PERMISSIBLE SUPPORT PATTERNS FOR IDENTIFYING THE SPREADING FUNCTION OF TIME-VARYING CHANNELS

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ABSTRACT

We study support patterns for covariance matrices that appear in the problem of stochastic time-varying channel identification. The problem reduces to solving a linear system that is associated with a matrix in the form of a Kronecker product of a Gabor system matrix with itself, and therefore solvability of the linear system depends on the choice of generating window for the Gabor system and the support pattern of the object vector. In this paper, we investigate support patterns that allows the linear system to be solvable with some window. We present several classes of permissible patterns and also provide how the corresponding windows need to be chosen.

Index Terms— Channel identification, covariance estimation, Kronecker product

1. INTRODUCTION

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_N)^{\mathrm{T}} \in \mathbb{C}^N$ be a vector of N independent zero-mean random variables with covariance $\boldsymbol{X} = \mathbb{E}\left[\boldsymbol{x}\boldsymbol{x}^*\right]$, and let $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ be the vector of $m \ll N$ linear measurements of $\boldsymbol{x}$ through a measurement matrix $\boldsymbol{A} \in \mathbb{C}^{m\times N}$. As the vector $\boldsymbol{x}$ is stochastic, our aim is not to recover $\boldsymbol{x}$ from $\boldsymbol{y}$ but to recover $\boldsymbol{X}$ by observing the covariance of $\boldsymbol{y}$ given by $\boldsymbol{Y} = \mathbb{E}[\boldsymbol{y}\boldsymbol{y}^*] = \boldsymbol{A}\,\mathbb{E}[\boldsymbol{x}\boldsymbol{x}^*]\boldsymbol{A}^* = \boldsymbol{A}\boldsymbol{X}\boldsymbol{A}^*$. Through a vectorization, $\boldsymbol{Y}$ can be expressed as

$$\operatorname{vec}(\boldsymbol{Y}) = (\overline{\boldsymbol{A}} \otimes \boldsymbol{A}) \operatorname{vec}(\boldsymbol{X}), \tag{1}$$

where $\overline{\boldsymbol{A}}$ is the conjugate matrix of $\boldsymbol{A}$ and $\otimes$ denotes the Kronecker product (see Sec. 3). Certainly, this linear system is underdetermined and therefore requires a priori knowledge in order to determine $\boldsymbol{X}$ from $\boldsymbol{Y}$, for example, sparsity or low rank assumptions on $\boldsymbol{X}$. If $\boldsymbol{X}$ is sparse and is even known to be supported in a certain set Γ, the question remains whether the submatrix $(\overline{\boldsymbol{A}} \otimes \boldsymbol{A})|_\Gamma$ (with columns restricted to the set Γ) is injective so that (1) can be actually solved. Clearly, characterization of support patterns Γ for which $(\overline{\boldsymbol{A}} \otimes \boldsymbol{A})|_\Gamma$ is injective will depend both on the matrix $\boldsymbol{A}$ and the Kronecker tensor structure.

Determining second order statistics is an important problem in many different applications like array signal processing [1], spectral analysis of stationary stochastic processes [2], prediction and estimation of communication channels [3], to mention only a few. In recent years, new ideas from compressive sampling has brought progress in this field, using sparse structures (patterns) of the covariance matrices [4, 5].

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In this paper we focus on a specific application, namely the identification of stochastic time-varying channels [3, 6] in which case $\boldsymbol{A}$ becomes a Gabor system matrix (see Sec. 2). Several support patterns of $\boldsymbol{X}$ for which the problem (1) is solvable/nonsolvable are found and discussed in [3], however, those patterns depend only on the structure of Kronecker tensor product and do not make use of the Gabor structure of $\boldsymbol{A}$ (with the exception of diagonal pattern). Furthermore, giving a full characterization of such patterns seems too challenging and may even be impossible. It should be noted that $\boldsymbol{X}$, by definition, is a positive semi-definite matrix and therefore its support has a certain symmetry (cf. (3)). This paper considers patterns that do not possess this symmetry, however, characterizing non-symmetric patterns that allow injectivity of $(\overline{\boldsymbol{A}} \otimes \boldsymbol{A})|_\Gamma$ is an interesting task from a theoretical viewpoint and may as well lead to substantial insights on the symmetric patterns.

2. MOTIVATION: CHANNEL IDENTIFICATION

A classical and frequently used model of a time-varying communication channel H with input signal x and output y is given by

$$y(t) = (Hx)(t) = \iint \eta(\tau,\nu)\,\big(M_\nu T_\tau x\big)(t)\,\mathrm{d}\tau\,\mathrm{d}\nu\ ,$$

wherein M_ν and T_τ are the continuous modulation and translation operators respectively. In applications like radar or communications, often the problem is to identify the *spreading function* η of the channel H from the channel response y of an appropriate test signal x. Often, the spreading function η can be considered as a zero-mean (two-dimensional) stochastic process. Then the goal is to estimate the covariance $R_\eta(\tau,\tau',\nu,\nu') = \mathbb{E}\big[\eta(\tau,\nu)\,\overline{\eta(\tau',\nu')}\big]$ of the spreading function from the covariance $\mathbb{E}[y(t)\,\overline{y(t')}]$ of the received signal. Using a sounding signal of the form

$$x(t) = \textstyle\sum_{n\in\mathbb{Z}} c_n\,\delta(t-kT) \tag{2}$$

with the Dirac delta function δ and with an L-periodic sequence $\{c_n\}_{n\in\mathbb{Z}}$, the described problem is reduced to a finite-dimensional problem of the form (1) where $\boldsymbol{A} = \boldsymbol{G}(\boldsymbol{c})$ is the full Gabor system matrix generated by the window $\boldsymbol{c} = (c_0,\ldots,c_L)^{\mathrm{T}}$, $\boldsymbol{X}$ corresponds to the covariance R_η of the channel, and $\boldsymbol{Y}$ corresponds to the covariance of the received signal y (see [3] for further details).

Often, the support pattern Γ for the covariance R_η of the channel (i.e. the support of $\boldsymbol{X}$ in (1)) is known. Then, in order to identify the channel, we have to choose $\boldsymbol{c}$ such that the matrix $\overline{\boldsymbol{G}(\boldsymbol{c})} \otimes \boldsymbol{G}(\boldsymbol{c})|_\Gamma$ is invertible. If this is possible, we say that Γ is *permissible*; otherwise Γ is said to be *defective*. In Section 4, we will identify several classes of permissible patterns and also present how the corresponding windows $\boldsymbol{c}$ have to be chosen.

3. NOTATIONS AND TERMINOLOGY

The Kronecker (tensor) product of two matrices $\boldsymbol{A} = [a_{k,\ell}]_{k=1}^{K}{}_{\ell=1}^{L}$ and $\boldsymbol{B} = [b_{m,n}]_{m=1}^{M}{}_{n=1}^{N}$ is the $KM \times LN$ matrix defined as the block matrix

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{bmatrix} a_{1,1}\boldsymbol{B} & \dots & a_{1,L}\boldsymbol{B} \\ \vdots & \ddots & \vdots \\ a_{K,1}\boldsymbol{B} & \dots & a_{K,L}\boldsymbol{B} \end{bmatrix}.$$

For a matrix $\boldsymbol{X} \in \mathbb{C}^{\Lambda\times\Lambda}$ with rows and columns indexed by a finite set Λ, its support set is defined as

$$\operatorname{supp} \boldsymbol{X} = \{(\lambda,\lambda') \in \Lambda\times\Lambda : X(\lambda,\lambda') \neq 0\},$$

which will be viewed as a (support) pattern in $\Lambda \times \Lambda$. The diagonal pattern is defined as $\operatorname{diag}(\Lambda) = \{(\lambda,\lambda) : \lambda \in \Lambda\}$. Note that if $\boldsymbol{X}$ is positive semi-definite, then $(\lambda,\lambda') \in \operatorname{supp}\boldsymbol{X}$ implies $(\lambda,\lambda), (\lambda',\lambda), (\lambda',\lambda') \in \operatorname{supp}\boldsymbol{X}$. For a support set $\Gamma \subseteq \Lambda\times\Lambda$, we say that Γ is a *positive semi-definite (psd) pattern* if

$$(\lambda,\lambda') \in \Gamma \;\Rightarrow\; (\lambda,\lambda),\,(\lambda',\lambda),\,(\lambda',\lambda') \in \Gamma. \qquad (3)$$

Given an ordered index set $\Lambda = \{\lambda_0,\dots,\lambda_{N-1}\}$ of size N, we define the vectorization of a matrix $\boldsymbol{X} \in \mathbb{C}^{\Lambda\times\Lambda}$ as $\operatorname{vec}(\boldsymbol{X}) = \{X(\lambda,\lambda')\}_{(\lambda,\lambda')\in\Lambda^2}$, where Λ^2 is the ordered set given by

$$\Lambda^2 = \{(\lambda_0,\lambda_0),(\lambda_1,\lambda_0),\dots,(\lambda_{N-1},\lambda_0),\dots,(\lambda_{N-1},\lambda_{N-1})\}.$$

Equivalently, $\operatorname{vec}(\boldsymbol{X})$ is the N^2-dimensional vector obtained by stacking up all the columns of $\boldsymbol{X}$ into a single column.

Gabor system Let $L \geq 2$ be an integer. The *translation* and *modulation* operators on $\mathbb{C}^L$ are defined as $\boldsymbol{T}\boldsymbol{x} = (x_{L-1}, x_0, \dots, x_{L-2})$ and $\boldsymbol{M}\boldsymbol{x} = (\omega^0 x_0, \omega^1 x_1, \dots, \omega^{L-1} x_{L-1})$ for $\boldsymbol{x} = (x_0,\dots,x_{L-1})$ in $\mathbb{C}^L$ respectively, where $\omega = e^{2\pi i/L}$. Note that $\boldsymbol{T}^L = \boldsymbol{M}^L = \boldsymbol{I}_L$. The time-frequency shift operators on $\mathbb{C}^L$ are defined as $\pi(k,\ell) = \boldsymbol{M}^\ell \boldsymbol{T}^k$ for $k,\ell \in \mathbb{Z}_L$.

For a window vector $\boldsymbol{c} = (c_0,\dots,c_{L-1}) \in \mathbb{C}^L$, the *full Gabor system matrix*, denoted by $\boldsymbol{G}(\boldsymbol{c})$, is the $L\times L^2$ matrix consisting of the column vectors $\pi(k,\ell)\boldsymbol{c}$, $k,\ell = 0,\dots,L-1$, that is,

$$\boldsymbol{G}(\boldsymbol{c}) = [\, \boldsymbol{D}_0\,\boldsymbol{W}_L \mid \boldsymbol{D}_1\,\boldsymbol{W}_L \mid \cdots \mid \boldsymbol{D}_{L-1}\,\boldsymbol{W}_L \,],$$

where $\boldsymbol{D}_k = \operatorname{diag}(\boldsymbol{T}^k c) = \operatorname{diag}(c_{L-k},\dots,c_{L-1},c_0,\dots,c_{L-k-1})$, and $\boldsymbol{W}_L = [\omega^{k\ell}]_{k,\ell=0}^{L-1}$ is the $L\times L$ discrete Fourier matrix. It is known that there exist vectors $\boldsymbol{c} \in \mathbb{C}^L$ for which every L column vectors of $\boldsymbol{G}(\boldsymbol{c})$ are linearly independent, in fact, such vectors form a dense, open subset of $\mathbb{C}^L$ with full measure [7, 8].

4. PERMISSIBLE / DEFECTIVE PATTERNS

As we are interested in patterns for which $(\overline{\boldsymbol{A}} \otimes \boldsymbol{A})|_\Gamma$ is injective, in the particular case where $\boldsymbol{A}$ is a Gabor matrix, we introduce the following definition as in [3].

Definition 1 ([3]): *A support pattern $\Gamma \subset (\mathbb{Z}_L\times\mathbb{Z}_L)\times(\mathbb{Z}_L\times\mathbb{Z}_L)$ is said to be* permissible *if there exists $\boldsymbol{c} \in \mathbb{C}^L$ such that $\overline{\boldsymbol{G}(\boldsymbol{c})} \otimes \boldsymbol{G}(\boldsymbol{c})|_\Gamma$ is injective; otherwise, we say that Γ is* defective*.*

Clearly, a permissible pattern is of cardinality at most L^2 and any pattern containing a defective pattern is again defective.

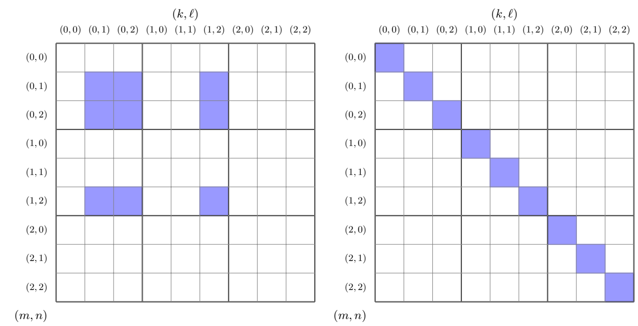

Fig. 1. Permissible patterns for $L = 3$. A tensor square (or rank-one) pattern (left) and the diagonal pattern (right).

4.1. Patterns that can be described by linear dependence of columns of $\boldsymbol{G}(\boldsymbol{c})$

There exist patterns whose permissibility is completely characterized by linear independence of the corresponding columns of $\boldsymbol{G}(\boldsymbol{c}) \in \mathbb{C}^{L\times L^2}$. We state the result for a general matrix of size $m\times N$ with $m \leq N$.

Theorem 1: *Let $\boldsymbol{A} \in \mathbb{C}^{m\times N}$ with $m \leq N$, and let $\Lambda \subseteq \{0,1,\dots,N-1\}$ with $|\Lambda| \geq 2$. The following are equivalent.*

- *(a) $\boldsymbol{A}|_\Lambda$ $(\in \mathbb{C}^{m\times|\Lambda|})$ is injective.*
- *(b) $\overline{\boldsymbol{A}} \otimes \boldsymbol{A}|_{\Lambda\times\Lambda}$ $(\in \mathbb{C}^{m^2\times|\Lambda|^2})$ is injective.*
- *(c) There exist nonempty disjoint subsets $\Lambda_1,\Lambda_2 \subset \Lambda$ with $\Lambda_1\cup\Lambda_2 = \Lambda$ such that $\overline{\boldsymbol{A}} \otimes \boldsymbol{A}|_{(\Lambda_1\times\Lambda_1)\cup(\Lambda_2\times\Lambda_2)}$ is injective.*
- *(d) There exist nonempty disjoint subsets $\Lambda_1,\Lambda_2 \subset \Lambda$ with $\Lambda_1\cup\Lambda_2 = \Lambda$ such that $\overline{\boldsymbol{A}} \otimes \boldsymbol{A}|_{(\Lambda_1\times\Lambda_2)\cup(\Lambda_2\times\Lambda_1)\cup\operatorname{diag}(\Lambda)}$ is injective.*
- *(e) There exists $\lambda \in \Lambda$ for which $\overline{\boldsymbol{A}} \otimes \boldsymbol{A}|_{(\{\lambda\}\times\Lambda)\cup(\Lambda\times\{\lambda\})\cup\operatorname{diag}(\Lambda)}$ is injective.*

Moreover in this case, $|\Lambda| \leq \operatorname{rank}\boldsymbol{A}$ $(\leq m)$ and $\overline{\boldsymbol{A}} \otimes \boldsymbol{A}|_\Gamma$ is injective for every (psd or non-psd) subpattern $\Gamma \subset \Lambda\times\Lambda$.

Recall that there exists $\boldsymbol{c} \in \mathbb{C}^L$ for which every L column vectors of $\boldsymbol{A} = \boldsymbol{G}(\boldsymbol{c}) \in \mathbb{C}^{L\times L^2}$ are linearly independent ($m = L$, $N = L^2$). By choosing such a vector $\boldsymbol{c}$, the condition (a) is fulfilled whenever the set Λ is of cardinality $\leq L$. Hence, all the patterns appearing in (b)-(e), namely the patterns $\Lambda\times\Lambda$, $(\Lambda_1\times\Lambda_1)\cup(\Lambda_2\times\Lambda_2)$, $(\Lambda_1\times\Lambda_2)\cup(\Lambda_2\times\Lambda_1)\cup\operatorname{diag}(\Lambda)$, $(\{\lambda\}\times\Lambda)\cup(\Lambda\times\{\lambda\})\cup\operatorname{diag}(\Lambda)$, where $\Lambda = \Lambda_1\dot\cup\Lambda_2$ and $\lambda \in \Lambda$, are permissible whenever $|\Lambda| \leq L$. Note that these patterns are all psd.

As every set of $L+1$ vectors in $\mathbb{C}^L$ are linearly dependent, Theorem 1 implies that all patterns appearing in (b)-(e) are defective whenever $|\Lambda| \geq L+1$.

Related Work *Permissible patterns*: The permissibility of the pattern $\Lambda\times\Lambda$ with $|\Lambda| \leq L$ (*tensor square* or *rank-one*) and the diagonal pattern of Proposition 2 below, was first proved in [3]. Fig.1 shows two permissible patterns.
Defective patterns: The pattern $(\{\lambda\}\times\Lambda)\cup(\Lambda\times\{\lambda\})\cup\operatorname{diag}(\Lambda)$ with $\lambda\in\Lambda$ and $|\Lambda| \geq L+1$ (*arrowhead*) is known to be defective (e.g., [4]). The defectiveness of the patterns $(\Lambda_1\times\Lambda_1)\cup(\Lambda_2\times\Lambda_2)$ (*two squares* or *rank-two*) and $(\Lambda_1\times\Lambda_2)\cup(\Lambda_2\times\Lambda_1)\cup\operatorname{diag}(\Lambda_1\cup\Lambda_2)$ (*butterfly*) for disjoint sets Λ_1,Λ_2 with $|\Lambda_1|+|\Lambda_2| \geq L+1$, was proved in [3]. Fig. 2 illustrates the first two defective patterns.

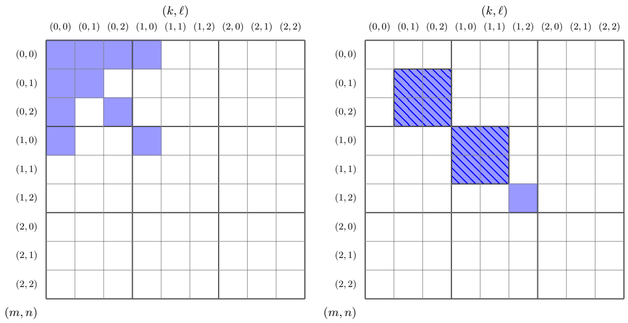

Fig. 2. Defective patterns for $L = 3$. An arrowhead pattern (left) and a pattern containing a rank-two defective pattern (right).

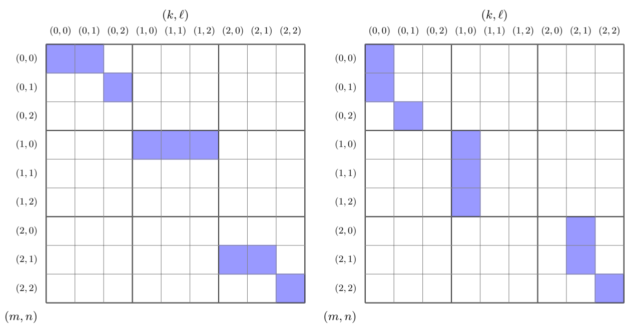

Fig. 3. Patterns of type (i) (left) and (ii) (right) for $L = 3$.

4.2. Generalized Diagonal Patterns

Proposition 2 ([3,6]): *Let* $\Gamma_{\text{diag}} = \{(\lambda, \widetilde{\lambda}) \in (\mathbb{Z}_L \times \mathbb{Z}_L) \times (\mathbb{Z}_L \times \mathbb{Z}_L) : \lambda = \widetilde{\lambda}\}$ *be the diagonal set. The matrix* $\overline{\boldsymbol{G}(\boldsymbol{c})} \otimes \boldsymbol{G}(\boldsymbol{c})|_{\Gamma_{\text{diag}}} = [\overline{\pi(\lambda)\boldsymbol{c}} \otimes \pi(\lambda)\boldsymbol{c}]_{\lambda \in \mathbb{Z}_L \times \mathbb{Z}_L}$ *is invertible if* $\boldsymbol{c} \in \mathbb{C}^L$ *satisfies* $\langle \boldsymbol{c}, \pi(\lambda)\boldsymbol{c}\rangle \neq 0$ *for all* $\lambda \in \mathbb{Z}_L \times \mathbb{Z}_L$*. The set of all such vectors* $\boldsymbol{c}$ *is a dense, open subset of* $\mathbb{C}^L$ *with full measure.*

As a generalization of the diagonal pattern $\Gamma_{\text{diag}} = \{(k, \ell, k, \ell) : k, \ell \in \mathbb{Z}_L\}$, we consider patterns of the form (cf. Fig. 3)

(i) $\Gamma = \cup_{k=0}^{L-1}\{(k, \ell, k, n_{k,\ell}) : \ell \in \mathbb{Z}_L\}$,

(ii) $\Gamma = \cup_{k=0}^{L-1}\{(k, \ell_{k,n}, k, n) : n \in \mathbb{Z}_L\}$,

where $n_{k,\ell} \in \mathbb{Z}_L$ (resp. $\ell_{k,n} \in \mathbb{Z}_L$) is an arbitrary sequence in k, ℓ (resp. k, n). It should be noted that a pattern of types (i) or (ii) is a psd pattern only when it is a diagonal pattern, i.e., $n_{k,\ell} = \ell$ for type (i) and $\ell_{k,n} = n$ for type (ii).

Theorem 3: *Let* $L \geq 2$ *be any integer and let* $\Gamma \subset (\mathbb{Z}_L \times \mathbb{Z}_L) \times (\mathbb{Z}_L \times \mathbb{Z}_L)$ *be any pattern of type (i) or (ii). The matrix* $\overline{\boldsymbol{G}(\boldsymbol{c})} \otimes \boldsymbol{G}(\boldsymbol{c})|_\Gamma = [\overline{\pi(\lambda)\boldsymbol{c}} \otimes \pi(\widetilde{\lambda})\boldsymbol{c}]_{(\lambda,\widetilde{\lambda})\in\Gamma} \in \mathbb{C}^{L^2 \times L^2}$ *is invertible for all* $\boldsymbol{c}$ *in a dense open subset of* $\mathbb{C}^L$ *with full measure.*

This theorem shows that if the channel is known to have a support pattern of type (i) or (ii), then the L-periodic sequence in the sounding signal (2) can be chosen randomly, because for almost all vectors $\boldsymbol{c} \in \mathbb{C}^L$ the matrix $\overline{\boldsymbol{G}(\boldsymbol{c})} \otimes \boldsymbol{G}(\boldsymbol{c})|_\Gamma$ is invertible.

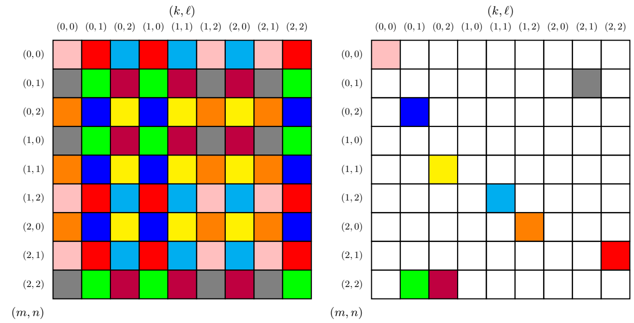

Fig. 4. The case with $L = 3$ and $p = 2$. All cosets $[V_p + (0, q)] \times [V_p + (0, q')]$, $q, q' = 0, \ldots, L-1$ are shown in different colors (left). A pattern of type (iii) is formed by choosing one element from each coset (right).

4.3. Scattered Patterns

If $L \geq 2$ is a prime, there exist exactly $L + 1$ additive subgroups of $\mathbb{Z}_L \times \mathbb{Z}_L$ with cardinality L, namely [9, 10],

$$V_p := \{(j, jp) : j = 0, \ldots, L-1\}, \quad p = 0, \ldots, L-1,$$
$$V_\infty := \{0\} \times \mathbb{Z}_L,$$

where all integers are understood as elements of $\mathbb{Z}_L$. For $p \in \{0, 1, \ldots, L-1, \infty\}$ fixed, the sets $V_p + (0, q)$, $q = 0, \ldots, L-1$ are all distinct and therefore constitute a full set of cosets of V_p in $\mathbb{Z}_L \times \mathbb{Z}_L$; moreover, the sets $[V_p + (0, q)] \times [V_p + (0, q')]$, $q, q' = 0, \ldots, L-1$ constitute a full set of cosets of $V_p \times V_p$ in $(\mathbb{Z}_L \times \mathbb{Z}_L) \times (\mathbb{Z}_L \times \mathbb{Z}_L)$.

We consider patterns of the form

(iii) $\Gamma = \{(\lambda_{q,q'}, \widetilde{\lambda}_{q,q'})\}_{q,q'=0}^{L-1} \subset (\mathbb{Z}_L \times \mathbb{Z}_L) \times (\mathbb{Z}_L \times \mathbb{Z}_L)$, where $\lambda_{q,q'} \in V_p + (0, q)$ and $\widetilde{\lambda}_{q,q'} \in V_p + (0, q')$ with fixed $p \in \{0, 1, \ldots, L-1, \infty\}$,

whose constructions are illustrated in Fig. 4. Notice that e.g., both $\lambda_{q,0}$ and $\lambda_{q,1}$ belong in the coset $V_p + (0, q) \subset \mathbb{Z}_L \times \mathbb{Z}_L$ but they are not necessarily the same.

Theorem 4: *Let* $L \geq 2$ *be a prime and let* $\Gamma \subset (\mathbb{Z}_L \times \mathbb{Z}_L) \times (\mathbb{Z}_L \times \mathbb{Z}_L)$ *be any pattern of type (iii). The matrix* $\overline{\boldsymbol{G}(\boldsymbol{c})} \otimes \boldsymbol{G}(\boldsymbol{c})|_\Gamma$ *is invertible for all* $\boldsymbol{c}$ *in a dense open subset of* $\mathbb{C}^L$ *with full measure. Moreover, there exists* $\boldsymbol{c} \in \mathbb{C}^L$ *for which* $\overline{\boldsymbol{G}(\boldsymbol{c})} \otimes \boldsymbol{G}(\boldsymbol{c})|_\Gamma$ *is unitary; explicitly,* $\boldsymbol{c}$ *can be chosen from the set* $\{\boldsymbol{D}^p \boldsymbol{u}_n\}_{n=0}^{L-1}$ *if* $p < \infty$ *and* $\{\boldsymbol{e}_n\}_{n=0}^{L-1}$ *if* $p = \infty$*, where* $\boldsymbol{D}$ *is the* $L \times L$ *diagonal matrix whose* n*-th diagonal entry is* $\omega^{0+1+\ldots+(n-1)} = \omega^{n(n-1)/2}$, $\boldsymbol{u}_n = (1, \omega^n, \ldots, \omega^{(L-1)n})^T$ *is the* n*-th column of* $\boldsymbol{W}_L$*, and* $\boldsymbol{e}_n$ *is the* n*-th canonical basis vector.*

So, if the channel is known to have a support pattern of type (iii), one can also pick $\boldsymbol{c} \in \mathbb{C}^L$ randomly like in the cases (i) and (ii). However, the strength of Theorem 4 is that it provides explicit vectors $\boldsymbol{c} \in \mathbb{C}^L$ for which $\overline{\boldsymbol{G}(\boldsymbol{c})} \otimes \boldsymbol{G}(\boldsymbol{c})|_\Gamma$ is not only invertible but also unitary; this obviously implies a stable recovery of $\text{vec}(\boldsymbol{X})$ in Eqn. (1).

We remark that there exist psd-patterns of type (iii). For example, the upper-left 3×3 and the lower-right 3×3 corners in Fig. 4 (left), are psd patterns.

5. APPENDIX – PROOFS

5.1. Proof of Theorem 1

We will use several times the fact that the linear map $\boldsymbol{X} \mapsto \boldsymbol{A}\boldsymbol{X}\boldsymbol{A}^*$ is injective on $\{\boldsymbol{X} \in \mathbb{C}^{N\times N} : \operatorname{supp}\boldsymbol{X} \subseteq \Gamma\}$ if and only if $\overline{\boldsymbol{A}}\otimes\boldsymbol{A}|_\Gamma$ is injective.

(a) $\Leftrightarrow$ (b): If $\boldsymbol{A}|_\Lambda$ is injective, i.e., $\operatorname{rank}\boldsymbol{A}|_\Lambda = |\Lambda| \le m$, then $\operatorname{rank}(\overline{\boldsymbol{A}}\otimes\boldsymbol{A}|_{\Lambda\times\Lambda}) = (\operatorname{rank}\boldsymbol{A}|_\Lambda)^2 = |\Lambda|^2$ and hence $\overline{\boldsymbol{A}}\otimes\boldsymbol{A}|_{\Lambda\times\Lambda}$ is injective. Conversely, if $\overline{\boldsymbol{A}}\otimes\boldsymbol{A}|_{\Lambda\times\Lambda}$ is injective, then for any vector $\boldsymbol{v} \in \mathbb{C}^N$ with $\operatorname{supp}\boldsymbol{v} \subseteq \Lambda$ and $\boldsymbol{A}\boldsymbol{v} = 0$, we have $\boldsymbol{A}\boldsymbol{v}\boldsymbol{v}^*\boldsymbol{A}^* = 0$ where $\operatorname{supp}\boldsymbol{v}\boldsymbol{v}^* \subseteq \Lambda\times\Lambda$, so that $\boldsymbol{v}\boldsymbol{v}^* = 0$, hence, $\boldsymbol{v} = 0$.

(b) $\Rightarrow$ (c), (d), (e): These are trivial because the respective patterns are all contained in $\Lambda\times\Lambda$.

(c) $\Rightarrow$ (a): Suppose that $\boldsymbol{v} \in \mathbb{C}^N$ is a nontrivial vector with $\operatorname{supp}\boldsymbol{v} \subseteq \Lambda$ and $\boldsymbol{A}\boldsymbol{v} = 0$. We write $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2 \neq 0$ with $\operatorname{supp}\boldsymbol{v}_1 \subseteq \Lambda_1$ and $\operatorname{supp}\boldsymbol{v}_2 \subseteq \Lambda_2$. Then $\boldsymbol{A}\boldsymbol{v}_1 = \boldsymbol{A}(-\boldsymbol{v}_2)$ implies $\boldsymbol{A}\boldsymbol{v}_1\boldsymbol{v}_1^*\boldsymbol{A}^* = \boldsymbol{A}\boldsymbol{v}_2\boldsymbol{v}_2^*\boldsymbol{A}^*$, so that $\boldsymbol{A}\mathbf{N}\boldsymbol{A}^* = 0$ where $\mathbf{N} = \boldsymbol{v}_1\boldsymbol{v}_1^* - \boldsymbol{v}_2\boldsymbol{v}_2^* \in \mathbb{C}^{N\times N}$ is a nontrivial matrix supported in $(\Lambda_1\times\Lambda_1)\cup(\Lambda_2\times\Lambda_2)$. This contradicts with $\overline{\boldsymbol{A}}\otimes\boldsymbol{A}|_{(\Lambda_1\times\Lambda_1)\cup(\Lambda_2\times\Lambda_2)}$ being injective. Therefore, $\boldsymbol{A}|_\Lambda$ is injective.

(e) $\Rightarrow$ (d): We may choose $\Lambda_1 = \{\lambda\}$ and $\Lambda_2 = \Lambda\backslash\{\lambda\}$.

(d) $\Rightarrow$ (a): Suppose that $\boldsymbol{v} \in \mathbb{C}^N$ is a nontrivial vector with $\operatorname{supp}\boldsymbol{v} \subseteq \Lambda$ and $\boldsymbol{A}\boldsymbol{v} = 0$. As before, we write $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2 \neq 0$ with $\operatorname{supp}\boldsymbol{v}_1 \subseteq \Lambda_1$ and $\operatorname{supp}\boldsymbol{v}_2 \subseteq \Lambda_2$. If $\boldsymbol{v}_1 \neq 0$ and $\boldsymbol{v}_2 \neq 0$, then $\boldsymbol{A}\boldsymbol{v}_1 = \boldsymbol{A}(-\boldsymbol{v}_2)$ yields $\boldsymbol{A}\mathbf{N}\boldsymbol{A}^* = 0$ where $\mathbf{N} = \boldsymbol{v}_1\boldsymbol{v}_2^* - \boldsymbol{v}_2\boldsymbol{v}_1^* \in \mathbb{C}^{N\times N}$ is a nontrivial matrix supported in $(\Lambda_1\times\Lambda_2)\cup(\Lambda_2\times\Lambda_1)$. If $\boldsymbol{v}_1 \neq 0$ and $\boldsymbol{v}_2 = 0$, then pick any $\lambda_2 \in \Lambda_2$ and let $\mathbf{R}$ be the $n\times n$ matrix whose λ_2-th column is $\boldsymbol{v}_1$ and all zero in other columns. Then $\boldsymbol{A}\mathbf{N}\boldsymbol{A}^* = 0$, where $\mathbf{N} = \mathbf{R} + \mathbf{R}^* \in \mathbb{C}^{N\times N}$ is a nontrivial matrix supported in $(\Lambda_1\times\{\lambda_2\})\cup(\{\lambda_2\}\times\Lambda_1)$. The case where $\boldsymbol{v}_1 = 0$ and $\boldsymbol{v}_2 \neq 0$ is similar. In all three cases, we get a contradiction with the fact that $\overline{\boldsymbol{A}}\otimes\boldsymbol{A}|_{(\Lambda_1\times\Lambda_2)\cup(\Lambda_2\times\Lambda_1)\cup\operatorname{diag}(\Lambda)}$ is injective. Therefore, $\boldsymbol{A}|_\Lambda$ is injective. This completes the proof.

5.2. Proof of Theorem 3

First, we write $\overline{\boldsymbol{G}(\boldsymbol{c})}\otimes\boldsymbol{G}(\boldsymbol{c})|_\Gamma = [\mathbf{A}^{(0)}\,|\,\mathbf{A}^{(1)}\,|\,\ldots\,|\,\mathbf{A}^{(L-1)}]$, where $\mathbf{A}^{(k)}$ is the $L^2\times L$ submatrix consisting of the columns indexed by $\{(k,\ell,k,n) : (n,\ell) \in S_k\}$, where $S_k = \{(n_{k,\ell},\ell)\}_{\ell=0}^{L-1}$ for type (i), and $S_k = \{(n,\ell_{k,n})\}_{n=0}^{L-1}$ for type (ii). Applying the extended Laplace expansion (see e.g., [7]), we obtain

$$\begin{aligned}&\det(\overline{\boldsymbol{G}(\boldsymbol{c})}\otimes\boldsymbol{G}(\boldsymbol{c})|_\Gamma) \\ &= \sum_B \operatorname{sgn}(B)\cdot\det(\mathbf{A}^{(0)}(B_0))\cdot\ldots\cdot\det(\mathbf{A}^{(L-1)}(B_{L-1})),\end{aligned} \tag{4}$$

where $B = (B_0,\ldots,B_{L-1})$ runs through all ordered partitions of row indices $\mathbb{Z}_L\times\mathbb{Z}_L$ with $|B_0| = \ldots = |B_{L-1}| = L$. Here $\operatorname{sgn}(B)$ denotes the sign (± 1) of the permutation $\left(\begin{smallmatrix}\{0\}\times\mathbb{Z}_L & \cdots & \{L-1\}\times\mathbb{Z}_L \\ B_0 & \cdots & B_{N-1}\end{smallmatrix}\right)$, and $\mathbf{A}^{(k)}(B_k)$ is the $L\times L$ submatrix of $\mathbf{A}^{(k)}$ formed with the rows indexed by B_k.

Since every entry of $\overline{\boldsymbol{G}(\boldsymbol{c})}\otimes\boldsymbol{G}(\boldsymbol{c})$ is of the form $\overline{c_i\omega^p}\,c_j\omega^q$ with $i,j,p,q \in \mathbb{Z}_L$, the $\det(\overline{\boldsymbol{G}(\boldsymbol{c})}\otimes\boldsymbol{G}(\boldsymbol{c})|_\Gamma)$ is a homogeneous polynomial in $\overline{c_i}\,c_j$, $i,j = 0,\ldots,L-1$, i.e., a linear combination of $\{\overline{c_0^{\alpha_0}c_1^{\alpha_1}\ldots c_{L-1}^{\alpha_{L-1}}}\,c_0^{\beta_0}c_1^{\beta_1}\ldots c_{L-1}^{\beta_{L-1}} : \alpha_i,\beta_i \in \mathbb{N}\cup\{0\}, \sum_{i=0}^{L-1}\alpha_i = L, \sum_{i=0}^{L-1}\beta_i = L\}$. We will show that for any pattern Γ of type (i) or (ii), $\det(\overline{\boldsymbol{G}(\boldsymbol{c})}\otimes\boldsymbol{G}(\boldsymbol{c})|_\Gamma)$ is not identically zero. Since there exist only finitely many such patterns, we then obtain a finite collection $\mathcal{F}$ of nonzero polynomials each representing $\det(\overline{\boldsymbol{G}(\boldsymbol{c})}\otimes\boldsymbol{G}(\boldsymbol{c})|_\Gamma)$ for a pattern Γ. Note that for any nonzero polynomial $p = p(\overline{c_0},\ldots,\overline{c_{L-1}},c_0,\ldots,c_{L-1})$, the set $Z(p) = \{(c_0,\ldots,c_{L-1}) \in \mathbb{C}^L : p = 0\}$ is a closed subset of $\mathbb{C}^L$ with empty interior and has zero Lebesgue measure. Therefore, by excluding the set $\cup_{p\in\mathcal{F}}Z(p)$ from $\mathbb{C}^L$ we obtain a desired set.

For patterns of type (i), we consider the partition $B = (\mathbb{Z}_L\times\{0\},\mathbb{Z}_L\times\{1\},\ldots,\mathbb{Z}_L\times\{L-1\})$ of $\mathbb{Z}_L\times\mathbb{Z}_L$. Then $\det(\mathbf{A}^{(0)}(B_0))\cdot\ldots\cdot\det(\mathbf{A}^{(L-1)}(B_{L-1})) = \overline{c_0^L c_1^L\ldots c_{L-1}^L}\,c_0^{L^2}\cdot\omega^{\sum_{k,\ell=0}^{L-1}kn_{k,\ell}}\cdot[\det(\overline{\boldsymbol{W}_L})]^L$. It is easily seen that the monomial $\overline{c_0^L c_1^L\ldots c_{L-1}^L}\,c_0^{L^2} = (\overline{c_0}c_0)^L(\overline{c_1}c_0)^L\ldots(\overline{c_{L-1}}c_0)^L$ is obtained only from this particular choice of partition. Therefore, this monomial appears with the nonzero coefficient $\omega^{\sum_{k,\ell=0}^{L-1}kn_{k,\ell}}\cdot[\det(\overline{\boldsymbol{W}_L})]^L$ (up to a $\pm$ sign) in $\det(\overline{\boldsymbol{G}(\boldsymbol{c})}\otimes\boldsymbol{G}(\boldsymbol{c})|_\Gamma)$, which shows that $\det(\overline{\boldsymbol{G}(\boldsymbol{c})}\otimes\boldsymbol{G}(\boldsymbol{c})|_\Gamma)$ is not identically zero.

For patterns of type (ii), we consider the partition $B' = (\{0\}\times\mathbb{Z}_L,\{1\}\times\mathbb{Z}_L,\ldots,\{L-1\}\times\mathbb{Z}_L)$ of $\mathbb{Z}_L\times\mathbb{Z}_L$. Similarly, we conclude that the monomial $\overline{c_0^{L^2}}\,c_0^L c_1^L\ldots c_{L-1}^L$ appears with the nonzero coefficient $\overline{\omega^{\sum_{k,\ell=0}^{L-1}k\ell_{k,n}}}\cdot(\det\boldsymbol{W}_L)^L$ in the determinant of $\overline{\boldsymbol{G}(\boldsymbol{c})}\otimes\boldsymbol{G}(\boldsymbol{c})|_\Gamma$, hence, $\det(\overline{\boldsymbol{G}(\boldsymbol{c})}\otimes\boldsymbol{G}(\boldsymbol{c})|_\Gamma)$ is not identically zero. This completes the proof.

5.3. Proof of Theorem 4

Proposition 5 ([10]): *Let $L \in \mathbb{N}$ be a prime and let $\boldsymbol{D}$, $\boldsymbol{u}_n$, $\boldsymbol{e}_n$ be as introduced in the statement of Theorem 4.*

(a) For any $j,p,q,n = 0,\ldots,L-1$, there exists $N(j,p,q,n) \in \mathbb{Z}_L$ such that $\boldsymbol{M}^{jp+q}\boldsymbol{T}^j\boldsymbol{D}^p\boldsymbol{u}_n = \omega^{N(j,p,q,n)}\boldsymbol{D}^p\boldsymbol{u}_{n+q}$. For any $j,n = 0,\ldots,L-1$, we have $\boldsymbol{M}^j\boldsymbol{e}_n = \omega^{jn}\boldsymbol{e}_n$.

(b) Let $S \subset \mathbb{Z}_L\times\mathbb{Z}_L$ with $|S| = L$. There exists $\boldsymbol{c} \in \mathbb{C}^L$ with $[\pi(\lambda)\boldsymbol{c}]_{\lambda\in S} \in \mathbb{C}^{L\times L}$ unitary if and only if exists $p \in \{0,1,\ldots,L-1,\infty\}$ such that $S + V_p = \mathbb{Z}_L\times\mathbb{Z}_L$. In this case, the vector $\boldsymbol{c} \in \mathbb{C}^L$ can be chosen from the set $\{\boldsymbol{D}^p\boldsymbol{u}_n\}_{n=0}^{L-1}$ if $p < \infty$, and $\{\boldsymbol{e}_n\}_{n=0}^{L-1}$ if $p = \infty$.

It should be noted that $\{\boldsymbol{D}^p\boldsymbol{u}_n\}_{n=0}^{L-1}$ forms an orthonormal basis for $\mathbb{C}^L$. Proposition 5(a) implies that this is a full set of eigenvectors for every operators in $\{\pi(\lambda) : \lambda \in V_p\}$. Moreover, for $\boldsymbol{c} = \boldsymbol{D}^p\boldsymbol{u}_n$ fixed we have that $\pi(\lambda)\boldsymbol{c} = \boldsymbol{D}^p\boldsymbol{u}_{n+q}$ up to a phase factor for every λ in $V_p + (0,q)$. Similarly, $\{\boldsymbol{e}_n\}_{n=0}^{L-1}$ is a full set of eigenvectors for every operators in $\{\pi(\lambda) : \lambda \in V_\infty\}$; moreover, for $\boldsymbol{c} = \boldsymbol{e}_n$ fixed we have that $\pi(\lambda)\boldsymbol{c} = \boldsymbol{e}_{n+q}$ up to a phase factor for every λ in $V_\infty + (q,0)$.

Proof (Theorem 4): If $p < \infty$, then fix any $\boldsymbol{c} = \boldsymbol{D}^p\boldsymbol{u}_n$ with $n \in \mathbb{Z}_L$. For $q,q' = 0,\ldots,L-1$, we have $\overline{\pi(\lambda_{q,q'})\boldsymbol{c}}\otimes\pi(\widetilde{\lambda}_{q,q'})\boldsymbol{c} = \overline{\boldsymbol{D}^p\boldsymbol{u}_{n+q}}\otimes\boldsymbol{D}^p\boldsymbol{u}_{n+q'}$ up to a phase factor. Since $\{\boldsymbol{D}^p\boldsymbol{u}_q\}_{q=0}^{L-1}$ is an orthonormal basis for $\mathbb{C}^L$, the vectors $\{\overline{\pi(\lambda_{q,q'})\boldsymbol{c}}\otimes\pi(\widetilde{\lambda}_{q,q'})\boldsymbol{c} : q,q' = 0,\ldots,L-1\}$ form an orthonormal basis for $\mathbb{C}^{L^2}$.

If $p = \infty$, then fix any $\boldsymbol{c} = \boldsymbol{e}_n$ with $n \in \mathbb{Z}_L$. For $q,q' = 0,\ldots,L-1$, we have $\overline{\pi(\lambda_{q,q'})\boldsymbol{c}}\otimes\pi(\widetilde{\lambda}_{q,q'})\boldsymbol{c} = \boldsymbol{e}_{n+q}\otimes\boldsymbol{e}_{n+q'}$ up to a phase factor. Since $\{\boldsymbol{e}_q\}_{q=0}^{L-1}$ is an orthonormal basis for $\mathbb{C}^L$, the vectors $\{\overline{\pi(\lambda_{q,q'})\boldsymbol{c}}\otimes\pi(\widetilde{\lambda}_{q,q'})\boldsymbol{c} : q,q' = 0,\ldots,L-1\}$ form an orthonormal basis for $\mathbb{C}^{L^2}$.

Moreover, the existence of a single vector $\boldsymbol{c} \in \mathbb{C}^L$ for which $\overline{\boldsymbol{G}(\boldsymbol{c})}\otimes\boldsymbol{G}(\boldsymbol{c})|_\Gamma$ is unitary implies that $\det(\overline{\boldsymbol{G}(\boldsymbol{c})}\otimes\boldsymbol{G}(\boldsymbol{c})|_\Gamma)$ is a nontrivial polynomial in the variable $c_0,\ldots,c_{L-1}$, therefore, the vectors $\boldsymbol{c} \in \mathbb{C}^L$ for which $\overline{\boldsymbol{G}(\boldsymbol{c})}\otimes\boldsymbol{G}(\boldsymbol{c})|_\Gamma$ is invertible constitute a dense open subset of $\mathbb{C}^L$ with full measure. ∎

6. REFERENCES

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