}(t)$ is just τ_i. That is, τ_j merely contributes to the nth harmonic component of the signal with period τ_i.

5. CONCLUDING REMARKS

We showed that the signal model (1) implies that $x_c(t)$ is periodic whenever $x(n)$ has an integer period. Since $\phi(t)$ need not be bandlimited, the results are widely applicable. It will be interesting to find more general models that allow similar conclusions, opening up other multirate period-estimation methods.

(a) (b) (c) (d)

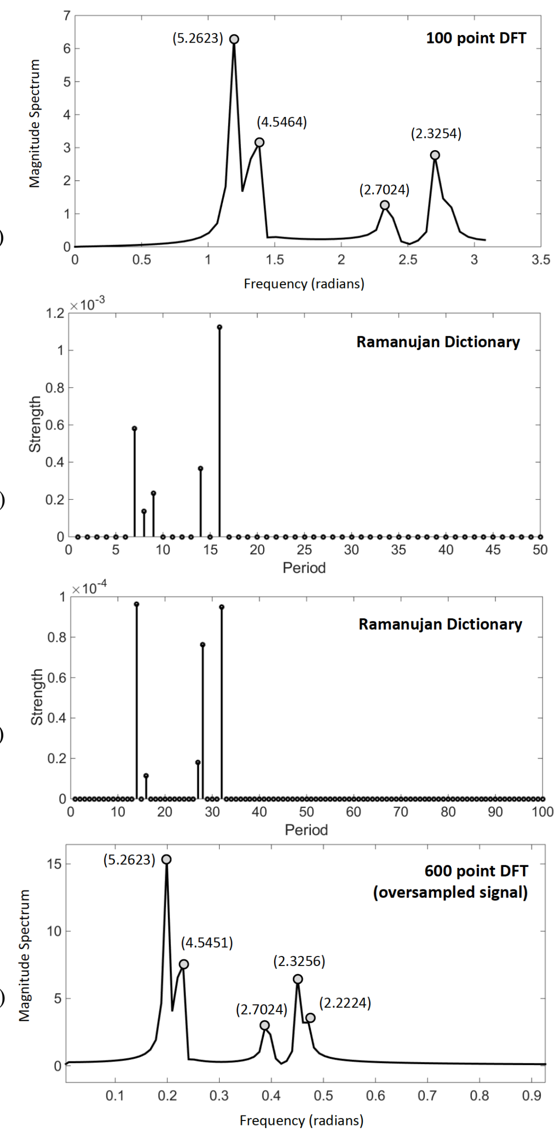

Fig. 6. (a) The 100-point DFT of $x(n)$, (b) the strength-period plot using Ramanujan dictionaries, (c) the strength-period plot using Ramanujan dictionaries, for the oversampled signal $x_L(n)$ (with $L = 6$) and (d) the 600-point DFT of the oversampled signal $x_L(n)$.

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