TENSOR SUBSPACE DETECTION WITH TUBAL-SAMPLING AND ELEMENTWISE-SAMPLING

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ABSTRACT

The problem of testing whether an incomplete tensor lies in a given tensor subspace, called *tensor matched subspace detection*, is significant when it is unavoidable to have missing entries. Compared with the matrix case, the tensor matched subspace detection problem is much more challenging due to the curse of dimensionality and the intertwinement between the sampling operator and the tensor product operation. In this paper, we investigate the subspace detection problem for the transform-based tensor models. Under this framework, tensor subspaces and the orthogonal projection onto a given subspace are defined, and the energies of a tensor outside the given subspace (also called *residual energy* in statistics) with tubal-sampling and elementwise-sampling are derived. We have proved that the residual energy of sampling signals is bounded with high probability. Based on the residual energy, the reliable detection is feasible.

Index Terms— Tensor subspace detection, transform-based tensor model, tubal-sampling, elementwise-sampling.

1. INTRODUCTION

Matched subspace detection is widely used in many applications, such as image representation [1], MIMO system [2, 3], compressive sensing [4, 5], shape detection [6], matrix and tensor completion [7, 8, 9], etc. However, in cases such as sensor networks, we can only obtain a signal with high loss rate [10]. Therefore, it is necessary to testing whether an incomplete signal lies within a given subspace. In [11], the matched subspace detector under linear model with missing data has been well studied. A nonlinear version of matched subspace detector, using kernel functions, is given in the presence of missing data in [12]. However, all these methods are based on the vector space [11, 12, 13], and do not apply to situations when signals are represented as multidimensional data arrys, i.e., tensors, which capture the spatial and temporal correlations within the data. Therefore, it is urgent to propose a more efficient approach for tensor matched subspace detection based on tensors.

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In this paper, we focus on the problem of tensor matched subspace detection for third-order transform-based tensors. We formulate a binary hypothesis test. Suppose we have signal $\mathcal{V} \in \mathbb{R}^{n_1 \times 1 \times n_3}$, and let $\mathcal{S} \subset \mathbb{R}^{n_1 \times r \times n_3}$ be a given subspace. Then the hypotheses are $\mathcal{H}_0 : \mathcal{V} \in \mathcal{S}$ and $\mathcal{H}_1 : \mathcal{V} \notin \mathcal{S}$. And we wish to decide whether $\mathcal{V} \in \mathcal{S}$ or not based on the samples of $\mathcal{V}$. Tests are usually based on the energy of $\mathcal{V}$ out of $\mathcal{S}$ (residual energy). However, the challenge of tensor matched subspace detection is the sampling process is intertwined with the underlying tensor structure.

We exploit the tensor algebraic framework of $\mathcal{L}$-product [14], which is defined in a so-called transform domain for any invertible linear transform. And the conventional t-product [15, 16] is a spacial case of $\mathcal{L}$-product. In this framework, a tensor column subspace is spanned by the columns of a tensor, and the orthogonal projection onto a tensor subspace is well defined. Furthermore, we proposed a scheme for tensor matched subspace detection with tubal-sampling and elementwise-sampling.

The main results of this paper show that the residual energies of a signal with tubal-sampling and elementwise-sampling are bounded with high probability. And based on the residual energy of a sampling signal, the reliable detection of whether an incomplete signal lies in given subspace is possible when the number of samples is slightly greater than r for tubal-sampling while $r \times n_3$ for elementwise-sampling, where r is the dimension of the given tensor subspace and n_3 is the size of the third dimensionality of the subspace.

2. ALGEBRAIC FRAMEWORK AND PROBLEM STATEMENT

2.1. Transform-based Tensor Model

Notations- A third-order tensor is denoted by calligraphic letters, e.g., $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$. A tube of a tensor is defined by fixing all indices but one, while a slice of a tensor defined by fixing all but two indices, i.e., $\mathcal{A}(:, j, k)$, $\mathcal{A}(i, :, k)$, $\mathcal{A}(i, j, :)$ denote mode-1, mode-2, mode-3 tubes of $\mathcal{A}$, and $\mathcal{A}(:, :, k)$, $\mathcal{A}(:, j, :)$, $\mathcal{A}(i, :, :)$ denote the frontal, lateral, and horizontal slices of $\mathcal{A}$. Furthermore, $\mathcal{A}(i, :, :)$ and $\mathcal{A}(:, j, :)$ are also called tensor row and tensor column. For easy representation, we use $\mathcal{A}^{(k)}$ to denote $\mathcal{A}(:, :, k)$. For an $n_1 \times$

$n_2 \times n_3$ tensor $\mathcal{A}$, its Frobenius norm is defined as $\|\mathcal{A}\|_F = \sqrt{\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3} \mathcal{A}_{ijk}^2}$. For a tensor column $\mathcal{X} \in \mathbb{R}^{n_1 \times 1 \times n_3}$, we define ℓ_{∞^*} norm as $\|\mathcal{X}\|_{\infty^*} = \max_i \|\mathcal{X}(i,1,:)\|_2$, and ℓ_∞ norm as $\|\mathcal{X}\|_\infty = \max_{ijk} |\mathcal{X}_{ijk}|$. The transpose of a vector or a matrix is denoted with a superscript ^{T}, and the transpose of a tensor is denoted with a superscript †. We use $[n]$ to denote the index set $\{1, 2, \ldots, n\}$, and $[n_1] \times [n_2]$ to denote the set $\{(1,1),(1,2),\ldots,(1,n_2),(2,1),\ldots,(n_1,n_2)\}$.

Definition 1. *[14] Given an invertible discrete transform $\mathcal{L} : \mathbb{R}^{1\times 1\times n} \to \mathbb{R}^{1\times 1\times n}$, the elementwise multiplication $\circ$, and $\boldsymbol{a}$, $\boldsymbol{b} \in \mathbb{R}^{1\times 1\times n}$, the tubal-scalar multiplication is defined as*

$$\boldsymbol{a} \bullet \boldsymbol{b} = \mathcal{L}^{-1}(\mathcal{L}(\boldsymbol{a}) \circ \mathcal{L}(\boldsymbol{b})),$$

where $\mathcal{L}^{-1}$ is the inverse of $\mathcal{L}$.

Definition 2. *[14] The $\mathcal{L}$-product $\mathcal{C} = \mathcal{A} \bullet \mathcal{B}$ of $\mathcal{A} \in \mathbb{R}^{n_1\times n_2\times k}$ and $\mathcal{B} \in \mathbb{R}^{n_2\times n_3\times k}$ is a tensor of size $n_1 \times n_3 \times k$, with $\mathcal{C}(i,j,:) = \sum_{s=0}^{n_2} \mathcal{A}(i,s,:) \bullet \mathcal{B}(s,j,:)$, for $i \in [n_1]$ and $j \in [n_3]$.*

Definition 3. *[14, 17] Let $\mathcal{A} \in \mathbb{R}^{n_1\times n_2\times n_3}$, then the transpose $\mathcal{A}^\dagger \in \mathbb{R}^{n_2\times n_1\times n_3}$ is such that $\mathcal{L}(\mathcal{A}^\dagger)^{(i)} = (\mathcal{L}(\mathcal{A})^{(i)})^T$, $i \in [n_3]$.*

Definition 4. *[14, 17] Identity tensor based on $\mathcal{L}$-product is defined as $\mathcal{I} \in \mathbb{R}^{m\times m\times n}$ with $\mathcal{L}(\mathcal{I})^{(i)}$, $i \in [n]$ are $m \times m$ identity matrices.*

Definition 5. *[14] A tensor $\mathcal{A} \in \mathbb{R}^{m\times m\times n}$ is invertible if there exists a tensor $\mathcal{A}^{-1} \in \mathbb{R}^{m\times m\times n}$ such that $\mathcal{A} \bullet \mathcal{A}^{-1} = \mathcal{A}^{-1} \bullet \mathcal{A} = \mathcal{I}$.*

Definition 6. *$\mathcal{A}$ is $\mathcal{L}$-orthogonal if $\mathcal{A}^\dagger \bullet \mathcal{A} = \mathcal{A} \bullet \mathcal{A}^\dagger = \mathcal{I}$.*

Definition 7. *[14] The $\mathcal{L}$-SVD of $\mathcal{A} \in \mathbb{R}^{n_1\times n_2\times n_3}$ is given by $\mathcal{A} = \mathcal{U} \bullet \Sigma \bullet \mathcal{V}^\dagger$, where $\mathcal{U}$ and $\mathcal{V}$ are $\mathcal{L}$-orthogonal tensors of size $n_1 \times n_1 \times n_3$ and $n_2 \times n_2 \times n_3$ respectively, and Σ is a diagonal tensor of size $n_1 \times n_2 \times n_3$. The entries in Σ are called the singular values of $\mathcal{A}$, and the number of non-zero tubal-scalars of Σ is called the $\mathcal{L}$-rank of $\mathcal{A}$.*

Definition 8. *[14] If $\mathcal{A} \in \mathbb{R}^{n_1\times n_2\times n_3}$ with $\mathcal{L}$-rank of r, the r-dimensional tensor-column subspace $\mathcal{S}$ spanned by the columns of $\mathcal{A}$ is defined as*

$$\mathcal{S} = \{\mathcal{X} | \mathcal{X} = \mathcal{A}_1 \bullet \boldsymbol{c}_1 + \mathcal{A}_2 \bullet \boldsymbol{c}_2 + \cdots + \mathcal{A}_{n_2} \bullet \boldsymbol{c}_{n_2}\}$$

where $\boldsymbol{c}_j, j \in [n_2]$, is an arbitrary tubal scalar of length n_3.

Remark - If S is spanned by the columns of $\mathcal{A} \in \mathbb{R}^{n_1\times n_2\times n_3}$, $\mathcal{P} \triangleq \mathcal{A} \bullet (\mathcal{A}^\dagger \bullet \mathcal{A})^{-1} \bullet \mathcal{A}^\dagger$ is an orthogonal projection onto $\mathcal{S}$ when $\mathcal{A}^\dagger \bullet \mathcal{A}$ is invertible.

Definition 9. *The incoherence of an r-dimensional subspace $\mathcal{S}$ is defined as*

$$\mu(\mathcal{S}) \triangleq \frac{n_1}{r} \max_j \|\mathcal{P}(:,j,:)\|_F^2,$$

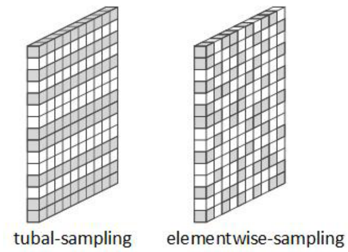


Fig. 1. An illustration of the tubal-sampling and elementwise-sampling patterns.

2.2. Problem Statement

Let $\mathcal{V} \in \mathbb{R}^{n_1\times 1\times n_3}$ denote a signal with its entries are sampled with replacement, and the given subspace $\mathcal{S}$ is spanned by the columns of $\mathcal{U} \in \mathbb{R}^{n_1\times n_2\times n_3}$. Then we want to find out that how many samples are required to decided whether a signal belongs to a given subspace with high probability. We assume the dimension of $\mathcal{S}$ is n_2. Here, for signal $\mathcal{V}$, we consider two types of sampling: tubal-sampling and elementwise-sampling, as showed in Fig. 1.

Tubal-sampling: Let Ω_1 be the index set of samples and $\Omega_1 \subset [n_1]$. If $i \in \Omega_1$, $\mathcal{V}(i,1,:)$ is a sample.

Elementwise-sampling: Let Ω_2 be the index set of samples and $\Omega_2 \subset [n_1] \times [n_3]$, i.e., if $(i,j) \in \Omega_2$, $\mathcal{V}(i,1,j)$ is a sample.

For $\Omega \in \{\Omega_1, \Omega_2\}$, and $m = |\Omega|$, where $|\Omega|$ denotes the cardinality of Ω, and we hope to determine the value of m so that we can decide wether $\mathcal{V}$ belongs $\mathcal{S}$ based on samples with high probability.

3. MAIN THEOREMS

Let $\mathcal{V}_\Omega$ denote the sampling signal, and the energies of $\mathcal{V}_\Omega$ outside $\mathcal{S}$ with tubal-sampling and elementwise-sampling are bounded with high probability.

3.1. Main Theorem with Tubal-sampling

When the sampling method is tubal-sampling, $\Omega = \Omega_1$. Then $\mathcal{V}_{\Omega_1} \in \mathbb{R}^{m\times 1\times n_3}$ denotes the sampling signal with its entries $\mathcal{V}_{\Omega_1}(i,1,:) = \mathcal{V}(\Omega_1(i),1,:)$. We define $\mathcal{P}_{\Omega_1} = \mathcal{U}_{\Omega_1} \bullet (\mathcal{U}_{\Omega_1}^\dagger \bullet \mathcal{U}_{\Omega_1})^{-1} \bullet \mathcal{U}_{\Omega_1}^\dagger$ as the projection where $\mathcal{U}_{\Omega_1} \in \mathbb{R}^{m\times n_2\times n_3}$ denotes the tensor organized by the horizontal slices of $\mathcal{U}$ indicated by Ω_1, that is $\mathcal{U}_{\Omega_1}(i,:,:) = \mathcal{U}(\Omega_1(i),:,:)$. If $\mathcal{V} \in \mathcal{S}$, $\|\mathcal{V}_{\Omega_1} - \mathcal{P}_{\Omega_1} \bullet \mathcal{V}_{\Omega_1}\|_F^2 = 0$. Then we have the following.

Theorem 1. *Let $\delta > 0$ and $m \geq \frac{8}{3} n_2 \mu(\mathcal{S}) \log(\frac{2n_2n_3}{\delta})$. Then with probability at least $1 - 4\delta$,*

$$\frac{m(1-\alpha) - cn_2\mu(\mathcal{S})\frac{\beta}{(1-\gamma)}}{n_1} \|\mathcal{V} - \mathcal{P} \bullet \mathcal{V}\|_F^2 \leq$$

$$\|\mathcal{V}_{\Omega_1} - \mathcal{P}_{\Omega_1} \bullet \mathcal{V}_{\Omega_1}\|_F^2 \le (1+\alpha)\frac{m}{n_1}\|\mathcal{V} - \mathcal{P} \bullet \mathcal{V}\|_F^2 \quad (1)$$

holds, where $\alpha = \sqrt{\frac{2(n_1\|\mathcal{Y}\|_{\infty*}^2 - \|\mathcal{Y}\|_F^2)}{m\|\mathcal{Y}\|_F^2}\log(\frac{1}{\delta})} + \frac{2(n_1\|\mathcal{Y}\|_{\infty*}^2 - \|\mathcal{Y}\|_F^2)}{3m\|\mathcal{Y}\|_F^2}\log(\frac{1}{\delta})$, $\beta = \left(1+2\sqrt{\log(\frac{1}{\delta})}\right)^2$, $\gamma = \sqrt{\frac{8n_2\mu(\mathcal{S})}{3m}\log(\frac{2n_2n_3}{\delta})}$, *and c is a constant satisfying* $\|\mathcal{L}(\mathcal{V})\|_F^2 = c\|\mathcal{V}\|_F^2$.

3.2. Main Theorem with Elementwise-sampling

For convenience of next discussion, we first introduce the operation unfold$(\cdot)$ and lmat$(\cdot)$ here. For $\mathcal{A} \in \mathbb{R}^{n_1\times n_2\times k}$,

$$\text{unfold}(\mathcal{A}) = \left[\begin{array}{cccc} \mathcal{A}^{(1)T} & \mathcal{A}^{(2)T} & \cdots & \mathcal{A}^{(n_3)T} \end{array}\right]^T,$$

and the operation fold$(\cdot)$ is the inverse operation of unfold$(\cdot)$.

Motivated by the definition of t-product in [15] and the cosine transform product in [17], we introduce the $\mathcal{L}$-product based on block matrix tools. For tensor $\mathcal{A} \in \mathbb{R}^{n_1\times n_2\times k}$, we use lmat$(\mathcal{A})$ to denote a spacial structured block matrix determined by the frontal slices of $\mathcal{A}$, such that the $\mathcal{L}$-product $\mathcal{V} = \mathcal{A} \bullet \mathcal{C}$, where $\mathcal{V} \in \mathbb{R}^{n_1\times 1\times k}$ and $\mathcal{C} \in \mathbb{R}^{n_2\times 1\times k}$, can be represented as unfold$(\mathcal{V})$ = lmat$(\mathcal{A})$unfold$(\mathcal{C})$. The form of the block matrix varies with the discrete transformation [15, 18, 19], i.e., when the transform $\mathcal{L}$ is discrete Fourier transform,

$$\text{lmat}(\mathcal{A}) = \begin{bmatrix} \mathcal{A}^{(1)} & \mathcal{A}^{(k)} & \cdots & \mathcal{A}^{(2)} \\ \mathcal{A}^{(2)} & \mathcal{A}^{(1)} & \cdots & \mathcal{A}^{(3)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}^{(k)} & \mathcal{A}^{(k-1)} & \cdots & \mathcal{A}^{(1)} \end{bmatrix},$$

and when the transform $\mathcal{L}$ is discrete cosine transform, lmat$(\mathcal{A}) = ((\boldsymbol{I}_{n_3} + \boldsymbol{Z}_{n_3}) \otimes \boldsymbol{I}_{n_1})^{-1}(\boldsymbol{T}+\boldsymbol{H})((\boldsymbol{I}_{n_3} + \boldsymbol{Z}_{n_3}) \otimes \boldsymbol{I}_{n_2})$, where $\otimes$ is the Kronecker product [20, 17], $\boldsymbol{I}_{n_i}$ denotes $n_i \times n_i, i \in [3]$, identity matrix, $\boldsymbol{Z}_{n_3}$ is the $n_3 \times n_3$ circular upshift matrix, and $\boldsymbol{T}+\boldsymbol{H}$ is the following $n_1n_3 \times n_2n_3$ block Toeplitz-plus-Hankel matrix [18, 19, 17].

$$\boldsymbol{T}+\boldsymbol{H} = \begin{bmatrix} \mathcal{A}^{(1)} & \mathcal{A}^{(2)} & \cdots & \mathcal{A}^{(k)} \\ \mathcal{A}^{(2)} & \mathcal{A}^{(1)} & \cdots & \mathcal{A}^{(k-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}^{(k)} & \mathcal{A}^{(k-1)} & \cdots & \mathcal{A}^{(1)} \end{bmatrix} + \begin{bmatrix} \mathcal{A}^{(2)} & \cdots & \mathcal{A}^{(k)} & \boldsymbol{0} \\ \vdots & \vdots & & \mathcal{A}^{(k)} \\ \mathcal{A}^{(k)} & \boldsymbol{0} & \cdots & \vdots \\ \boldsymbol{0} & \mathcal{A}^{(k)} & \cdots & \mathcal{A}^{(2)} \end{bmatrix}.$$

For elementwise-sampling, $\Omega = \Omega_2$, and the subspace $\mathcal{S} \subset \mathbb{R}^{n_1\times 1\times n_3}$ should be mapped into a vector subspace $\boldsymbol{S} \subset \mathbb{R}^{n_1n_3}$. Then the vector subspace $\boldsymbol{S}$ be spanned by the columns of lmat$(\mathcal{U})$, and for all $\mathcal{V} \in \mathcal{S}$, unfold$(\mathcal{V}) \in \boldsymbol{S}$. Let $\mathcal{V}_{\Omega_2} \in \mathbb{R}^{n_1\times 1\times n_3}$ denotes the sampling signal with its entries satisfied

$$\mathcal{V}_{\Omega_2}(i,1,j) = \begin{cases} \mathcal{V}(i,1,j), & (i,j) \in \Omega_2; \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

Let $\boldsymbol{v}_{\Omega_2}$ = unfold$(\mathcal{V}_{\Omega_2})$ and $\boldsymbol{P}_{\Omega_2} = \boldsymbol{U}_{\Omega_2}(\boldsymbol{U}_{\Omega_2}^T\boldsymbol{U}_{\Omega_2})^{-1}\boldsymbol{U}_{\Omega_2}^T$, where $\boldsymbol{U}_{\Omega_2}$ satisfies that $\boldsymbol{U}_{\Omega_2}((j-1)n_1+i,:)$ = lmat$(\mathcal{U})((j-1)n_1+i,:)$, for $(i,j) \in \Omega_2$ while $\boldsymbol{U}_{\Omega_2}((j-1)n_1+i,:) = 0$ for $(i,j) \notin \Omega_2$. If $\mathcal{V} \in \mathcal{S}$, $\|\boldsymbol{v}_{\Omega_2} - \boldsymbol{P}_{\Omega_2}\boldsymbol{v}_{\Omega_2}\|_2^2 = 0$. Then we have the following.

Theorem 2. *Let* $\delta > 0$, $m \ge \frac{8}{3}n_2n_3\mu(\boldsymbol{S})\log(\frac{2n_2n_3}{\delta})$. *Then with probability at least* $1-4\delta$

$$\frac{m(1-\alpha) - n_2n_3\mu(\boldsymbol{S})\frac{\beta}{(1-\gamma)}}{n_1n_3}\cos^2(\theta)\|\mathcal{V} - \mathcal{P} \bullet \mathcal{V}\|_F^2 \le$$

$$\|\boldsymbol{v}_{\Omega_2} - \boldsymbol{P}_{\Omega_2}\boldsymbol{v}_{\Omega_2}\|_2^2 \le (1+\alpha)\frac{m}{n_1n_3}\cos^2(\theta)\|\mathcal{V} - \mathcal{P} \bullet \mathcal{V}\|_F^2 \quad (3)$$

holds, where $\alpha = \sqrt{\frac{2(n_1n_3\|\mathcal{Y}\|_\infty^2 - \|\mathcal{Y}\|_F^2)}{m\|\mathcal{Y}\|_F^2}\log(\frac{1}{\delta})} + \frac{2(n_1n_3\|\mathcal{Y}\|_\infty^2 - \|\mathcal{Y}\|_F^2)}{3m\|\mathcal{Y}\|_F^2}\log(\frac{1}{\delta})$, $\beta = \left(1+2\sqrt{\log(\frac{1}{\delta})}\right)^2$, $\gamma = \sqrt{\frac{8n_2n_3b\mu(\boldsymbol{S})}{3m}\log(\frac{2n_2n_3}{\delta})}$, $\mu(\boldsymbol{S})$ *is the coherence of* $\boldsymbol{S}$ *[11], and* θ *is the angle between unfold*$(\mathcal{Y})$ *and* $\boldsymbol{S}^\perp$. *When the transform* $\mathcal{L}$ *is the discrete Fourier transform, we have* $\mu(\boldsymbol{S}) = \mu(\mathcal{S})$ *and* $\theta = 0$.

The proofs of Theorem 1 and Theorem 2 can be found in [21].

4. MATCHED SUBSPACE DETECTION

We adopt discrete Fourier transform (DFT) as the transform $\mathcal{L}$, and the tensor product used in Theorem 1 and Theorem 2 changes into t-product (represented as $*$). Then we have the detection set up as followings. Our hypotheses are $\mathcal{H}_0 : \mathcal{V} \in \mathcal{S}$ and $\mathcal{H}_1 : \mathcal{V} \notin \mathcal{S}$.

4.1. Matched Subspace Detection with Tubal-sampling

Under tubal-sampling, the test statistic is

$$t(\mathcal{V}_{\Omega_1}) = \|\mathcal{V}_{\Omega_1} - \mathcal{P}_{\Omega_1} * \mathcal{V}_{\Omega_1}\|_F^2 \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \eta. \quad (4)$$

In the noiseless case, the detection threshold $\eta = 0$. Theorem 1 shows that for $\delta > 0$ and $m \ge \frac{8}{3}n_2\mu(\mathcal{S})\log(\frac{2n_2}{\delta})$, the probability of detection is $P_D = \mathbb{P}[t(\mathcal{V}_{\Omega_1}) > 0|\mathcal{H}_1] \ge 1-4\delta$, and the probability of false alarm is $P_{FA} = \mathbb{P}[t(\mathcal{V}_{\Omega_1}) > 0|\mathcal{H}_0] = 0$, since when $\mathcal{V} \in \mathcal{S}$, $\|\mathcal{V}_{\Omega_1} - \mathcal{P}_{\Omega_1} * \mathcal{V}_{\Omega_1}\|_F^2 = 0$.

When there is Gaussian white noise $\mathcal{N} \in \mathbb{R}^{n_1\times 1\times n_3}$ with its entries $\mathcal{N}(i,1,k) \sim N(0,1),\ i \in [n_1], j \in [n_3]$, the observed signal can be represented as $\mathcal{W} = \mathcal{V}_{\Omega_1} + \mathcal{N}_{\Omega_1}$ where

$\mathcal{N}_{\Omega_1}$ obtained by the same sampling as $\mathcal{V}_{\Omega_1}$. And the test statistic is represented as following

$$t(\mathcal{W}) = \|\mathcal{W} - \mathcal{P}_{\Omega_1} * \mathcal{W}\|_F^2 \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \eta_p. \tag{5}$$

Then from [22], we have $t(\mathcal{W})$ is distributed as non-central χ^2-distribution with degree of freedom $(m - n_2)n_3$ and non-centrality parameter $\lambda_2^2 = \|\mathcal{V}_{\Omega_1} - \mathcal{P}_{\Omega_1} * \mathcal{V}_{\Omega_1}\|_F^2$. For a requested probability of false alarm $P_{FA} = p$, the detection threshold η_p can be obtained according to $\mathbb{P}[t(\mathcal{W}) > \eta_p|\mathcal{H}_0] \leq p$, and it can be simplified as $\mathbb{P}[\chi^2_{(m-n_2)n_3}(0) \leq \eta_p] \geq 1 - p$. Then the detection probability $P_D = 1 - \mathbb{P}[\chi^2_{(m-n_2)n_3}(\lambda_2^2) \leq \eta_p]$.

4.2. Matched Subspace Detection with Elementwise-sampling

For elementwise-sampling, the test statistic is represented as

$$t(\mathcal{V}_{\Omega_2}) = \|\boldsymbol{v}_{\Omega_2} - \boldsymbol{P}_{\Omega_2}\boldsymbol{v}_{\Omega_2}\|_2^2 \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \eta. \tag{6}$$

In the noiseless case, the detection threshold $\eta = 0$. Theorem 2 shows that for $\delta > 0$ and $m \geq \frac{8}{3}n_2 n_3 \mu(\mathcal{S}) \log(\frac{2n_2n_3}{\delta})$, the probability of detection is $P_D = \mathbb{P}[t(\mathcal{V}_{\Omega_2}) > 0|\mathcal{H}_1] \geq 1 - 4\delta$, and the probability of false alarm is $P_{FA} = \mathbb{P}[t(\mathcal{V}_{\Omega_2}) > 0|\mathcal{H}_0] = 0$, since when $\mathcal{V} \in \mathcal{S}$, $\|\boldsymbol{v}_{\Omega_2} - \boldsymbol{P}_{\Omega_2}\boldsymbol{v}_{\Omega_2}\|_2^2 = 0$.

When there is Gaussian white noise $\mathcal{N} \in \mathbb{R}^{n_1 \times 1 \times n_3}$, the observed signal can be represented as $\mathcal{W} = \mathcal{V}_{\Omega_2} + \mathcal{N}_{\Omega_2}$ where $\mathcal{N}_{\Omega_2}$ obtained by the same sampling as $\mathcal{V}_{\Omega_2}$. Let $\boldsymbol{w} = \text{unfold}(\mathcal{W})$, and the test statistic can be represented as follows

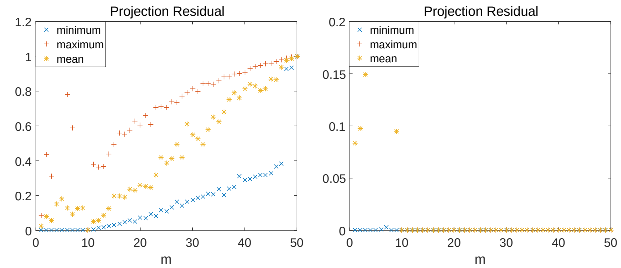
$$t(\mathcal{W}) = \|\boldsymbol{w} - \boldsymbol{P}_{\Omega_2}\boldsymbol{w}\|_2^2 \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \eta_p. \tag{7}$$

And from [22], we have $t(\mathcal{W})$ is distributed as a non-central χ^2-distribution with degree of freedom $m - n_2n_3$ and non-centrality parameter $\lambda_1^2 = \|\boldsymbol{v}_{\Omega_2} - \boldsymbol{P}_{\Omega_2}\boldsymbol{v}_{\Omega_2}\|_2^2$. Then for a requested probability of false alarm $P_{FA} = p$, the detection threshold η_p can be obtained according to $\mathbb{P}[t(\mathcal{W}) > \eta_p|\mathcal{H}_0] \leq p$, and it can be rewritten as $\mathbb{P}[\chi^2_{m-n_2n_3}(0) \leq \eta_p] \geq 1 - p$. Then the detection probability $P_D = \mathbb{P}[t(\mathcal{W}) > \eta_p|\mathcal{H}_1] = 1 - \mathbb{P}[\chi^2_{m-n_2n_3}(\lambda_1^2) \leq \eta_p]$.

5. NUMERICAL EXPERIMENTS

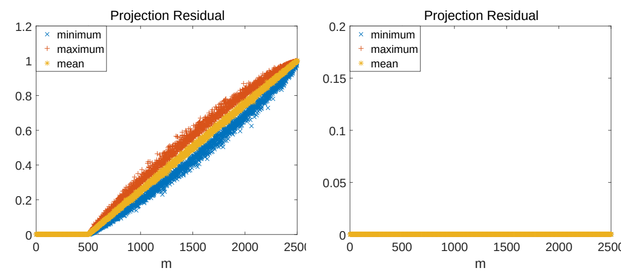
In this section, we examine our main results with simulations based on t-product. We take $\mathcal{U} \in \mathbb{R}^{50\times10\times50}$ with tubal-rank of 10 to span the subspace $\mathcal{S}$. The signal is represented as $\mathcal{V} \in \mathbb{R}^{50\times1\times50}$. Let m denote the number of samples, then then we compute the projection residual of $\mathcal{V}$ based on the samples for both $\mathcal{V} \in \mathcal{S}$ and $\mathcal{V} \in \mathcal{S}^\perp$. And Fig. 2 and Fig. 3 are the examination of our main results based on t-product with tubal-sampling and elementwise-sampling, respectively.

Fig. 2(a) and Fig. 3(a) show that the projection residual is always positive when $\mathcal{V} \in \mathcal{S}^\perp$, as long as the number of



(a) $\mathcal{V} \in \mathcal{S}^\perp$ (b) $\mathcal{V} \in \mathcal{S}$

Fig. 2. Simulation results for tubal-sampling over 20 runs with $n_1 = 50$, $n_2 = 10$, $n_3 = 50$ and $\mu(\mathcal{S}) \approx 1.1$. (a) is the the projection residual $\|\mathcal{V}_{\Omega_1} - \mathcal{P}_{\Omega_1} * \mathcal{V}_{\Omega_1}\|_F^2$ with $\mathcal{V} \in \mathcal{S}$, and (b) shows the the projection residual with $\mathcal{V} \in \mathcal{S}^\perp$.



(a) $\mathcal{V} \in \mathcal{S}^\perp$ (b) $\mathcal{V} \in \mathcal{S}$

Fig. 3. Simulation results for elementwise-sampling over 20 runs with $n_1 = 50$, $n_2 = 10$, $n_3 = 50$ and $\mu(\mathcal{S}) \approx 1.1$. (a) is the the projection residual $\|\boldsymbol{v}_{\Omega_2} - \boldsymbol{P}_{\Omega_2}\boldsymbol{v}_{\Omega_2}\|_2^2$ with $\mathcal{V} \in \mathcal{S}$, and (b) shows the the projection residual with $\mathcal{V} \in \mathcal{S}^\perp$.

samples $m > r\mu(\mathcal{S})\log(r)$ with tubal-sampling, and $m > r \times n_3(\mathcal{S})\log(rn_3)$ with elementwise-sampling where r is the dimensionality and n_3 is the size of the third dimension of $\mathcal{S}$. When $\mathcal{V} \in \mathcal{S}$, Fig. 2(b) and Fig. 3(b) show that the projection residual is zero.

6. CONCLUSION

In this paper, we extend conventional matched subspace detection to tensor matched subspace detection based on $\mathcal{L}$-product. We have shown that it is possible to detect whether a highly incomplete tensor belongs to a subspace when the number of samples is slightly greater than r for tubal-sampling while $r \times n_3$ for elementwise-sampling. Simulations have demonstrate the effectiveness of our methods.

7. REFERENCES

[1] Yeqing Li, Chen Chen, and Junzhou Huang, "Transformation-invariant collaborative sub-representation," in *IEEE 22nd International Conference on Pattern Recognition*. IEEE, 2014, pp. 3738–3743.

[2] Hadi Sariedeen, Mohammad M. Mansour, and Ali Chehab, "Efficient subspace detection for high-order mimo systems," in *2016 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*. IEEE, 2016, pp. 1001–1005.

[3] Mohammad M. Mansour, "A near-ml mimo subspace detection algorithm," *IEEE Signal Processing Letters*, vol. 22, pp. 408–412, April 2015.

[4] Zhongmin Wang, Gonzalo R. Arce, and Brian M. Sadler, "Subspace compressive detection for sparse signals," in *IEEE International Conference on Acoustics, Speech and Signal Processing*, 2008, pp. 3873–3876.

[5] Le Zheng, Arian Maleki, Haolei Weng, Xiaodong Wang, and Teng Long, "Does lp-minimization outperform l1-minimization?," *IEEE Trans. on Information Theory*, vol. 63, no. 11, pp. 6896–6935, November 2017.

[6] Jun Wang and Kai Xu, "Shape detection from raw lidar data with subspace modeling," *IEEE Transactions on Visualization and Computer Graphics*, vol. 23, no. 9, pp. 2137–2150, August 2017.

[7] Akshay Krishnamurthy and Aarti Singh, "Low-rank matrix and tensor completion via adaptive sampling," *Advances in Neural Information Processing Systems*, pp. 836–844, 2013.

[8] Xiao-Yang Liu, Shuchin Aeron, Vaneet Aggarwal, and Xiaodong Wang, "Tensor completion via adaptive sampling of tensor fibers: Application to efficient indoor rf fingerprinting," in *2016 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*. IEEE, 2016, pp. 2529–2533.

[9] Xiao-Yang Liu, Shuchin Aeron, Vaneet Aggarwal, Xiaodong Wang, and Min-You Wu, "Adaptive sampling of rf fingerprints for fine-grained indoor localization," *IEEE Transactions on Mobile Computing*, vol. 15, no. 10, pp. 2411–2423, October 2016.

[10] Xiao-Yang Liu and Xiaodong Wang, "Ls-decomposition for robust recovery of sensory big data," *IEEE Transactions on Big Data*, vol. PP, no. 99, pp. 1–1, 2017.

[11] Laura Balzano, Benjamin Recent, and Robert Nowak, "High-dimentional matched subspace detection when data are missing," *IEEE International Symposium on Information Theory*, pp. 1638–1642, June 2010.

[12] Tong Wu and Waheed U. Bajwa, "Subspace detection in a kernel space: the missing data case," in *2014 IEEE Workshop on Statistical Processing (SSP)*. IEEE, 2014, pp. 93–96.

[13] Martin Azizyan and Aarti Singh, "Subspace detection of high-dimensional vectors using compressive sampling," in *2012 IEEE Workshop on Statistical Processing (SSP)*. IEEE, 2012, pp. 724–727.

[14] Xiao-Yang Liu and Xiaodong Wang, "Fourth-order tensors with multidimensional discrete transforms," in *arXiv preprint arXiv:1705.01576*, 2017, pp. 1–37.

[15] Misha E. Kilmer, Karen Braman, Ning Hao, and Randy C. Hoover, "Third-order tensors as operators on matrices: a theoretical and computational framework with applications in imaging," *Siam Journal on Matrix Analysis and Applications*, vol. 34, no. 1, pp. 148–172, 2013.

[16] Misha E. Kilmer and Carla D. Martin, "Factorization strategies for third-order tensors," *Linear Algebra and Its Applications*, vol. 435, no. 3, pp. 641–658, 2011.

[17] Eric Kernfeld, Misha Kilmer, and Shuchin Aeron, "Tensor-tensor products with invertible linear transforms," *Linear Algebra and Its Applications*, vol. 485, pp. 545–570, 2015.

[18] Victoria Sanchez, Pedro Garcia, Antonio M. Peinado, Jose C. Segura, and Antonio J. Rubio, "Diagonalized properties of the discrete cosine transforms," *IEEE Transactions on Signal Processing*, vol. 43, no. 11, pp. 2631–2641, 1995.

[19] Thomas Kailath and Vadim Olshevsky, "Displacement structure approach to discrete-trigonometric-transform based preconditioners of g.strang type and of t.chan type," *Calcolo*, vol. 33, no. 3-4, pp. 191–208, 1996.

[20] Tamara G Kolda and Brett W Bader, "Tensor decompositions and applications," *Siam Review*, vol. 51, no. 3, pp. 455–500, 2009.

[21] Cuiping Li, Xiao-Yang Liu, and Yue Sun, "Tensor subspace detection," in *arXiv preprint arXiv:1710.08308*, 2017.

[22] Louis L Scharf and Benjamin Friedlander, "Matched subspace detectors," *IEEE Transactions on Signal Processing*, vol. 42, pp. 2146–2157, August 1994.