UNCERTAINTY PRINCIPLE FOR RATIONAL FUNCTIONS IN HARDY SPACES

*Dan Xiong, Li Chai**

Engineering Research Center of Metallurgical Automation and Measurement Technology
Wuhan University of Science and Technology
Wuhan, China
{xiongdan, chaili}@wust.edu.cn

Jingxin Zhang

School of Software & Electrical Eng.
Swinburne University of Technology
Melbourne, Australia.
jingxinzhang@swin.edu.au

ABSTRACT

Uncertainty principles in finite dimensional vector space have been studied extensively, however they cannot be applied to sparse representation of rational functions. This paper considers the sparse representation for a rational function under a pair of orthonormal rational function bases. We prove the uncertainty principle concerning pairs of compressible representation of rational functions in the infinite dimensional function space. The uniqueness of compressible representation using such pairs is provided as a direct consequence of uncertainty principle.

Index Terms— uncertainty principle, rational functions, orthonormal rational function basis, infinite dimensional function space, sparse representation

1. INTRODUCTION

Uncertainty principles generally express the impossibility for a function (or a vector in the discrete case) to be simultaneously sharply concentrated in two different representations, provided the latter are incoherent enough [1]. Its first statement in quantum mechanics was the so-called Robertson-Schrödinger inequality, which established a lower bound for the product of variances of any two self-adjoint operators on a generic Hilbert space. For various purposes, many such settings using different measures of "concentration" have been proposed in the literature. For example, the uncertainty principles in the setting of discrete sequences, continuous and discrete-time functions for L_2 and L_1 measures were presented in [2]. Uncertainty principles on the unit sphere have been established in [3]. A very general uncertainty principle for operators on Banach spaces was given in [4] and, in more abstract settings, on compact Riemannian manifolds [5]. The uncertainty principle from an information theory point of view was discussed in [6] and bounds for the entropic uncertainty principle are derived in [7], using entropies as concentration measures. [8] provided a fundamental tradeoff between a signal's localization on a graph and in its spectral domain.

The uncertainty principle has encountered many parallel evolutions and generalizations in different domains. Heisenberg's celebrated uncertainty principle, as a cornerstone of developments in signal processing, can be generalized in inequality $\Delta_t^2\Delta_\omega^2 \geq \frac{1}{4}$, in which Δ_t^2 and Δ_ω^2 measure the "time spread" and "frequency spread", which provides a tradeoff between signal localization in time and frequency, i.e. a signal cannot be concentrated in both time and frequency. The analogous results for discrete-time signals were given in [9, 10] as well. A complete account of classical uncertainty relations focused on time-frequency uncertainty can be found in [11].

In the field of signal and systems, the uncertainty principle in the discrete setting has gained increasing attention recently due to its connection with sparse representation and compressive sensing [1]. In [12], Donoho and Huo discussed the signal represented by a highly sparse superposition of atoms from time-frequency dictionary $[\Phi\ \Psi]$, where Φ and Ψ are the spike basis and the Fourier basis, respectively. If a signal S is expressed in each basis respectively

$$S = \sum_{i=1}^{n} \alpha_i \phi_i = \sum_{i=1}^{n} \beta_i \psi_i,$$

then $\|\alpha\|_0 + \|\beta\|_0 \geq 1 + M^{-1}$, where $\alpha := [\alpha_1, \alpha_2, \cdots, \alpha_n]^T$, $\beta := [\beta_1, \beta_2, \cdots, \beta_n]^T$ and $M = \sup\limits_{1\leq i,j\leq n} |\langle \phi_i, \psi_j\rangle|$ is the mutual coherence of the two bases. The mutual coherence measures how different two representations are. The more different the representations, the more constraining the bounds.

The direct consequence of uncertainty principle is the uniqueness of sparse representation of the signal S in pair of such two bases $[\Phi\ \Psi]$ if the representation coefficient γ satisfies $\|\gamma\|_0 < \frac{1}{2}(1 + \frac{1}{M})$. And the solution can be obtained by minimizing the 1-norm of the coefficients among all decompositions. Elad and Bruckstein extended the result in [12] to arbitrary orthonormal bases and presented an "improved" uncertainty principle with better bounds $\|\gamma\|_0 < \frac{1}{M}$ yield-

**Corresponding author. Thanks to the National Science Foundation of China (grant 61625305 and 61471275) and Hubei Provincial Education Department (grant T201302) for funding.

ing uniqueness of the sparse representation in [13]. Further studied in, e.g. [14, 15], the uncertainty principles led to the uniqueness of the sparse solution and laid the foundation for sparse signal recovery from partial measurements.

The key idea of uncertainty principle in finite dimensional settings is that if two orthonormal bases are mutually incoherent, no nonzero signal can have a sparse representation in both bases simultaneously. As analyzed in [13] and [16], the basic uncertainty principle concerning pairs of representations of finite dimensional vectors in different orthonormal bases has a direct impact on the uniqueness property of the sparse representation of such vectors using pairs of orthonormal bases as overcomplete dictionaries. Hence the uncertainty principle and uniqueness are fundamentally instrumental to the sparse representation of signals under the pairs of orthonormal bases.

It should be noted that uncertainty principle in finite dimensional vector space cannot work for the sparse representation and reconstruction of rational functions which have infinite impulse response. In this paper, we establish analogous uncertainty principles for rational functions represented by orthonormal rational function (**ORF**) basis, which is a new version for infinite dimensional function space. Orthonormal rational functions have been widely studied over the last twenty years, since the rationality improves efficiency of the representation of linear systems, and orthonormality leads to a great simplification of the analysis and synthesis involved in using the basis functions, see for instance [17–24]. On the basis of the uncertainty principle for rational function, we investigate the sparse representation of a rational function under a pair of ORF bases.

The contributions of this paper are: the proposal of a new definition of sparsity for infinite sequence, and establishment of the uncertainty principle and uniqueness of sparse representation for rational transfer functions, using the uniform bound of maximal absolute inner product of the pair of the ORFs as an index, which extends the results of [13] to infinite dimensional function space and compressible representation.

The rest of this paper is organized as follows. In Section II, the uncertainty principle for sparse representation of rational functions is given. The uniqueness of the representation of rational functions using two ORF bases is presented in Section III. Section IV concludes the paper and discusses the future work. Given the space limitations, we only give the proof outlines of the theorems, see [25] for the proofs in the full version.

2. UNCERTAINTY PRINCIPLE FOR SPARSE REPRESENTATION OF RATIONAL FUNCTIONS

Rational functions, which are widely used in signal processing and control to model both signals and dynamic systems, can be put in a Hilbert space framework. The Hardy space H_2 is a Hilbert space with the inner product between two rational functions $F(z)$ and $G(z)$ defined as

$$\langle F(z), G(z)\rangle = \frac{1}{2\pi i}\oint_{\mathbb{T}} F(z)\overline{G(z)}\frac{dz}{z} = \frac{1}{2\pi}\int_0^{2\pi} F(e^{i\omega})\overline{G(e^{i\omega})}d\omega, \quad (1)$$

where $\mathbb{T} = \{z \mid |z| = 1\}$.

Given a rational function $H(z) \in H_2$, it has a unique representation in every ORF basis of this space. If $\{\phi_k(z)\}_{k=1}^{\infty}$ is an ORF basis, then we have $H(z) = \sum_{k=1}^{\infty} \alpha_k \phi_k(z)$, where $\alpha_k = \langle H(z), \phi_k(z)\rangle$.

Suppose we have two different ORF bases $\{\phi_k(z)\}_{k=1}^{\infty}$, $\{\psi_l(z)\}_{l=1}^{\infty}$. Then every rational function $H(z)$ has a unique representation under the two bases, respectively, denoted as

$$H(z) = \sum_{k=1}^{\infty} \alpha_k \phi_k(z) = \sum_{l=1}^{\infty} \beta_l \psi_l(z). \quad (2)$$

Obviously, if $H(z)$ has a stable pole, then the representation of $H(z)$ using impulse response is infinite, which requires a large number of sampling data to guarantee the approximation performance. While if the pole of the selected rational bases is exactly the same as that of $H(z)$, then the representation will be much shorter. A natural idea is to get a much sparser representation of $H(z)$ in a joint, overcomplete set of ORF bases, say

$$\{\Phi(z), \Psi(z)\} = \{\phi_1(z), \phi_2(z), \cdots, \psi_1(z), \psi_2(z), \cdots\}.$$

Notice that rational functions have infinite impulse coefficients, the sparsity defined in [13] is no longer applicable. We first present a new definition of sparsity, called ε-sparsity and then establish the uncertainty principle for rational functions that leads to the bound yielding uniqueness for the sparse representation of rational function in pairs of ORF bases.

Definition 1. *For a fixed threshold $\varepsilon > 0$ and an infinite sequence $\alpha = [\alpha_1, \alpha_2, \cdots]^T$ in l_1, i.e. $\|\alpha\|_1 = \sum_{k=1}^{\infty} |\alpha_k| < \infty$. Let*

$$N_\varepsilon(\alpha) = \min\{K : \sum_{k=K}^{\infty} |\alpha_k| \le \varepsilon\}.$$

The ε-support of α is defined as

$$\Gamma_\varepsilon(\alpha) = \{k : |\alpha_k| \ne 0, 1 \le k < N_\varepsilon(\alpha)\},$$

and the cardinality of $\Gamma_\varepsilon(\alpha)$ as the ε-0 norm of α, denoted by $\|\alpha\|_{0(\varepsilon)}$.

Remark 1. *α in l_1 guarantees the existence of $N_\varepsilon(\alpha)$. If $\varepsilon = 0$, then ε-support $\Gamma_\varepsilon(\alpha)$ for α is the support of α in the general sense, i.e. $\{k : |\alpha_k| \ne 0\}$.*

Definition 2. *For a given positive integer s, the coefficient α is (ε, s)-sparse in the sense of ε-0 norm if $\|\alpha\|_{0(\varepsilon)} \le s$. For brevity, we call the coefficient α ε-sparse if the value of s is not concerned.*

Definition 3. *A rational function is (ε, s)-sparse if the representation coefficient under an ORF basis is (ε, s)-sparse.*

If $\varepsilon = 0$, then $\|\alpha\|_{0(\varepsilon)} = \|\alpha\|_0$, which is the number of nonzeros in α, and the (ε, s)-sparsity becomes the s-sparsity in compressed sensing. However it should be noted that for a rational function with poles away from zero, its impulse response is usually infinite and cannot be $(0, s)$-sparse corresponding to $\varepsilon = 0$, which shows that the definition of the sparsity in traditional compressed sensing is not applicable to the sparsity of the rational function. Hence the uncertainty principle discussed in the sparse representation in dictionaries cannot apply for the sparse representation of rational functions. In the following, based on the (ε, s)-sparsity, we present the uncertainty principle for the representation of rational functions under two ORF bases.

Theorem 1. *(Uncertainty Principle) Let $H(z) \in H_2$ be a rational function that can be represented in (2). For a fixed threshold ε, $\|\alpha\|_{0(\varepsilon)}$ and $\|\beta\|_{0(\varepsilon)}$ are the ε-0 norm of α and β, respectively, then for all such pairs of representation we have*

$$\left(\sqrt{\|\alpha\|_{0(\varepsilon)}} + \varepsilon\right)^2 + \left(\sqrt{\|\beta\|_{0(\varepsilon)}} + \varepsilon\right)^2 \geq \frac{2}{\mu},$$

where

$$\mu = \sup_{k,l} |\langle \phi_k(z), \psi_l(z) \rangle|$$

and $\langle \phi_k(z), \psi_l(z) \rangle$ is the inner product of $\phi_k(z)$ and $\psi_l(z)$ defined in (1). Following the terminology of compressed sensing, we call μ the mutual coherence of two ORF bases $\{\phi_k(z)\}$ and $\{\psi_l(z)\}$.

Proof outline of Theorem 1. Notice that the transfer function considered is in H_2 space satisfying

$$H(z) = \sum_{k=1}^{\infty} \alpha_k \phi_k(z) = \sum_{l=1}^{\infty} \beta_l \psi_l(z).$$

For $\{\alpha_k\}$ and $\{\beta_l\}$, denote $\Gamma_\varepsilon(\alpha)$ and $\Gamma_\varepsilon(\beta)$ as the ε-support of α and β, respectively. Then we have $\sum_{k \notin \Gamma_\varepsilon(\alpha)} |\alpha_k| \leq \varepsilon$ and $\sum_{l \notin \Gamma_\varepsilon(\beta)} |\beta_l| \leq \varepsilon$.

Without loss of generality, we assume $\langle H(z), H(z) \rangle = 1$. Note that

$$\begin{aligned} 1 &= |\langle H(z), H(z) \rangle| \\ &= |\langle \sum_{k=1}^{\infty} \alpha_k \phi_k(z), \sum_{l=1}^{\infty} \beta_l \psi_l(z) \rangle| \\ &= |\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \alpha_k \langle \phi_k(z), \psi_l(z) \rangle \bar{\beta}_l| \\ &\leq \mu \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |\alpha_k||\beta_l| \\ &\leq \mu (\sum_{k \in \Gamma_\varepsilon(\alpha)} |\alpha_k| + \varepsilon)(\sum_{l \in \Gamma_\varepsilon(\beta)} |\beta_l| + \varepsilon). \end{aligned}$$

Similarly, we have

$$\begin{aligned} 1 &= \langle H(z), H(z) \rangle \\ &= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \alpha_k \langle \phi_k(z), \phi_l(z) \rangle \bar{\alpha}_l \\ &= \sum_{k=1}^{\infty} |\alpha_k|^2 = \sum_{k \in \Gamma_\varepsilon(\alpha)} |\alpha_k|^2 + \sum_{k \notin \Gamma_\varepsilon(\alpha)} |\alpha_k|^2 \\ &\leq \sum_{k \in \Gamma_\varepsilon(\alpha)} |\alpha_k|^2 + \sum_{k \notin \Gamma_\varepsilon(\alpha)} |\alpha_k| \varepsilon \leq \sum_{k \in \Gamma_\varepsilon(\alpha)} |\alpha_k|^2 + \varepsilon^2 \end{aligned}$$

and

$$1 = \sum_{l \in \Gamma_\varepsilon(\beta)} |\beta_l|^2 + \sum_{l \notin \Gamma_\varepsilon(\beta)} |\beta_l|^2 \leq \sum_{l \in \Gamma_\varepsilon(\beta)} |\beta_l|^2 + \varepsilon^2.$$

The bound of the above expression can be solved by the optimization problem

$$\max_{\alpha_k, \beta_l} \quad (\sum_{k \in \Gamma_\varepsilon(\alpha)} |\alpha_k| + \varepsilon)(\sum_{l \in \Gamma_\varepsilon(\beta)} |\beta_l| + \varepsilon) \tag{3}$$

$$\begin{aligned} \text{s.t.} \quad & |\alpha_k| > 0, |\beta_l| > 0, \\ & \sum_{k \in \Gamma_\varepsilon(\alpha)} |\alpha_k|^2 \geq 1 - \varepsilon^2, \sum_{l \in \Gamma_\varepsilon(\beta)} |\beta_l|^2 \geq 1 - \varepsilon^2. \end{aligned}$$

This can be separated into two optimization problems:

$$\max_{\alpha_k} \sum_{k \in \Gamma_\varepsilon(\alpha)} |\alpha_k| \text{ s.t. } |\alpha_k| > 0, \sum_{k \in \Gamma_\varepsilon(\alpha)} |\alpha_k|^2 \geq 1 - \varepsilon^2 \tag{4}$$

$$\max_{\beta_l} \sum_{l \in \Gamma_\varepsilon(\beta)} |\beta_l| \quad \text{s.t. } |\beta_l| > 0, \sum_{l \in \Gamma_\varepsilon(\beta)} |\beta_l|^2 \geq 1 - \varepsilon^2. \tag{5}$$

The optimization (4) can be solved by

$$\max_{\alpha_k} \sum_{k \in \Gamma_\varepsilon(\alpha)} |\alpha_k| \quad \text{s.t. } |\alpha_k| > 0, \sum_{k \in \Gamma_\varepsilon(\alpha)} |\alpha_k|^2 = C, \tag{6}$$

where $C \in [1 - \varepsilon^2, 1]$ is a constant.

By using Lagrangian multiplier method, we have Lagrangian function

$$F(|\alpha_k|, \lambda) = \sum_{k \in \Gamma_\varepsilon(\alpha)} |\alpha_k| + \lambda (\sum_{k \in \Gamma_\varepsilon(\alpha)} |\alpha_k|^2 - C).$$

Let the partial differentiation

$$\frac{\partial F(|\alpha_k|, \lambda)}{\partial |\alpha_k|} = 1 + 2\lambda |\alpha_k| = 0.$$

Then we have all the $|\alpha_k|$ are equal when $k \in \Gamma_\varepsilon(\alpha)$.

Denote $\|\alpha\|_{0(\varepsilon)} = A$. Then $|\alpha_k| = \sqrt{\frac{C}{A}}$. Hence the maxima of (6) is $A\sqrt{\frac{C}{A}} = \sqrt{AC}$. Since $C \in [1-\varepsilon^2, 1]$, then the maxima of (4) is $\sqrt{A} = \sqrt{\|\alpha\|_{0(\varepsilon)}}$.

Similarly, the maxima of (5) is $\sqrt{\|\beta\|_{0(\varepsilon)}}$. Therefore, the maxima of (3) is $(\sqrt{\|\alpha\|_{0(\varepsilon)}} + \varepsilon) \cdot (\sqrt{\|\beta\|_{0(\varepsilon)}} + \varepsilon)$, and

$$\begin{aligned} 1 &\leq \mu \left(\sum_{k \in \Gamma_\varepsilon(\alpha)} |\alpha_k| + \varepsilon \right) \left(\sum_{l \in \Gamma_\varepsilon(\beta)} |\beta_l| + \varepsilon \right) \\ &\leq \mu \left(\sqrt{\|\alpha\|_{0(\varepsilon)}} + \varepsilon \right) \cdot \left(\sqrt{\|\beta\|_{0(\varepsilon)}} + \varepsilon \right). \end{aligned}$$

Using the inequality between the geometric and arithmetic means, we have

$$1 \leq \mu \frac{\left(\sqrt{\|\alpha\|_{0(\varepsilon)}} + \varepsilon\right)^2 + \left(\sqrt{\|\beta\|_{0(\varepsilon)}} + \varepsilon\right)^2}{2}.$$

That is

$$\left(\sqrt{\|\alpha\|_{0(\varepsilon)}} + \varepsilon \right)^2 + \left(\sqrt{\|\beta\|_{0(\varepsilon)}} + \varepsilon \right)^2 \geq \frac{2}{\mu}.$$

□

3. UNIQUENESS FOR SPARSE REPRESENTATION OF RATIONAL FUNCTIONS USING TWO ORF BASES

Uncertainty principle reveals that in the perspective of ε-0 norm, a rational function $H(z)$ having a sparse representation in the joint set of two ORF bases will have highly nonsparse representation in either of these bases alone. And the uncertainty principle directly determines the bound which guarantees the uniqueness of the sparse representation. Before we present the uniqueness of the sparse representation, we give the definition for the sparse representation in pairs of ORF bases.

Definition 4. *For a rational function $H(z)$ represented under a pair of orthonormal rational function bases $\{\phi_k(z)\}$ and $\{\psi_l(z)\}$ as*

$$H(z) = \sum_{k=1}^{\infty} \theta_k^\phi \phi_k(z) + \sum_{l=1}^{\infty} \theta_l^\psi \psi_l(z), \tag{7}$$

denote $\theta_1 = [\theta_1^\phi, \theta_2^\phi, \cdots]^T$ and $\theta_2 = [\theta_1^\psi, \theta_2^\psi, \cdots]^T$, respectively. $H(z)$ is called (ε, s)-sparse if $\|\theta_1\|_{0(\varepsilon)} + \|\theta_2\|_{0(\varepsilon)} \leq s$.

Theorem 2. *(Uniqueness) For $\varepsilon > 0$, assume $H(z)$ is (ε, s)-sparse under a pair of ORF bases $\{\phi_k(z)\}$ and $\{\psi_l(z)\}$. Then the representation (7) is unique if*

$$\left(\sqrt{\|\theta_1\|_{0(\varepsilon)}} + \varepsilon \right)^2 + \left(\sqrt{\|\theta_2\|_{0(\varepsilon)}} + \varepsilon \right)^2 < \frac{1}{\mu},$$

where θ_1 and θ_2 are as defined in Definition 4.

Proof outline of Theorem 2. Suppose there are two different sparse representations of transfer function $H(z)$, that is

$$\begin{aligned} H(z) &= \sum_{k=1}^{\infty} \theta_k^\phi \phi_k(z) + \sum_{l=1}^{\infty} \theta_k^\psi \psi_l(z) \\ &= \sum_{k=1}^{\infty} \xi_k^\phi \phi_k(z) + \sum_{l=1}^{\infty} \xi_l^\psi \psi_l(z) \end{aligned}$$

and

$$\left(\sqrt{\|\theta_1\|_{0(\varepsilon)}} + \varepsilon \right)^2 + \left(\sqrt{\|\theta_2\|_{0(\varepsilon)}} + \varepsilon \right)^2 < \frac{1}{\mu},$$

$$\left(\sqrt{\|\xi_1\|_{0(\varepsilon)}} + \varepsilon \right)^2 + \left(\sqrt{\|\xi_2\|_{0(\varepsilon)}} + \varepsilon \right)^2 < \frac{1}{\mu},$$

where $\xi_1 = [\xi_1^\phi, \xi_2^\phi, \cdots]^T$ and $\xi_2 = [\xi_1^\psi, \xi_2^\psi, \cdots]^T$. Then

$$\sum_{k=1}^{\infty} (\theta_k^\phi - \xi_k^\phi) \phi_k(z) = \sum_{l=1}^{\infty} (\xi_l^\psi - \theta_l^\psi) \psi_l(z).$$

According to the uncertainty principle, we have

$$\left(\sqrt{\|\theta_1 - \xi_1\|_{0(\varepsilon)}} + \varepsilon \right)^2 + \left(\sqrt{\|\theta_2 - \xi_2\|_{0(\varepsilon)}} + \varepsilon \right)^2 \geq \frac{2}{\mu}. \tag{8}$$

However, based on the sparsity assumption of Theorem 2

$$\begin{aligned} &\left(\sqrt{\|\theta_1 - \xi_1\|_{0(\varepsilon)}} + \varepsilon \right)^2 + \left(\sqrt{\|\theta_2 - \xi_2\|_{0(\varepsilon)}} + \varepsilon \right)^2 \\ < &\left(\sqrt{\|\theta_1\|_{0(\varepsilon)}} + \varepsilon \right)^2 + \left(\sqrt{\|\xi_1\|_{0(\varepsilon)}} + \varepsilon \right)^2 \\ + &\left(\sqrt{\|\theta_2\|_{0(\varepsilon)}} + \varepsilon \right)^2 + \left(\sqrt{\|\xi_2\|_{0(\varepsilon)}} + \varepsilon \right)^2 < \frac{2}{\mu}, \end{aligned}$$

which contradicts (8). □

Remark 2. *Theorems 2 shows that there cannot be two different $\theta^T = [\theta_1^T \quad \theta_2^T]$ obeying $(\sqrt{\|\theta_1\|_{0(\varepsilon)}} + \varepsilon)^2 + (\sqrt{\|\theta_2\|_{0(\varepsilon)}} + \varepsilon)^2 < \frac{1}{\mu}$ that represent the same rational function. And if $\varepsilon = 0$ and the sequence is finite, then the results of Theorems 1 and 2 are parallel to the results of [13].*

4. CONCLUSION

A novel uncertainty principle for sparse representation of rational functions in infinite dimensional function space is presented in this paper. The bound which guarantees the uniqueness of the sparse representation is presented using mutual coherence as a measure. Since rational functions are widely used to model both signals and dynamic systems, the consequence in this paper can be applicable to sparse representation of signal and systems using pairs of ORF bases.

5. REFERENCES

[1] B. Ricaud and B. Torrésani, "A survey of uncertainty principles and some signal processing applications," *Advances in Computational Mathematics*, vol. 40, no. 3, pp. 629–650, June 2014.

[2] D. L. Donoho and P. B. Stark, "Uncertainty principles and signal recovery," *SIAM Journal on Applied Mathematics*, vol. 49, no. 3, pp. 906–931, 1989.

[3] S. S. Goh and T. N.T Goodman, "Uncertainty principles and asymptotic behavior," *Applied & Computational Harmonic Analysis*, vol. 16, no. 1, pp. 19–43, 2004.

[4] S. S. Goh and T. N.T Goodman, "Uncertainty principles in banach spaces and signal recovery," *Journal of Approximation Theory*, vol. 143, no. 1, pp. 26–35, November 2006.

[5] W. Erb, "Uncertainty principles on compact riemannian manifolds," *Applied and Computational Harmonic Analysis*, vol. 29, no. 2, pp. 182–197, 2010.

[6] A. Dembo, T. M. Cover, and J. A. Thomas, "Information theoretic inequalities," *IEEE Transactions on Information Theory*, vol. 37, no. 6, pp. 1501–1518, 2002.

[7] B. Ricaud and B. Torrésani, "Refined support and entropic uncertainty inequalities," *IEEE Transactions on Information Theory*, vol. 59, no. 7, pp. 4272–4279, 2013.

[8] A. Agaskar and Y. M. Lu, "A spectral graph uncertainty principle," *IEEE Transactions on Information Theory*, vol. 59, no. 7, pp. 4338–4356, July 2013.

[9] R. Ishii and K. Furukawa, "The uncertainty principle in discrete signals," *IEEE Transactions on Circuits & Systems*, vol. 33, no. 10, pp. 1032–1034, 1986.

[10] L. C. Calves and P. Vilbe, "On the uncertainty principle in discrete signals," *IEEE Transactions on Circuits & Systems II Analog & Digital Signal Processing*, vol. 39, no. 6, pp. 394–395, 2002.

[11] G. B. Folland and A. Sitaram, "The uncertainty principle: A mathematical survey," *Journal of Fourier Analysis & Applications*, vol. 3, no. 3, pp. 207–238, 1997.

[12] D. L. Donoho and X. Huo, "Uncertainty principles and ideal atomic decomposition," *IEEE Transactions on Information Theory*, vol. 47, no. 7, pp. 2845–2862, November 2001.

[13] M. Elad and A. M. Bruckstein, "A generalized uncertainty principle and sparse representation in pairs of bases," *IEEE Transactions on Information Theory*, vol. 48, no. 9, pp. 2558–2567, September 2002.

[14] E. J. Candès and J. Romberg, "Quantitative robust uncertainty principles and optimally sparse decompositions," *Foundations of Computational Mathematics*, vol. 7, no. 4, pp. 529–531, 2007.

[15] E. J. Candès, J. Romberg, and T. Tao, "Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information," *IEEE Transactions on Information Theory*, vol. 52, no. 2, pp. 489–509, January 2006.

[16] D. L. Donoho and M. Elad, "Optimally sparse representation in general (nonorthogonal) dictionaries via ℓ^1 minimization," vol. 100, no. 5, pp. 2197–2202, 2003.

[17] B. Ninness and F. Gustafsson, "A unifying construction of orthonormal bases for system identification," *IEEE Transactions on Automatic Control*, vol. 42, no. 4, pp. 515–521, April 1997.

[18] H. Akcay and B. Ninness, "Rational basis functions for robust identification from frequency and time-domain measurements," *American Control Conference*, pp. 1101–1117, 1998.

[19] P. Van gucht and A. Bultheel, "Orthogonal rational functions for system identification: numerical aspects," *IEEE Transactions on Automatic Control*, vol. 48, no. 4, pp. 705–709, April 2003.

[20] M. Wen and Q. Tao, "Frequency-domain identification: an algorithm based on an adaptive rational orthogonal system," *Automatica*, vol. 48, no. 6, pp. 1154–1162, 2012.

[21] Q. Chen, W. Mai, L. Zhang, and W. Mi, "System identification by discrete rational atoms," *Automatica*, vol. 56, no. C, pp. 53–59, June 2015.

[22] Y. Ohta, "Stochastic system transformation using generalized orthonormal basis functions with applications to continuous-time system identification," *Automatica*, vol. 47, no. 5, pp. 1001–1006, May 2011.

[23] K. Tiels and J. Schoukens, "Wiener system identification with generalized orthonormal basis functions," *Automatica*, vol. 50, no. 12, pp. 3147–3154, 2014.

[24] E. Hidayat and A. Medvedev, "Laguerre domain identification of continuous linear time-delay systems from impulse response data," *Automatica*, vol. 48, no. 11, pp. 2902–2907, November 2012.

[25] D. Xiong, L. Chai, and J. Zhang, "Uncertainty principle and sparse reconstruction in pairs of orthonormal rational function bases," *arXiv:1712.02981*, 2017.