HIGHER ORDER EXPONENTIAL SPLITTINGS FOR THE FAST NON-LINEAR FOURIER TRANSFORM OF THE KORTEWEG-DE VRIES EQUATION

Peter J. Prins *Sander Wahls*

Delft Center for Systems and Control, Delft University of Technology, 2628 CD Delft, The Netherlands

ABSTRACT

Non-linear Fourier Transforms (NFTs) enable the analysis of signals governed by certain non-linear evolution equations in a way that is analogous to how the conventional Fourier transform is used to analyse linear wave equations. Recently, fast numerical algorithms have been derived for the numerical computation of certain NFTs. In this paper, we are primarily concerned with fast NFTs with respect to the Korteweg-de Vries equation (KdV), which describes e.g. the evolution of waves in shallow water. We find that in the KdV case, the fast NFT can be more sensitive to numerical errors caused by an exponential splitting. We present higher order splittings that reduce these errors and are compatible with the fast NFT. Furthermore we demonstrate for the NSE case that using these splittings can make the accuracy of the fast NFT match that of the conventional NFT.

Index Terms— Signal processing algorithms, non-linear Fourier transform, exponential splittings, Korteweg-de Vries equation

1. INTRODUCTION

The Korteweg-de Vries equation (KdV) for a function $q = q(x,t)$,

$$q_t + 6qq_x + q_{xxx} = 0\,, \tag{1}$$

where the subscripts x and t denote partial derivatives, is a well known non-linear differential equation. It describes a large class of nearly hyperbolic mathematical systems, including water waves, lattice waves, and hydromagnetic and ion-acoustic waves in a plasma [11, 24, 25]. The initial value problem for the KdV can be solved with a Non-linear Fourier Transform (NFT) in a way that is similar to the way Fourier solved the heat equation with the linear Fourier transform [9]. Like the linear Fourier transform, NFTs can be used to analyse data. The KdV NFT has for example been used to analyse water waves [4, 6, 15]. (NFTs are also known as scattering transforms in the literature.) Another evolution equation that can be solved using NFTs is the Non-linear Schrödinger Equation (NSE) [1, 13, 26]. It has recently attracted attention for applications in fibre-optic communication [21]. Unfortunately, most naive numerical implementations of NFTs have a computational complexity of at least $\mathcal{O}(D^2)$, making it unattractive for large numbers of samples D. For the NSE, recently a Fast Non-linear Fourier Transform (FNFT), an algorithm with a computational complexity of $\mathcal{O}(D\log^2 D)$, has been introduced [22].[1] Formally, the approach used for the NSE can also be used to obtain an FNFT for the KdV. However, we observed unexpectedly large numerical errors when we tried this. The FNFT algorithm requires an exponential splitting to become fast. (The matrix exponential $\mathrm{e}^{\mathbf{A}} \equiv \exp(\mathbf{A}) \coloneqq \sum_{k=0}^{\infty} \mathbf{A}^k/k!$ in general does not satisfy $\exp\big((\mathbf{A}+\mathbf{B})\varepsilon\big) = \exp(\mathbf{A}\varepsilon)\exp(\mathbf{B}\varepsilon)$. Exponential splittings approximate $\exp\big((\mathbf{A}+\mathbf{B})\varepsilon\big)$ up to an error of $\mathcal{O}(\varepsilon^{n+1})$, where n is the order of accuracy.) We found that removing the exponential splitting made these errors disappear. A splitting with a higher order of accuracy can reduce the error, but most spittings known in the literature unfortunately do not have the special structure that is needed for the FNFT. In this paper, we therefore present higher order splittings that have this structure and can thus be used to obtain FNFTs with reduced errors. More precisely, we are interested in n-th order accurate exponential splitting schemes of the general form

$$\mathrm{e}^{(\mathbf{A}+\mathbf{B})\varepsilon} = \sum_{k=0}^{k_{\max}} \gamma_k \prod_{j=0}^{j_{\max,k}} \mathrm{e}^{\alpha_{j,k}\mathbf{A}\varepsilon}\mathrm{e}^{\beta_{j,k}\mathbf{B}\varepsilon} + \mathcal{O}\Big(\varepsilon^{n+1}\Big)\,. \tag{2}$$

As will be explained in Section 3, if all $\alpha_{j,k}$ (or all $\beta_{j,k}$) are rational numbers, we obtain a suitable form for the FNFT. Two widely used splittings that fulfil these conditions are the symmetrically weighted sequential splitting and the symmetric Strang splitting, which are both second order accurate [18, 19]. Suzuki [20] presented an algorithm to find the parameters for splittings with any desired order of accuracy, of the form of Eq. (2) with $k_{\max} = 0$, but this algorithm results in irrational or even complex coefficients and is not suitable for our purpose. We remark that the stability of a splitting scheme can be guaranteed if all the parameters $\alpha_{j,k}$, $\beta_{j,k}$ and γ_k in Eq. (2) are real and non-negative. Unfortunately, then the order of accuracy is two at most [10, 17]. Higher order schemes may be stable, but that needs to be determined on a per application basis.

The paper is structured as follows. In Section 2, we summarise the part of the theory behind the (F)NFTs that is relevant for the purpose of this paper. In Section 3, we establish a sufficient condition for a splitting to be suitable for the FNFT and present several higher order exponential splitting schemes that fulfil this condition. We demonstrate the value of these splitting schemes numerically in Section 4. The paper is concluded in Section 5.

2. PRELIMINARIES

In this section we describe the numerical calculation of the reflection coefficient of the Schrödinger equation. This is an intermediate step in the calculation of the NFT of a potential $q(x;t_0)$ that evolves according to a suitable non-linear differential equation, like the KdV or the NSE. We will omit the dependence on the fixed time t_0.

The following differential equation is the basis for the (F)NFT:

$$\boldsymbol{v}_x(x,\zeta) = \mathbf{C}(x,\zeta)\cdot\boldsymbol{v}(x,\zeta) = \begin{bmatrix} -\mathrm{j}\zeta & q(x) \\ r(x) & \mathrm{j}\zeta \end{bmatrix}\cdot\boldsymbol{v}(x,\zeta)\ \text{[1]}, \tag{3}$$

where j is the imaginary unit. For the KdV the boundary condition

$$\lim_{x\to-\infty} \boldsymbol{v}(x,\zeta)\cdot\exp(\mathrm{j}\zeta x) = \begin{bmatrix} 2\mathrm{j}\zeta & 1 \end{bmatrix}^\top \tag{4}$$

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[1] To be precise: This complexity holds for the so-called reflection coefficient when $\zeta \in \mathbb{R}$ (see Eq. (3)), the case we consider in this paper.

is used and $r(x) \equiv -1$ is chosen in Eq. (3) [1].[2] This choice reduces Eq. (3) to $v_{2xx} + (\zeta^2 + q(x))v_2 = 0$: the Schrödinger equation associated with Eq. (1) [1]. In this standard form of the KdV all variables have been made unit-less by normalization. Note that Eq. (3) is a generalised scattering problem [2, Sec. 6]. Similar problems appear in many other signal processing applications [3].

The NFT consists of multiple parts, of which we only need the so-called reflection coefficient $R(\zeta)$ in this paper. It is defined in terms of the solution of Eqs. (3) and (4), denoted as

$$\boldsymbol{v}(x,\zeta) \to \frac{1}{T(\zeta)} \begin{bmatrix} 2\mathrm{j}\zeta \mathrm{e}^{-\mathrm{j}\zeta x} \\ \mathrm{e}^{-\mathrm{j}\zeta x} + R(\zeta)\,\mathrm{e}^{\mathrm{j}\zeta x} \end{bmatrix} \text{ as } x \to \infty \quad [1]. \tag{5}$$

$$\text{Hence,} \quad R(\zeta) = \lim_{x\to\infty} \left(2\mathrm{j}\zeta \cdot \frac{v_2(x,\zeta)}{v_1(x,\zeta)} - 1 \right) \mathrm{e}^{-2\mathrm{j}\zeta x}. \tag{6}$$

The quantity $T(\zeta)$ is called the transmission coefficient.

As in for example [14], we apply two approximations to calculate Eq. (6) numerically:

1. The interval $x \in (-\infty, \infty)$ is replaced by $x \in [L_-, L_+]$, with $L_\pm$ 'close' to $\pm\infty$. The approximation is exact if $q(x) = 0$ for all $x \notin [L_-, L_+]$.
2. The potential $q(x)$ is approximated by a piecewise constant function, a staircase $\hat{q}(x)$. That is, the interval $[L_-, L_+]$ is divided into D subintervals of width $\varepsilon = (L_+ - L_-)/D$, and in each of these subintervals $\hat{q}(x) = q_i \coloneqq q(x_i)$, where x_i is the midpoint of the i-th subinterval. (In the general case, the same is done for $r(x)$. For the KdV it is already constant.)

Because of Approximation 2, Eq. (3) becomes a linear differential equation for each subinterval, which is solved as

$$\boldsymbol{v}\big(x_i + \varepsilon/2, \zeta\big) = \mathbf{G}(x_i,\zeta) \cdot \boldsymbol{v}\big(x_i - \varepsilon/2, \zeta\big), \tag{7}$$

$$\text{with} \quad \mathbf{G}(x_i,\zeta) \coloneqq \mathrm{e}^{\mathbf{C}(x_i,\zeta)\cdot\varepsilon}. \tag{8}$$

Continuity of $\boldsymbol{v}(x,\zeta)$ at each of the boundaries of the subintervals allows us to write

$$\boldsymbol{v}(L_+,\zeta) = \mathbf{H}(\zeta) \cdot \boldsymbol{v}(L_-,\zeta)\ \text{, with} \tag{9}$$

$$\mathbf{H}(\zeta) \coloneqq \mathbf{G}(x_D,\zeta) \cdots \mathbf{G}(x_2,\zeta) \cdot \mathbf{G}(x_1,\zeta)\ . \tag{10}$$

We substitute the result of Eq. (9) with the boundary condition $\boldsymbol{v}(L_-,\zeta) = \begin{bmatrix} 2\mathrm{j}\zeta & 1 \end{bmatrix}^\top \cdot \exp(-\mathrm{j}\zeta L_-)$ in Eq. (6), to approximate the reflection coefficient as[3]

$$\hat{R}(\zeta) = \left(2\mathrm{j}\zeta \cdot \frac{2\mathrm{j}\zeta H_{21}(\zeta) + H_{22}(\zeta)}{2\mathrm{j}\zeta H_{11}(\zeta) + H_{12}(\zeta)} - 1 \right) \mathrm{e}^{-2\mathrm{j}\zeta L_+}. \tag{11}$$

The calculation of $\mathbf{H}(\zeta)$ according to Eq. (10) requires $D-1$ matrix multiplications for each value of ζ. To reduce the computational complexity, Wahls and Poor [22, 23] proposed to approximate the entries of $\mathbf{G}(x_i,\zeta)$ by rational functions with respect to a transformed variable $z(\zeta)$. Then, after evaluation of Eq. (10) with fast tree-wise polynomial multiplication, the entries of $\mathbf{H}(\zeta)$ are rational approximations as well, which can be evaluated using a fast polynomial evaluation method for each desired value of ζ. This is the idea behind the FNFT. In this paper, we express this third approximation as follows:

3. Find a rational approximation

$$\hat{\mathbf{G}}(x_i,\zeta) = \sum_{k=-m}^{m} \mathbf{M}_{k+m}(q_i,r_i) \cdot \big(z(\zeta)\big)^k = \big(z(\zeta)\big)^{-m} \cdot \sum_{k=0}^{2m} \mathbf{M}_k(q_i,r_i) \cdot \big(z(\zeta)\big)^k, \tag{12}$$

for some set of matrix valued coefficients $\mathbf{M}_k(q_i,r_i)$, and substitute it in Eq. (10) to find

$$\hat{\mathbf{H}}(\zeta) = \big(z(\zeta)\big)^{-d} \cdot \sum_{k=0}^{2d} \mathbf{N}_k(q_i,r_i) \cdot \big(z(\zeta)\big)^k. \tag{13}$$

This result can be used in Eq. (11).

How can a rational approximation like Eq. (12) be obtained? — Feced, Zervas and Muriel [8] simplified Eq. (8) by applying a symmetric Strang splitting (see Eq. (18)) with $\mathbf{C}(x_i,\zeta) = \mathbf{A}(\zeta) + \mathbf{B}(x_i)$, where

$$\mathbf{A}(\zeta) \coloneqq \begin{bmatrix} -\mathrm{j}\zeta & 0 \\ 0 & \mathrm{j}\zeta \end{bmatrix}; \quad \mathbf{B}(x_i) \coloneqq \begin{bmatrix} 0 & q(x_i) \\ r(x_i) & 0 \end{bmatrix}. \tag{14}$$

The result of their approximation, which expresses $\mathbf{G}(x_i,\zeta)$ as a product of matrices that depend either only on x_i or only on ζ, is rational in $z(\zeta) \coloneqq \exp\big(-\mathrm{j}\zeta\epsilon/2\big)$ and can be written like Eq. (12). This can for example be seen from Lemma 1 in Section 3.

3. HIGHER ORDER SPLITTINGS

Modification of the NSE FNFT for the KdV seems straightforward: Just use $r(x_i) \equiv -1$ instead of $r(x_i) \equiv \pm q^*(x_i)$ in Eq. (14), as described in Section 2. However, as we will see in the numerical examples in Section 4, then the error in $\hat{R}$ caused by Approximation 3 can be much higher than what one would expect from the NSE case.[3] To reduce this error (without reducing the step size ε), we want to use splitting schemes with a higher order of accuracy. Lemma 1 below states that a splitting scheme of the form of Eq. (2) is suitable for the FNFT if all parameters $\alpha_{j,k}$ are positive rational numbers. (We will omit the dependencies on ζ and x_i from here.)

Lemma 1 *If* $\mathbf{G}$ *is approximated by a splitting scheme of the form of Eq.* (2) *with* $\mathbf{A}$ *and* $\mathbf{B}$ *defined in Eq.* (14) *and* $\sum_j \alpha_{j,k} = 1\ \forall k$ *and* $0 < \alpha_{j,k} \in \mathbb{Q}\ \forall j,k$*, then the approximation can be written as Eq.* (12) *with* $2m \in \mathbb{N}$.

PROOF Write $\alpha_{j,k}$ for all j and k as an irreducible fraction. Let $2m$ be the least common multiple of their denominators and rewrite

$$\mathrm{e}^{\alpha_{j,k}\mathbf{A}\varepsilon} = \mathrm{e}^{-\mathrm{j}\zeta\varepsilon\alpha_{j,k}} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \exp\big(\mathrm{j}\zeta\varepsilon/m\big) \end{bmatrix}^{2m\alpha_{j,k}}. \tag{15}$$

Define $z \coloneqq \exp\big(\mathrm{j}\zeta\epsilon/m\big)$. Equation (2) should hold for any arbitrary step size ε. For $\epsilon = 0$ it yields $\sum_k \gamma_k = 1$, so substitution of Eq. (15) in Eq. (2) results in

$$\hat{\mathbf{G}} = z^{-m} \cdot \sum_{k=0}^{k_{\max}} \gamma_k \cdot \prod_{j=0}^{j_{\max,k}} \begin{bmatrix} 1 & 0 \\ 0 & z^{2m\alpha_{j,k}} \end{bmatrix} \cdot \mathrm{e}^{\beta_{j,k}\mathbf{B}\varepsilon}. \tag{16}$$

Since $2m\alpha_{j,k} \in \mathbb{N}\ \forall j,k$, Eq. (16) can be written as Eq. (12). ∎

[2] We remark that the NSE is obtained for the boundary condition $\lim_{x\to-\infty} \boldsymbol{v}(x,\zeta) \cdot \exp(\mathrm{j}\zeta x) = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$ and $r(x) \equiv \pm q^*(x)$, where the star denotes the complex conjugate [1].

[3] While this paper was being reviewed we noted that we could use $\bar{R}(-\zeta) = R(\zeta)$ in [1, p. 299], to obtain an alternative to Eq. (11): $\hat{R}(\zeta) = H_{12}(-\zeta) \cdot \big(2\mathrm{j}\zeta H_{11}(-\zeta) - H_{12}(-\zeta)\big)^{-1} \cdot \mathrm{e}^{-2\mathrm{j}\zeta L_+}$. This equation appears to be less sensitive to numerical errors in $\mathbf{H}(\zeta)$ than Eq. (11).

The following splitting schemes are suitable for the FNFT, because each of them fulfils the requirements of Lemma 1. The same holds for the dual schemes that can be obtained by changing every $\mathbf{A}$ to a $\mathbf{B}$ and vice versa.

$$\mathrm{e}^{\mathbf{C}\varepsilon} = \mathrm{e}^{\mathbf{A}\varepsilon}\mathrm{e}^{\mathbf{B}\varepsilon} + \mathcal{O}(\varepsilon^2)\,; \tag{17}$$

$$\mathrm{e}^{\mathbf{C}\varepsilon} = \mathrm{e}^{\frac{1}{2}\mathbf{A}\varepsilon}\mathrm{e}^{\mathbf{B}\varepsilon}\mathrm{e}^{\frac{1}{2}\mathbf{A}\varepsilon} + \mathcal{O}(\varepsilon^3)\,; \tag{18}$$

$$\mathrm{e}^{\mathbf{C}\varepsilon} = \tfrac{9}{8}\mathrm{e}^{\frac{1}{3}\mathbf{A}\varepsilon}\mathrm{e}^{\frac{2}{3}\mathbf{B}\varepsilon}\mathrm{e}^{\frac{2}{3}\mathbf{A}\varepsilon}\mathrm{e}^{\frac{1}{3}\mathbf{B}\varepsilon} - \tfrac{1}{8}\mathrm{e}^{\mathbf{A}\varepsilon}\mathrm{e}^{\mathbf{B}\varepsilon} + \mathcal{O}(\varepsilon^4)\,; \tag{19}$$

$$\mathrm{e}^{\mathbf{C}\varepsilon} = \tfrac{4}{3}\mathrm{e}^{\frac{1}{4}\mathbf{A}\varepsilon}\mathrm{e}^{\frac{1}{2}\mathbf{B}\varepsilon}\mathrm{e}^{\frac{1}{2}\mathbf{A}\varepsilon}\mathrm{e}^{\frac{1}{2}\mathbf{B}\varepsilon}\mathrm{e}^{\frac{1}{4}\mathbf{A}\varepsilon} + \ldots - \tfrac{1}{3}\mathrm{e}^{\frac{1}{2}\mathbf{A}\varepsilon}\mathrm{e}^{\mathbf{B}\varepsilon}\mathrm{e}^{\frac{1}{2}\mathbf{A}\varepsilon} + \mathcal{O}(\varepsilon^5)\,; \tag{20}$$

$$\mathrm{e}^{\mathbf{C}\varepsilon} = \tfrac{625}{384}\mathrm{e}^{\frac{1}{5}\mathbf{A}\varepsilon}\left(\mathrm{e}^{\frac{2}{5}\mathbf{B}\varepsilon}\mathrm{e}^{\frac{2}{5}\mathbf{A}\varepsilon}\right)^2\mathrm{e}^{\frac{1}{5}\mathbf{B}\varepsilon} + \ldots - \tfrac{81}{128}\mathrm{e}^{\frac{1}{3}\mathbf{A}\varepsilon}\mathrm{e}^{\frac{2}{3}\mathbf{B}\varepsilon}\mathrm{e}^{\frac{2}{3}\mathbf{A}\varepsilon}\mathrm{e}^{\frac{1}{3}\mathbf{B}\varepsilon} + \tfrac{1}{192}\mathrm{e}^{\mathbf{A}\varepsilon}\mathrm{e}^{\mathbf{B}\varepsilon} + \mathcal{O}(\varepsilon^6)\,; \tag{21}$$

$$\mathrm{e}^{\mathbf{C}\varepsilon} = \tfrac{81}{40}\mathrm{e}^{\frac{1}{6}\mathbf{A}\varepsilon}\left(\mathrm{e}^{\frac{1}{3}\mathbf{B}\varepsilon}\mathrm{e}^{\frac{1}{3}\mathbf{A}\varepsilon}\right)^2\mathrm{e}^{\frac{1}{3}\mathbf{B}\varepsilon}\mathrm{e}^{\frac{1}{6}\mathbf{A}\varepsilon} + \ldots - \tfrac{16}{15}\mathrm{e}^{\frac{1}{4}\mathbf{A}\varepsilon}\mathrm{e}^{\frac{1}{2}\mathbf{B}\varepsilon}\mathrm{e}^{\frac{1}{2}\mathbf{A}\varepsilon}\mathrm{e}^{\frac{1}{2}\mathbf{B}\varepsilon}\mathrm{e}^{\frac{1}{4}\mathbf{A}\varepsilon} + \ldots + \tfrac{1}{24}\mathrm{e}^{\frac{1}{2}\mathbf{A}\varepsilon}\mathrm{e}^{\mathbf{B}\varepsilon}\mathrm{e}^{\frac{1}{2}\mathbf{A}\varepsilon} + \mathcal{O}(\varepsilon^7)\,; \tag{22}$$

$$\mathrm{e}^{\mathbf{C}\varepsilon} = \tfrac{117649}{46080}\mathrm{e}^{\frac{1}{7}\mathbf{A}\varepsilon}\left(\mathrm{e}^{\frac{2}{7}\mathbf{B}\varepsilon}\mathrm{e}^{\frac{2}{7}\mathbf{A}\varepsilon}\right)^3\mathrm{e}^{\frac{1}{7}\mathbf{B}\varepsilon} + \ldots - \tfrac{15625}{9216}\mathrm{e}^{\frac{1}{5}\mathbf{A}\varepsilon}\left(\mathrm{e}^{\frac{2}{5}\mathbf{B}\varepsilon}\mathrm{e}^{\frac{2}{5}\mathbf{A}\varepsilon}\right)^2\mathrm{e}^{\frac{1}{5}\mathbf{B}\varepsilon} + \ldots + \tfrac{729}{5120}\mathrm{e}^{\frac{1}{3}\mathbf{A}\varepsilon}\mathrm{e}^{\frac{2}{3}\mathbf{B}\varepsilon}\mathrm{e}^{\frac{2}{3}\mathbf{A}\varepsilon}\mathrm{e}^{\frac{1}{3}\mathbf{B}\varepsilon} - \tfrac{1}{9216}\mathrm{e}^{\mathbf{A}\varepsilon}\mathrm{e}^{\mathbf{B}\varepsilon} + \mathcal{O}(\varepsilon^8)\,; \tag{23}$$

$$\mathrm{e}^{\mathbf{C}\varepsilon} = \tfrac{1024}{315}\mathrm{e}^{\frac{1}{8}\mathbf{A}\varepsilon}\left(\mathrm{e}^{\frac{1}{4}\mathbf{B}\varepsilon}\mathrm{e}^{\frac{1}{4}\mathbf{A}\varepsilon}\right)^3\mathrm{e}^{\frac{1}{4}\mathbf{B}\varepsilon}\mathrm{e}^{\frac{1}{8}\mathbf{A}\varepsilon} + \ldots - \tfrac{729}{280}\mathrm{e}^{\frac{1}{6}\mathbf{A}\varepsilon}\left(\mathrm{e}^{\frac{1}{3}\mathbf{B}\varepsilon}\mathrm{e}^{\frac{1}{3}\mathbf{A}\varepsilon}\right)^2\mathrm{e}^{\frac{1}{3}\mathbf{B}\varepsilon}\mathrm{e}^{\frac{1}{6}\mathbf{A}\varepsilon} + \ldots + \tfrac{16}{45}\mathrm{e}^{\frac{1}{4}\mathbf{A}\varepsilon}\mathrm{e}^{\frac{1}{2}\mathbf{B}\varepsilon}\mathrm{e}^{\frac{1}{2}\mathbf{A}\varepsilon}\mathrm{e}^{\frac{1}{2}\mathbf{B}\varepsilon}\mathrm{e}^{\frac{1}{4}\mathbf{A}\varepsilon} + \ldots - \tfrac{1}{360}\mathrm{e}^{\frac{1}{2}\mathbf{A}\varepsilon}\mathrm{e}^{\mathbf{B}\varepsilon}\mathrm{e}^{\frac{1}{2}\mathbf{A}\varepsilon} + \mathcal{O}(\varepsilon^9)\,. \tag{24}$$

Equations (17) and (18) are well-known, as sequential splitting and symmetric Strang splitting [19] respectively. Equation (19) is reported in [5]. Equation (20) is known as the Strang-Richardson scheme (see e.g. [7]). We remark that only the symmetric Strang splitting has been used for FNFTs so far. We could not find the other schemes in literature, so they may be new. We derived these as follows. We imposed the number of terms $k_{\max}$ and factors $j_{\max,k}$ and the order of accuracy n in Eq. (2). In case n is even we set $\beta_{j_{\max,k},k} = 0\,\forall k$. (In words: Every term in an even order accurate splitting has to start and end with an exponential of $\mathbf{A}$.) Then we replaced every matrix exponential by its Taylor series expansion and used algebraic computational software (Wolfram Mathematica) to find the parameter values $\alpha_{j,k}$, $\beta_{j,k}$ and γ_k by equating the appropriate left and right hand side terms. That is, terms that contain the same product of matrices $\mathbf{A}$ and $\mathbf{B}$. These values appear to be unique, with respect to the particular choices for $k_{\max}$, $j_{\max,k}$ and n corresponding to Eqs. (17) to (24) respectively. We do not know whether these splittings are part of a family that extends up to arbitrary order of accuracy; neither do we have direct formulas for the parameters.

Remark 1 Equations (12) and (16) for $2m \in \mathbb{N}$ are rational in $\sqrt{z}$, because the denominator z^m is a positive integer power of $\sqrt{z}$. Yet, the numerator is a polynomial in z (with matrix-valued coefficients).□

Remark 2 If the conditions for Lemma 1 hold, except for the positivity of $\alpha_{j,k}$, we can still obtain a rational approximation that is suitable for the FNFT. The constructive proof becomes more complicated in that case and since Lemma 1 suffices for the purpose of this paper, we favoured the less general, but clearer result. □

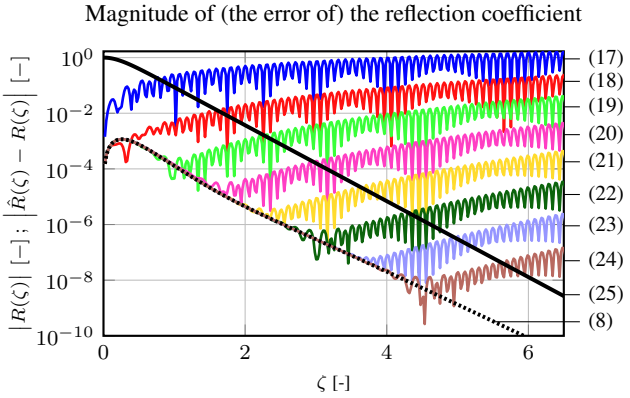

Fig. 1: Magnitude of (the error of) the reflection coefficient for a squared hyperbolic secant potential for the KdV. The numbers between brackets refer to the corresponding equation: (25): ground truth, (8): error without splitting (due to staircase approximation), (17)–(24): error with splitting.

Remark 3 The constructive proof of Lemma 1 is based on finding a number m such that $2m\alpha_{j,k} \in \mathbb{Z}\;\forall j,k$. One could hypothesise that such a number m may still exist when the condition $\alpha_{j,k} \in \mathbb{Q}$ does not hold. However, if $2m\alpha_{j,k} \in \mathbb{Z}\;\forall j,k$ and $\sum_j \alpha_{j,k} = 1$ it follows that $2m \in \mathbb{Z}$ and $\alpha_{j,k} \in \mathbb{Q}\;\forall j,k$: $1 = \sum_j \alpha_{j,k} \Rightarrow 2m = \sum_j 2m\alpha_{j,k} \in \mathbb{Z} \Rightarrow \alpha_{j,k} \in \mathbb{Q}\;\forall j,k$. □

4. NUMERICAL EXAMPLES

In this section we demonstrate the application of the proposed splitting schemes for two potential functions $q(x)$ for the KdV. As a comparison, we also include an example for the NSE. For these examples the reflection coefficient is known analytically; the respective equations provide the ground truth. For the KdV examples, we divide the interval $x \in [-16, 16]$ in $D = 256$ sections of width $\varepsilon = 1/8$ and sample $q(x)$ at the midpoints of each section to obtain a staircase approximation. (The interval is large enough to make Approximation 1 negligible for the chosen potentials.) We apply each of the proposed splitting schemes (Eqs. (17) to (24)) as well as the un-split matrix exponential (Eq. (8)), and calculate the error of the approximated reflection coefficient (compared to the ground truth) with Eqs. (10) and (11). Because of the page limit, we omit the transmission coefficient, the reflection coefficient for imaginary frequencies ζ and the dual splitting schemes with the roles of $\mathbf{A}$ and $\mathbf{B}$ reversed, but the shown results are representative.

4.1. KdV with a squared hyperbolic secant potential

Consider the potential function $q(x) = \mathring{q}\,\mathrm{sech}^2(x)$ with $\mathring{q} = 9$. The reflection coefficient for this function is given by

$$R(\zeta) = \frac{\cos(\pi\delta)}{\pi}\frac{\Gamma(\mathrm{j}\zeta)}{\Gamma(-\mathrm{j}\zeta)}\Gamma\left(\tfrac{1}{2} - \mathrm{j}\zeta + \delta\right)\Gamma\left(\tfrac{1}{2} - \mathrm{j}\zeta - \delta\right), \tag{25}$$

where $\delta \coloneqq \sqrt{\mathring{q} + 1/4}$ and Γ is the gamma function [12].

The magnitude of this reflection coefficient is shown in Figure 1, as well as the magnitude of the error for the various numerical approximations. We see that the error due to Approximation 2 is below

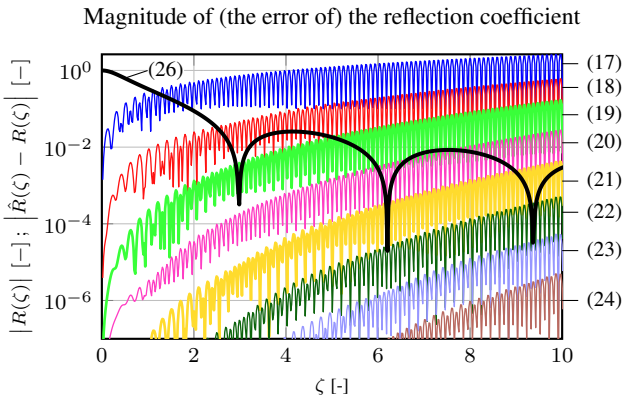

Fig. 2: Magnitude of (the error of) the reflection coefficient for a rectangular potential for the KdV. The numbers between brackets refer to the corresponding equation: (26): ground truth, (17)–(24): error due to splitting. The error without splitting (Eq. (8)) is zero here, because this potential equals its staircase approximation exactly.

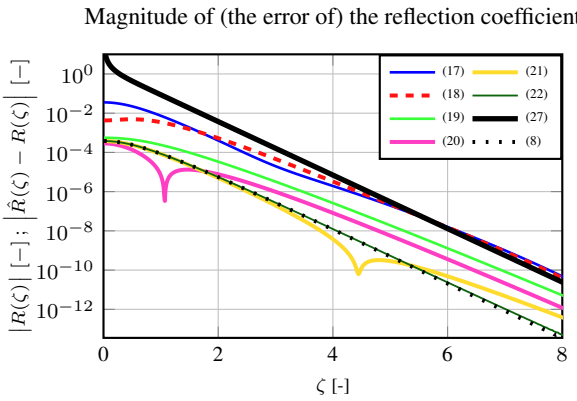

Fig. 3: Magnitude of (the error of) the reflection coefficient for a hyperbolic secant potential for the NSE. The numbers between brackets refer to the corresponding equation: (27): ground truth, (8): error without splitting (due to staircase approximation), (17)–(22): error with splitting.

approximately 1% for each value of ζ. Above some frequency, Approximation 3 becomes the dominant error source for the splitting schemes. One can reduce this error and increase this frequency by choosing a splitting scheme with a higher order of accuracy.

4.2. KdV with a Rectangular potential

The rectangular potential $q(x) = \mathring{q} = 1$ for $|x| < l/2 = 1/2$ and $q(x) = 0$ else, is represented exactly by its staircase approximation for the x grid we chose. Hence, the errors due to Approximations 1 and 2 will be zero in this case. The reflection coefficient for this function is given by

$$R(\zeta) = \frac{\mathrm{j}\gamma_- \cdot \sin(\varsigma l) \cdot \exp(-\mathrm{j}\zeta l)}{\cos(\varsigma l) - \mathrm{j}\gamma_+ \cdot \sin(\varsigma l)}, \tag{26}$$

where $\varsigma := \sqrt{\mathring{q} + \zeta^2}$ and $\gamma_\pm := \frac{1}{2}\left(\varsigma/\zeta \pm \zeta/\varsigma\right)$ [14].

The magnitude of this reflection coefficient is shown in Figure 2, as well as the magnitude of the error for the various numerical approximations. The error due to Approximation 2 is zero, as expected. The error due to Approximation 3 is reduced by choosing a splitting scheme with a higher order of accuracy, which increases the frequency up to which the approximated reflection coefficient is accurate.

4.3. NSE with a hyperbolic secant potential

As a comparison, we have included a numerical example for the NSE. We take the potential function $q(x) = \mathring{q}\,\mathrm{j}\,\mathrm{sech}(x)$ with $\mathring{q} = 5.5$ and $r(x) = -q^*(x)$. For this example we need a larger interval to prevent significant truncation errors (Approximation 1) and choose $x \in [-32, 32]$ while keeping $\varepsilon = 1/8$ the same, so $D = 512$. The ground truth reflection coefficient for this example is given by

$$R(\zeta) = \frac{\mathrm{j}\sin(\pi\mathring{q})}{\cosh(\pi\zeta)} \cdot \frac{\Gamma\left(\frac{1}{2} - \mathrm{j}\zeta + \mathring{q}\right)\Gamma\left(\frac{1}{2} - \mathrm{j}\zeta - \mathring{q}\right)}{\Gamma^2\left(\frac{1}{2} - \mathrm{j}\zeta\right)}, \tag{27}$$

where Γ is the gamma function [16].

The magnitude of this reflection coefficient is shown in Figure 3, as well as the error for some of the proposed numerical approximations. Different from the KdV examples, all approximation errors decay for higher frequencies. Although the absolute error is low at high frequencies for every splitting scheme, the relative error for the first and second order scheme is high. We see that the error can be reduced significantly by using one of the schemes between third and sixth order accuracy, Eq. (19) to Eq. (22) respectively. The seventh and eighth order accurate scheme (Eqs. (23) and (24)) are omitted in Figure 3, because the result is on this scale indistinguishable from the error without splitting, Eq. (8). That is, the error due to Approximation 3 in this example becomes negligible compared to the error due to Approximation 2, which means that by choosing a splitting scheme of sufficient order, the FNFT can reach the same accuracy as any other NFT that uses a staircase approximation of the potential.

5. CONCLUSION AND DISCUSSION

We have presented several exponential splitting schemes that can be incorporated into Fast Non-linear Fourier Transforms and investigated their performance for two examples for the KdV NFT and one example for the NSE NFT, respectively. The presented higher order splitting schemes allow to increase the numerical accuracy of the NFT without having to decrease the step size (which might not always be feasible in applications).

Since the calculation time of the FNFT depends partly on the degrees of the rational approximations, one would like to obtain the maximum accuracy for a certain degree. Our method results in rational expressions, but their degrees are not monotonically increasing in the order of accuracy of the splitting schemes. For example, Eq. (21) with an order of accuracy of five results in a higher degree than Eq. (22) with an order of accuracy of six. This obscures the trade-off between the calculation time of the FNFT and its accuracy.

Remark 4 (Post-review) The numerical errors in the calculation of $\hat{\mathbf{H}}(\zeta)$ with the FNFT lead to much larger errors in the reflection coefficient in the KdV case compared to the NSE case. While this paper was being reviewed, we noted that an alternative to Eq. (11) exists, as mentioned in Footnote 3. When we use it, the two cases show comparable error behaviour: Figure 1 then looks very similar to Figure 3, with the results from the highest order splitting schemes overlaying the error without splitting. This shows that also in the KdV case FNFTs can reach the same accuracies as the conventional NFT based on a staircase approximation of the potential. □

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