A FAMILY OF MATRICES FOR GENERATING HERMITE-GAUSSIAN-LIKE DFT EIGENVECTORS

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ABSTRACT

A generating matrix is a matrix such that, when multiplied by an eigenvector of a discrete transform, a new eigenvector is obtained. In this paper, we introduce a family of generating matrices of DFT eigenvectors. We demonstrate that, if a specific initial set of eigenvectors is chosen, using the referred family of matrices, a Hermite-Gaussian-like DFT eigenbasis is obtained. Such an eigenbasis is then employed to define a discrete fractional Fourier transform which numerically approximates the corresponding continuous transform.

Index Terms— Generating matrices, eigenstructure of discrete Fourier transform, Hermite-Gaussian-like eigenvectors, fractional Fourier transform

1. INTRODUCTION AND RELATED WORK

Techniques for constructing eigenvectors of the discrete Fourier transform (DFT) have been widely investigated in the last decades. In the field of signal processing, the main motivation for developing such techniques is related to the definition of discrete fractional Fourier transforms (DFrFT), which have been employed in several applications [1–11] and can be established from the spectral expansion of the DFT matrix operator $\mathbf{F}$. To be more specific, the vectors $\{\mathbf{e}_m\}$, $m = 0, 1, \ldots, N-1$, of an orthonormal DFT eigenbasis can be used as the columns of $\mathbf{E}$ in

$$\mathbf{F} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}^T; \tag{1}$$

the diagonal matrix $\mathbf{\Lambda}$ contains the eigenvalues of $\mathbf{F}$, given by the fourth complex roots of unity. This allows us to compute

$$\mathbf{F}^a = \mathbf{E}\mathbf{\Lambda}^a\mathbf{E}^T, \tag{2}$$

the DFrFT matrix operator with fractional order $a \in \mathbb{R}$.

Among the techniques for constructing DFT eigenvectors, those based on closed formulae have received special attention in the last years. Such methods may provide some computational advantages compared, for instance, to those based on matrices commuting with $\mathbf{F}$. In this context, the work published by F. N. Kong in 2008 has particular relevance [12]; in that paper, closed-form zeroth and first order Hermite-Gaussian-like (HGL) vectors were proposed [1]. Recently, in [14] and [15], Kong's approach was generalized and dilated versions of the referred vectors were constructed.

More specifically, in [14], the dilated zeroth and first order HGL vectors have components respectively given by

$$u_m(k) = N2^{L+m}\prod_{s=L-m+1}^{2L}\left[\cos\left(k\frac{2\pi}{N}\right) - \cos\left(s\frac{2\pi}{N}\right)\right] \tag{3}$$

$$= S(3L+m+k)S(3L+m-k) \tag{4}$$

and

$$v_m(k) = \sin\left(k\frac{2\pi}{N}\right)u_m(k), \tag{5}$$

where $0 \leq m \leq N-1$, $N = 4L+1$[2] and $k \in I_N$, $I_N := \{-M+1, -M+2, \ldots, -M+N\}$ and $M = \lfloor\frac{N+1}{2}\rfloor$; the sequence $\{S(k)\}_{k\geq 0}$ is defined by $S(0) := 1$ and $S(k) := \prod_{j=1}^{k} 2\sin(\pi j/N)$, for $k \geq 1$. The author obtained orthonormal HGL DFT eigenbases by linearly combining such dilated vectors and their DFT, and applying an orthogonalization algorithm. In [15], the authors employed a generating matrix method [16] to obtain sets of HGL DFT eigenvectors for $N \equiv 1 \pmod 4$. In [13], the first two authors of this paper considered the sets given in [14] and [15]; using some creative strategies, they removed restrictions of the cited approaches and explained how the respective sets of DFT eigenvectors can be used to define a DFrFT.

In this paper, we introduce an alternative procedure for constructing the HGL DFT eigenbases given in [14]. For each eigenspace of $\mathbf{F}$, we use an initial eigenvector and obtain the remaining ones employing a certain family of generating matrices. The procedure finishes after the orthogonalization of

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[1]In order to the DFrFT numerically approximate the continuous fractional Fourier transform, the eigenvectors applied as columns of $\mathbf{E}$ in (2) should be discrete analogs of continuous Hermite-Gaussian functions [13].

[2]Expressions for $u_m(k)$ and $v_m(k)$ for other values of N are omitted in this paper, but can be found in [14].

the resulting eigenvector sets. Due its recursion, the proposed method is of simple implementation and can be used in the definition of a DFrFT.

In the next section, we describe our method and demonstrate that it produces the mentioned HGL DFT eigenbases. In Section 3, we employ the generated bases to define DFrFT and provide an application example. The concluding remarks of this paper are presented in Section 4.

2. MATRICES FOR GENERATING HERMITE-GAUSSIAN-LIKE DFT EIGENVECTORS

In this paper, we consider a centered version of the discrete Fourier transform. The matrix $\mathbf{F}$ of such a DFT has in its $(k+M)$-th row and $(n+M)$-th column the entry $F(k,n) = e^{-i\frac{2\pi}{N}kn}$, $n \in I_N$.

It has been shown that, if $\mathbf{v}$ is an eigenvector of $\mathbf{F}$ with eigenvalue λ, the vector $\mathbf{v_A} = \mathbf{S}(\mathbf{A})\mathbf{v}$, where $\mathbf{S}(\mathbf{A}) = \alpha^{1/2}\mathbf{F}^{-1}\mathbf{A}\mathbf{F} + \mathbf{A}$ and $\mathbf{A}$ satisfies $\mathbf{F}^2\mathbf{A}\mathbf{F}^2 = \alpha\mathbf{A}$, is an eigenvector of $\mathbf{F}$ with eigenvalue $\alpha^{1/2}\lambda$ [16]; $\mathbf{S}(\mathbf{A})$ is identified as a generating matrix. We then define a diagonal matrix

$$\mathbf{G}_m = \mathrm{diag}\left(2\cos\left(\frac{2\pi n}{N}m\right)\right),\ n \in I_N,\ 0 \le m \le N-1,$$

satisfying $\mathbf{F}^2\mathbf{G}_m\mathbf{F}^2 = \mathbf{A}$, so that $\mathbf{v}$ and $\mathbf{v}_{\mathbf{G}_m} = \mathbf{S}(\mathbf{G}_m)\mathbf{v}$ are eigenvectors of $\mathbf{F}$ with the same eigenvalue λ.

In what follows, $\mathbf{w}_0$, $\mathbf{x}_0$, $\mathbf{y}_0$ and $\mathbf{z}_0$ denote the first eigenvectors of the eigenspaces related to eigenvalues 1, $-i$, -1 and i of $\mathbf{F}$, respectively, obtained in [14] by the linear combination of dilated zeroth and first order HGL vectors; $\#\{\lambda\}$ denotes the multiplicity of eigenvalue λ. From this point forward, we restrict our results to $N \equiv 1(\mathrm{mod}4)$. Similar results can be obtained for other values of N.

Definition 1. *Let* $\mathbf{g}_0 = \mathbf{w}_0$, $\mathbf{g}_1 = \mathbf{x}_0$, $\mathbf{g}_2 = \mathbf{y}_0$ *and* $\mathbf{g}_3 = \mathbf{z}_0$. *The set* $\{\mathbf{g}_m\}_{0\le m\le N-1}$ *is constituted of DFT eigenvectors obtained according to*

$$\begin{aligned}
\mathbf{g}_{4m} &= \mathbf{S}(\mathbf{G}_m)\mathbf{g}_0,\ \ m = 1,2,\dots,\#\{1\}-1,\\
\mathbf{g}_{4m+1} &= \mathbf{S}(\mathbf{G}_m)\mathbf{g}_1,\ \ m = 1,2,\dots,\#\{-i\}-1,\\
\mathbf{g}_{4m+2} &= \mathbf{S}(\mathbf{G}_m)\mathbf{g}_2,\ \ m = 1,2,\dots,\#\{-1\}-1,\\
\mathbf{g}_{4m+3} &= \mathbf{S}(\mathbf{G}_m)\mathbf{g}_3,\ \ m = 1,2,\dots,\#\{i\}-1.
\end{aligned}$$

In Definition 1, each subset $\{\mathbf{g}_{4m+l}\}_{0\le m\le\#\{(-i)^l\}-1}$ is composed by eigenvectors related to eigenvalue $(-i)^l$, $0 \le l \le 3$. The next theorem is the main result of this paper.

Theorem 1. *The set* $\{\mathbf{g}_m^{\perp}\}_{0\le m\le N-1}$, *obtained after the orthogonalization of each eigenvector subset constructed using Definition 1, is the HGL DFT eigenbasis* $\{\phi_m^{\perp}\}_{0\le m\le N-1}$ *described in [14].*

Theorem 1 can be proved by demonstrating that the eigenvectors of each subset constructed according to Definition 1 are linear combinations of the vectors of the corresponding eigenspace basis described in [14] (before the orthogonalization process). In order to develop this idea, we first rename vectors $\{\mathbf{g}_m\}_{0\le m\le N-1}$ as

$$\begin{aligned}
\{\mathbf{w}'_m\}_{0\le m\le(\#\{1\}-1)} &= \{\mathbf{g}_{4m}\}_{0\le m\le(\#\{1\}-1)},\\
\{\mathbf{x}'_m\}_{0\le m\le(\#\{-i\}-1)} &= \{\mathbf{g}_{4m+1}\}_{0\le m\le(\#\{-i\}-1)},\\
\{\mathbf{y}'_m\}_{0\le m\le(\#\{-1\}-1)} &= \{\mathbf{g}_{4m+2}\}_{0\le m\le(\#\{-1\}-1)},\\
\{\mathbf{z}'_m\}_{0\le m\le(\#\{i\}-1)} &= \{\mathbf{g}_{4m+3}\}_{0\le m\le(\#\{i\}-1)}.
\end{aligned}$$

We then introduce the following lemma, which can be easily proved by using Definition 1 and the fact that $\mathbf{S}(\mathbf{G}_m) = \mathbf{F}^{-1}\mathbf{G}_m\mathbf{F} + \mathbf{G}_m$.

Lemma 1. *The vectors of the set* $\{\mathbf{g}_m\}_{0\le m\le N-1}$ *can be expressed as*

$$\begin{aligned}
\mathbf{w}'_m &= \mathbf{G}_m\mathbf{w}_0 + \mathbf{F}\mathbf{G}_m\mathbf{w}_0,\\
\mathbf{x}'_m &= \mathbf{G}_m\mathbf{x}_0 + i\mathbf{F}\mathbf{G}_m\mathbf{x}_0,\\
\mathbf{y}'_m &= \mathbf{G}_m\mathbf{y}_0 - \mathbf{F}\mathbf{G}_m\mathbf{y}_0,\\
\mathbf{z}'_m &= \mathbf{G}_m\mathbf{z}_0 - i\mathbf{F}\mathbf{G}_m\mathbf{z}_0.
\end{aligned}$$

Lemma 2. *The vectors* $\mathbf{G}_m\mathbf{u}_n$ *and* $\mathbf{G}_m\mathbf{v}_n$, $n < m$, *can be expressed respectively as*

$$\mathbf{G}_m\mathbf{u}_n = \mathbf{u}_{m+n} + \alpha_{m+n-1}\mathbf{u}_{m+n-1} + \cdots + \alpha_n\mathbf{u}_n \qquad (6)$$

and

$$\mathbf{G}_m\mathbf{v}_n = \mathbf{v}_{m+n} + \alpha_{m+n-1}\mathbf{v}_{m+n-1} + \cdots + \alpha_n\mathbf{v}_n, \qquad (7)$$

where the $\{\alpha_k\}_{n\le k\le m+n-1}$ *are constants.*

Proof. We first assume that $m = n+1$ and use the fact that

$$u_m(k) = \left(2\cos\left(\frac{2\pi k}{N}\right) - 2\cos\left(\frac{2\pi(3L+m)}{N}\right)\right)u_{m-1}(k),$$

which has been demonstrated in [14]. The last equation can be written as

$$u_m(k) = (G_1(k) - c_m)u_{m-1}(k),$$

where $G_m(k) = 2\cos\left(\frac{2\pi k}{N}m\right)$ and $c_m = 2\cos\left(\frac{2\pi(3L+m)}{N}\right)$. Therefore

$$G_1(k)u_{m-1}(k) = u_m(k) + c_m u_{m-1}(k) \qquad (8)$$

and, since $v_m(k) = \sin\left(\frac{2\pi k}{N}\right)u_m(k)$, one has

$$G_1(k)v_{m-1}(k) = v_m(k) + c_m v_{m-1}(k). \qquad (9)$$

Similarly, for $m = n+2$, one obtains

$$\begin{aligned}
u_m(k) &= (G_1(k) - c_m)(G_1(k) - c_{m-1})u_{m-2}(k)\\
&= (G_2(k) - G_1(k)(c_m + c_{m-1}) + c_m c_{m-1} + 2)u_{m-2}(k).
\end{aligned}$$

Using (8) in the last equality, one has

$$\begin{aligned}
u_m(k) &= G_2(k)u_{m-2}(k) - (c_m + c_{m-1})\\
&\quad (u_{m-1}(k) + c_m u_{m-2}(k)) + (c_m c_{m-1} + 2)u_{m-2}(k),\\
G_2(k)u_{m-2}(k) &= u_m(k) + (c_m + c_{m-1})u_{m-1}(k) +\\
&\quad + (c_m c_{m-1} - 2)u_{m-2}(k).
\end{aligned}$$

Recursively applying the product of two cosines identity and (8) (resp. (9)), $\mathbf{G}_m\mathbf{u}_n$ (resp. $\mathbf{G}_m\mathbf{v}_n$), $n < m$, can be written in terms of the vectors in the set $\{\mathbf{u}_k\}_{n\leq k\leq m+n}$ (resp. $\{\mathbf{v}_k\}_{n\leq k\leq m+n}$), as shown in (6) (resp. (7)). ∎

Proposition 1. *Let* $d_n = N^{-\frac{1}{2}}S(2L+2n)$*. For* $0 \leq n \leq 2L$*, (i)* $d_n d_{-n} = 1$ *and (ii)* $d_0 = 1$*.*

Proof. (i) For $0 \leq n \leq 2L$, one has

$$\begin{aligned} d_n d_{-n} &= N^{-\frac{1}{2}}S(2L+2n)N^{-\frac{1}{2}}S(2L-2n) \\ &= N^{-1}S(2L+2n)S(2L-2n). \end{aligned}$$

Using the substitution $k = 2L + 2n$ and the fact that $S(k)S(N-k-1) = N$ [14], one obtains

$$d_n d_{-n} = N^{-1}S(k)S(N-k-1) = N^{-1}N = 1.$$

(ii) For $0 \leq n \leq 2L$, using $S(k)S(N-k-1) = N$ with $k = 2L$, one concludes that $S(2L) = N^{\frac{1}{2}}$. Therefore $d_0 = N^{-\frac{1}{2}}S(2L) = N^{-\frac{1}{2}}N^{\frac{1}{2}} = 1$. ∎

Proposition 2. *Let* $\delta_n = N^{-\frac{1}{2}}S(2L+2n+1)$*. For* $0 \leq n \leq 2L$*,* $\delta_n\delta_{-n-1} = 1$*.*

Proof. One may write

$$\begin{aligned} \delta_n\delta_{-n-1} &= N^{-\frac{1}{2}}S(2L+2n+1)N^{-\frac{1}{2}}S(2L-2n-2+1) \\ &= N^{-1}S(2L+2n+1)S(2L-2n-1). \end{aligned}$$

Using the substitution $k = 2L + 2n + 1$, one has

$$\delta_n\delta_{-n-1} = N^{-1}S(k)S(N-k-1) = N^{-1}N = 1.$$ ∎

Expressions for $\mathbf{w}_0$, $\mathbf{x}_0$, $\mathbf{y}_0$ and $\mathbf{z}_0$ can be rewritten using d_n and δ_n as

$$\begin{aligned} \mathbf{w}_0 = \mathbf{u}_0 + d_0\mathbf{u}_0 = 2\mathbf{u}_0, \quad \mathbf{x}_0 = \mathbf{v}_0 + \delta_0\mathbf{v}_{-1}, \\ \mathbf{y}_0 = -\mathbf{u}_1 + d_1\mathbf{u}_{-1}, \quad \mathbf{z}_0 = -\mathbf{v}_0 + \delta_0\mathbf{v}_{-1}. \end{aligned}$$

Using Lemma 1, $\mathbf{y}'_m$ can be rewritten as

$$\begin{aligned} \mathbf{y}'_m &= \mathbf{G}_m\mathbf{y}_0 - \mathbf{F}\mathbf{G}_m\mathbf{y}_0 \\ &= \mathbf{G}_m(-\mathbf{u}_1 + d_1\mathbf{u}_{-1}) - \mathbf{F}\mathbf{G}_m(-\mathbf{u}_1 + d_1\mathbf{u}_{-1}). \end{aligned}$$

Using (6) in the last equality, one obtains

$$\begin{aligned} \mathbf{y}'_m = &-(\mathbf{u}_{m+1} + \alpha_m\mathbf{u}_m + \cdots + \alpha_1\mathbf{u}_1) + \\ &+ d_1(\mathbf{u}_{m-1} + \beta_{m-2}\mathbf{u}_{m-2} + \cdots + \beta_{-1}\mathbf{u}_{-1}) + \\ &- \mathbf{F}\left(-(\mathbf{u}_{m+1} + \alpha_m\mathbf{u}_m + \cdots + \alpha_1\mathbf{u}_1) + \right. \\ &\left. + d_1(\mathbf{u}_{m-1} + \beta_{m-2}\mathbf{u}_{m-2} + \cdots + \beta_{-1}\mathbf{u}_{-1})\right), \\ \mathbf{y}'_m = &-\mathbf{u}_{m+1} - \alpha_m\mathbf{u}_m + (d_1 - \alpha_{m-1})\mathbf{u}_{m-1} + \cdots + \\ &+ (d_1\beta_k - \alpha_k)\mathbf{u}_k + \cdots + d_1\beta_0\mathbf{u}_0 + d_1\beta_{-1}\mathbf{u}_{-1} + \\ &- \mathbf{F}(-\mathbf{u}_{m+1} - \alpha_m\mathbf{u}_m + (d_1 - \alpha_{m-1})\mathbf{u}_{m-1} + \cdots + \\ &+ (d_1\beta_k - \alpha_k)\mathbf{u}_k + \cdots + d_1\beta_0\mathbf{u}_0 + d_1\beta_{-1}\mathbf{u}_{-1}). \end{aligned}$$

Carrying out the multiplication by $\mathbf{F}$ in the second part of the above equation, one has

$$\begin{aligned} \mathbf{y}'_m = &-\mathbf{u}_{m+1} - \alpha_m\mathbf{u}_m + (d_1 - \alpha_m)\mathbf{u}_{m-1} + \cdots + \\ &+ (d_1\beta_k - \alpha_k)\mathbf{u}_k + \cdots + d_1\beta_0\mathbf{u}_0 + d_1\beta_{-1}\mathbf{u}_{-1} + \\ &+ d_{m+1}\mathbf{u}_{-m-1} - \alpha_m d_m\mathbf{u}_{-m} + \cdots + \\ &- (d_1\beta_k - \alpha_k)d_k\mathbf{u}_{-k} + \cdots + d_1\beta_0 d_0\mathbf{u}_0 + d_1\beta_{-1}d_{-1}\mathbf{u}_1. \end{aligned}$$

Using Proposition 1(i) and 1(ii), the terms related to $\mathbf{u}_k$ and $\mathbf{u}_{-k}$, $0 \leq k \leq m$, can be grouped as

$$\begin{aligned} \mathbf{y}'_m = &(-\mathbf{u}_{m+1} + d_m\mathbf{u}_{-m-1}) - \alpha_m(-\mathbf{u}_m + d_m\mathbf{u}_{-m}) + \\ &\cdots + (d_1\beta_k - \alpha_k)(-\mathbf{u}_k + d_k\mathbf{u}_{-k}) + \\ &\cdots + (d_1\beta_1 - \alpha_1)(\mathbf{u}_1 - d_1\mathbf{u}_{-1}) + (d1\beta_0 - d1\beta_0)\mathbf{u}_0 + \\ &+ (d_1\beta_{-1}\mathbf{u}_{-1} - \beta_{-1}\mathbf{u}_1), \\ \mathbf{y}'_m = &(-\mathbf{u}_{m+1} + d_m\mathbf{u}_{-m-1}) - \alpha_m(-\mathbf{u}_m + d_m\mathbf{u}_{-m}) + \\ &\cdots + (d_1\beta_k - \alpha_k)(-\mathbf{u}_k + d_k\mathbf{u}_{-k}) + \\ &\cdots + (\alpha_1 - \beta_{-1} - d_1\beta_1)(-\mathbf{u}_1 + d_1\mathbf{u}_{-1}), \\ \mathbf{y}'_m = &\mathbf{y}_m - \alpha_m\mathbf{y}_{m-1} + \cdots + (d_1\beta_{k+1} - \alpha_{k+1})\mathbf{y}_k + \\ &\cdots + (\alpha_1 - \beta_{-1} - d_1\beta_1)\mathbf{y}_0. \end{aligned}$$

In the last equation, the vector $\mathbf{y}'_m$ is expressed as a linear combination of vectors $\{\mathbf{y}_k\}_{0\leq k\leq m}$. Using the intermmediate results we have derived and performing steps analogous to those we have presented, one obtains an equivalent result for vectors $\mathbf{w}'_m$, $\mathbf{x}'_m$ and $\mathbf{z}'_m$. This concludes the proof of Theorem 1.

In Fig. 1, we illustrate the convergence of HGL eigenvectors $\{\mathbf{g}_m^{\perp}\}_{0\leq m\leq N-1}$ to continuous Hermite-Gaussian functions $\{\psi_m\}_{0\leq m\leq N-1}$ as N grows, by exhibiting the root-mean-square error between $\mathbf{g}_{24}^{\perp}$ and samples of ψ_{24}, for $N = 20R+5$, $R = 1, 2, ..., 20$. A similar curve is obtained if other eigenvectors are considered.

3. DISCRETE FRACTIONAL FOURIER TRANSFORM BASED ON HGL EGENVECTORS

The HGL eigenbasis $\{\mathbf{g}_m^{\perp}\}_{0\leq m\leq N-1}$ constructed using the family $\mathbf{S}(\mathbf{G}_m)$ of generating matrices can be employed in place of vectors $\{\mathbf{e}_m\}_{0\leq m\leq N-1}$ in (1) to spectrally expand the DFT matrix $\mathbf{F}$. This allows us to compute the DFrFT matrix operator $\mathbf{F}^a$ as in (2). In this section, we consider an illustrative application scenario of such a DFrFT: filtering in fractional Fourier domain [2].

We have created a signal by adding to the Gaussian pulse

$$x(t) = e^{-\frac{(t-30)^2}{20}} \tag{10}$$

the chirp signal

$$c(t) = 0.2\cos\left(\frac{t^2}{10} - 2t\right). \tag{11}$$

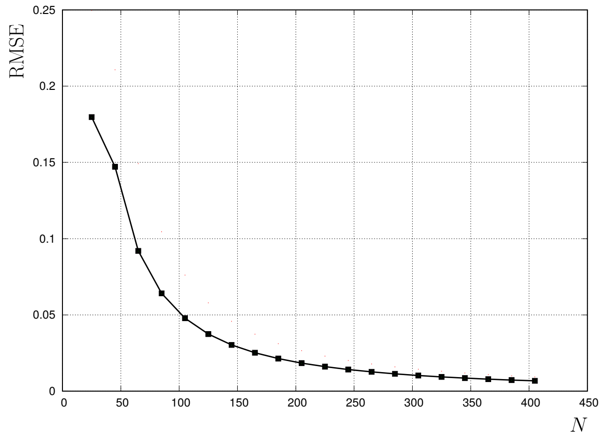

Fig. 1. Root-mean-square error between the HGL eigenvector $\mathbf{g}_{24}^{\perp}$ and samples of the continuous Hermite-Gaussian ψ_{24}, for $N = 20R + 5, R = 1, 2, ..., 20$.

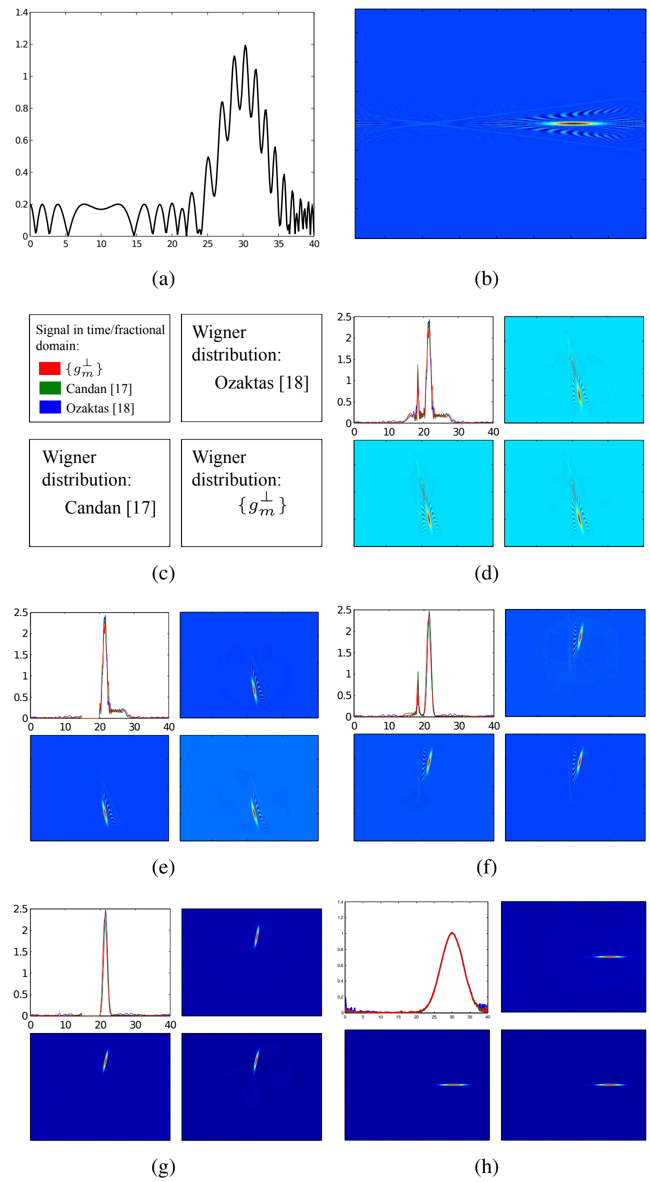

Fig. 2. (a) Gaussian signal plus chirp signal and (b) its Wigner distribution; (c) labels for subfigures (d)-(h); (d)-(g) signal in different DFrFT domains and corresponding Wigner distributions before and after a stop-band filtering; (h) recovered Gaussian signal in time-domain and its Wigner distribution.

The resulting signal in time-domain is shown in Fig. 2(a). In Fig. 2(b), where its Wigner distribution can be viewed, the Gaussian component is related to the stronger colored element with the shape of a vertically compressed ellipse. We want to remove the chirp component by computing succesive DFrFT of the signal; this takes into account the fact that such a computation produces rotations in the Wigner distribution [1, 4]. In order to apply this strategy, we have discretized the signal by collecting $N = 315$ samples in the interval $0 \leq t \leq 40$.

All steps described from this point forward were peformed by considering the DFrFT defined using (i) the basis $\{\mathbf{g}_m^{\perp}\}_{0 \leq m \leq N-1}$ and (ii) the eigenbasis obtained from a matrix commuting with **F** [17]; (iii) the method proposed in [18] was also employed. We first apply a DFrFT with fractional order $a = 3.1$. The result (signal in fractional-domain and Wigner distribution) is shown in Fig. 2(d), where one can see that, by means of a stop-band filter, part of the undesired components can be removed [13]. This produces the signal shown in Fig. 2(e). We then apply a DFrFT with fractional order $a = -2.2$ and obtain the signal shown in Fig. 2(f). Another stop-band filtering is performed, producing the signal presented in Fig. 2(g). Finally, by computing another DFrFT, the signal is returned to the original domain. The final result can be seen in Fig. 2(h), where we show the recovered Gaussian signal and its Wigner distribution.

Except for slight fluctuations whose levels vary from one method to another (see Fig. 2(h)), the final result is basically the same for (i), (ii) and (iii). This can be better evaluated by computing the signal-to-noise ratio (SNR) between recovered and original Gaussian signals; the SNR for (i) proposed, (ii) commuting matrix [17] and (iii) digital computation [18] methods are 199 dB, 203 dB and 195 dB, respectively. This suggests that, in the application focused in this section, the considered methods provide equivalent results and, therefore, the DFrFT based on the eigenbasis $\{\mathbf{g}_m^{\perp}\}_{0 \leq m \leq N-1}$ may be employed in practical scenarios.

4. CONCLUDING REMARKS

In this paper, a family of generating matrices for constructing a DFT eigenbasis formed by HGL eigenvectors was provided. Such an eigenbasis was employed in the definition of a DFrFT whose effectiveness in performing signal filtering was confirmed. We are currently investigating the possibility of developing fast algorithms for computing the proposed DFrFT and studying details related to its potential application scenarios.

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