

MARGINAL BAYESIAN BHATTACHARYYA BOUNDS FOR DISCRETE-TIME FILTERING

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ABSTRACT

In this paper, marginal versions of the Bayesian Bhattacharyya lower bound (BBLB), which is a tighter alternative to the classical Bayesian Cramér-Rao bound, for discrete-time filtering are proposed. Expressions for the second and third-order marginal BBLBs are obtained and it is shown how these can be approximately calculated using particle filtering. A simulation example shows that the proposed bounds predict the achievable performance of the filtering algorithms better.

Index Terms— Performance bounds, Bayesian estimation, Bhattacharyya bounds, nonlinear filtering, particle filter

1. INTRODUCTION

In discrete-time filtering, one is interested in estimating the state of a dynamic system at the current time instant, given all available measurements up to that time instance. If the dynamic system has a linear, additive Gaussian structure, then the celebrated Kalman filter [1] is the optimal filter (in mean-square error (MSE) sense). The case of a nonlinear dynamic system is much more challenging and a plethora of nonlinear filters have been proposed, see e.g. [2–4].

Assessing the best achievable performance of nonlinear filters is a challenging problem. In the last few years, a variety of Bayesian bounds, see e.g. [5–9], on the MSE performance for discrete-time filtering have been proposed [10–16]. The approach proposed by Tichavský et al. to compute the Bayesian Cramér-Rao lower bound (BCRLB) is perhaps the most widely used today. It is based on recursively computing the information matrix of the joint density of the state and measurement sequence, which is called joint BCRLB (J-BCRLB). In [15] a BCRLB that operates on the marginal density of the current state and the measurement sequence (M-BCRLB) was proposed that is tighter or equal to the J-BCRLB.

In this paper, we propose marginal Bayesian Bhattacharyya bounds (M-BBLBs) which, compared to the M-BCRLB, additionally account for the information contained

in higher-order derivatives of the marginal density, see also [17, 18] for application of the BLB to other problems. A particle filter (PF) approach is proposed to approximate these bounds numerically. The paper investigates only scalar, possibly nonlinear dynamic systems, since higher-dimensional systems require the computation of higher-order (mixed) derivatives of the current state vector elements making the computation of the bound eventually too complex to be used in practice. Further, the PF approach requires a huge amount of particles to approximate the marginal bounds in high-dimensional systems. A convincing example of a scalar dynamic system with a moderate number of particles shows that the proposed bounds achieve a better prediction performance for the filtering algorithms.

2. WEISS-WEINSTEIN FAMILY OF BOUNDS

We aim at providing a lower bound for the MSE of an arbitrary estimator $\hat{x}(z)$ of the random variable $x \in \mathbb{R}$ based on the measurements $z \in \mathbb{R}$. The lower bounds in Weiss-Weinstein family [19] solves this problem as follows.

$$\mathbb{E}_{x,z}\{[x - \hat{x}(z)]^2\} \geq VG^{-1}V^T, \quad (1)$$

where $\mathbb{E}_x[\cdot]$ denotes the expectation operator with respect to the variable x and the elements of the vector $V \in \mathbb{R}^{1 \times r}$ and the matrix $G \in \mathbb{R}^{r \times r}$ are defined as

$$V_j \triangleq \mathbb{E}_{x,z}[x\psi_j(x, z)], \quad G_{ij} \triangleq \mathbb{E}_{x,z}[\psi_i(x, z)\psi_j(x, z)].$$

Here, A_{ij} and x_i denote the i, j th element of the matrix A and the i th element of the vector x , respectively. The *score functions* $\{\psi_i(x, z)\}_{i=1}^r$ used in the definitions above must satisfy the property $\mathbb{E}_x[\psi_i(x, z)] = 0$ for $i = 1, \dots, r$ and for all z . In this study, we consider BBLBs which are in Weiss-Weinstein family of lower bounds.

3. GENERAL BHATTACHARYYA BOUNDS

The r -th order ($r \geq 1$) BBLB is obtained using the following specific selection of the score functions

$$\psi_i(x, z) = \frac{1}{p(x, z)} \frac{\partial^i p(x, z)}{\partial x^i} = \frac{1}{p(x|z)} \frac{\partial^i p(x|z)}{\partial x^i} \quad (2)$$

This work was partly supported by ELLIIT.

for $i = 1, \dots, r$. A lower bound for the MSE can be written as

$$\mathbb{E}_{x,z}\{[x - \hat{x}(z)]^2\} \geq V_r G^{-1} V_r^T \triangleq B_r, \quad (3)$$

where $V_r \triangleq -[1, \overbrace{0, \dots, 0}^{r-1 \text{ times}}]$ and the elements of the matrix $G \in \mathbb{R}^{r \times r}$ are defined as follows

$$G_{ij} \triangleq \mathbb{E}_{x,z} \left\{ \frac{1}{p^2(x,z)} \frac{\partial^i p(x,z)}{\partial x^i} \frac{\partial^j p(x,z)}{\partial x^j} \right\}. \quad (4)$$

The bound expression presented in (3) holds, given that suitable regularity are satisfied, see [19] for details.

Let us define the sub-matrix $G_{i_1:i_2, j_1:j_2}$ of the matrix G as $G_{i_1:i_2, j_1:j_2} \triangleq [G_{ij}]_{i=i_1, \dots, i_2, j=j_1, \dots, j_2}$. We can see that

$$\begin{aligned} B_r &\triangleq V_r G^{-1} V_r^T = V_r G_{1:r, 1:r}^{-1} V_r^T \\ &= V_{r-1} (G_{1:r-1, 1:r-1} - G_{1:r-1, r} G_{rr}^{-1} G_{r, 1:r-1})^{-1} V_{r-1}^T \\ &= \underbrace{V_{r-1} G_{1:r-1, 1:r-1}^{-1} V_{r-1}^T}_{\triangleq B_{r-1}} \\ &\quad + V_{r-1} G_{1:r-1, 1:r-1}^{-1} G_{1:r-1, r} S_{rr}^{-1} G_{r, 1:r-1} G_{1:r-1, 1:r-1}^{-1} V_{r-1}^T \end{aligned} \quad (5)$$

for $r > 1$ where $S_{rr} \triangleq G_{rr} - G_{r, 1:r-1} G_{1:r-1, 1:r-1}^{-1} G_{1:r-1, r}$ is the Schur complement of G_{rr} . Since S_{rr} is positive semi-definite (since G is positive semi-definite), the second term on the right hand side above is always non-negative. Hence we have $B_r \geq B_{r-1}$ for $r > 1$, i.e., BBLBs are monotonically non-decreasing as the order r increases. Since B_1 is also BCRLB, the second and higher order BBLBs are at least as tight as BCRLB. In this paper, we only consider BBLBs of orders $r = 2$ and $r = 3$.

4. MARGINAL BHATTACHARYYA BOUND

In contrast to the J-BCRLB and the joint BBLB (J-BBLB), which are based on the information in the joint density $p(X_k, Z_k)$ of the state sequence $X_k \triangleq [x_0, \dots, x_k]$ and measurement sequence $Z_k \triangleq [z_1, \dots, z_k]$, see [10, 12] for a detailed derivation, the marginal versions of these bounds extract information from the marginal density $p(x_k, Z_k)$ [or alternatively the posterior $p(x_k|Z_k)$]. Computation of the M-BBLB for the case of general linear and nonlinear dynamic systems is investigated in the following.

4.1. Linear Systems

In this section, we consider a linear scalar dynamic system with additive Gaussian noise, i.e.,

$$x_k = F_k x_{k-1} + v_{k-1}, \quad (6a)$$

$$z_k = H_k x_k + w_k, \quad (6b)$$

where $v_{k-1} \sim \mathcal{N}(0, Q_{k-1})$, $w_k \sim \mathcal{N}(0, R_k)$ and $x_0 \sim \mathcal{N}(\hat{x}_{0|0}, P_{0|0})$. For such systems, the posterior density is available in closed-form $p(x_k|Z_k) = \mathcal{N}(x_k; \hat{x}_{k|k}(Z_k), P_{k|k})$, where $\hat{x}_{k|k}(Z_k)$ and $P_{k|k}$ are computed from the well-known Kalman filter recursions. In particular, for the error variance we have

$$P_{k|k} = (1 - K_k H_k)(F_k^2 P_{k-1|k-1} + Q_{k-1}), \quad (7a)$$

$$K_k = \frac{(F_k^2 P_{k-1|k-1} + Q_{k-1}) H_k}{H_k^2 (F_k^2 P_{k-1|k-1} + Q_{k-1}) + R_k}, \quad (7b)$$

and the recursion is initiated with $P_{0|0}$. The M-BBLB for this case can be computed analytically, where the following theorem holds:

Theorem 1. *For linear additive Gaussian systems, the M-BBLB of order $r = 2, 3$ is equal to the (M-)BCRLB, and is given by the error covariance $P_{k|k}$ of the Kalman filter.*

Proof. See appendix. \square

We can conclude that for linear Gaussian systems, additionally taking into account (non-zero) higher-order derivatives cannot improve the tightness of the bound compared to the BCRLB, which is known to be the tightest bound in this setting [10, 15].

4.2. Nonlinear Systems

If the dynamic system is nonlinear, i.e.,

$$x_k = f_k(x_{k-1}, v_{k-1}), \quad (8a)$$

$$z_k = h_k(x_k, w_k), \quad (8b)$$

a closed-form expression for the posterior $p(x_k|Z_k)$ (and thus the M-BBLB) is generally not available. Still, it is possible to evaluate the expectations appearing in G_{ij} numerically by making use of sequential Monte Carlo techniques, a.k.a. particle filtering, and thus compute an approximate marginal bound. For this purpose, we decompose the marginal density $p(x_k, Z_k)$ as

$$p(x_k, Z_k) = p(z_k|x_k) p(x_k|Z_{k-1}) p(Z_{k-1}) \quad (9)$$

and introduce the following abbreviations: $p_k \triangleq p(x_k|Z_{k-1})$, $g_k \triangleq p(z_k|x_k)$. Then, the score functions can be written as

$$\psi_1 = \frac{1}{g_k} \frac{\partial g_k}{\partial x_k} + \frac{1}{p_k} \frac{\partial p_k}{\partial x_k}, \quad (10a)$$

$$\psi_2 = \frac{1}{g_k} \frac{\partial^2 g_k}{\partial x_k^2} + \frac{2}{g_k p_k} \frac{\partial g_k}{\partial x_k} \frac{\partial p_k}{\partial x_k} + \frac{1}{p_k} \frac{\partial^2 p_k}{\partial x_k^2}, \quad (10b)$$

$$\begin{aligned} \psi_3 &= \frac{1}{g_k} \frac{\partial^3 g_k}{\partial x_k^3} + \frac{3}{g_k p_k} \frac{\partial^2 g_k}{\partial x_k^2} \frac{\partial p_k}{\partial x_k} + \frac{3}{g_k p_k} \frac{\partial g_k}{\partial x_k} \frac{\partial^2 p_k}{\partial x_k^2} \\ &\quad + \frac{1}{p_k} \frac{\partial^3 p_k}{\partial x_k^3}. \end{aligned} \quad (10c)$$

Inserting (10) into (4) and performing straightforward manipulations, the elements of the matrix G can be expressed as

$$G_{11} = \mathbb{E} \left\{ \frac{1}{g_k^2} \left(\frac{\partial g_k}{\partial x_k} \right)^2 \right\} + \mathbb{E} \left\{ \frac{1}{p_k^2} \left(\frac{\partial p_k}{\partial x_k} \right)^2 \right\}, \quad (11a)$$

$$G_{12} = \mathbb{E} \left\{ \frac{1}{g_k^2} \frac{\partial g_k}{\partial x_k} \frac{\partial^2 g_k}{\partial x_k^2} \right\} + 2 \mathbb{E} \left\{ \frac{1}{g_k^2 p_k} \left(\frac{\partial g_k}{\partial x_k} \right)^2 \frac{\partial p_k}{\partial x_k} \right\} + \mathbb{E} \left\{ \frac{1}{p_k^2} \frac{\partial p_k}{\partial x_k} \frac{\partial^2 p_k}{\partial x_k^2} \right\}, \quad (11b)$$

$$G_{22} = \mathbb{E} \left\{ \frac{1}{g_k^2} \left(\frac{\partial^2 g_k}{\partial x_k^2} \right)^2 \right\} + 4 \mathbb{E} \left\{ \frac{1}{g_k^2 p_k} \frac{\partial g_k}{\partial x_k} \frac{\partial^2 g_k}{\partial x_k^2} \frac{\partial p_k}{\partial x_k} \right\} + 4 \mathbb{E} \left\{ \frac{1}{g_k^2 p_k^2} \left(\frac{\partial g_k}{\partial x_k} \frac{\partial p_k}{\partial x_k} \right)^2 \right\} + \mathbb{E} \left\{ \frac{1}{p_k^2} \left(\frac{\partial^2 p_k}{\partial x_k^2} \right)^2 \right\}, \quad (11c)$$

$$G_{13} = \mathbb{E} \left\{ \frac{1}{g_k^2} \frac{\partial g_k}{\partial x_k} \frac{\partial^3 g_k}{\partial x_k^3} \right\} + 3 \mathbb{E} \left\{ \frac{1}{g_k^2 p_k} \frac{\partial g_k}{\partial x_k} \frac{\partial^2 g_k}{\partial x_k^2} \frac{\partial p_k}{\partial x_k} \right\} + 3 \mathbb{E} \left\{ \frac{1}{g_k^2 p_k} \left(\frac{\partial g_k}{\partial x_k} \right)^2 \frac{\partial^2 p_k}{\partial x_k^2} \right\} + \mathbb{E} \left\{ \frac{1}{p_k^2} \frac{\partial p_k}{\partial x_k} \frac{\partial^3 p_k}{\partial x_k^3} \right\}, \quad (11d)$$

$$G_{23} = \mathbb{E} \left\{ \frac{1}{g_k^2} \frac{\partial^2 g_k}{\partial x_k^2} \frac{\partial^3 g_k}{\partial x_k^3} \right\} + 3 \mathbb{E} \left\{ \frac{1}{g_k^2 p_k} \left(\frac{\partial^2 g_k}{\partial x_k^2} \right)^2 \frac{\partial p_k}{\partial x_k} \right\} + 3 \mathbb{E} \left\{ \frac{1}{g_k^2 p_k} \frac{\partial g_k}{\partial x_k} \frac{\partial^2 g_k}{\partial x_k^2} \frac{\partial^2 p_k}{\partial x_k^2} \right\} + 2 \mathbb{E} \left\{ \frac{1}{g_k^2 p_k} \frac{\partial g_k}{\partial x_k} \frac{\partial^3 g_k}{\partial x_k^3} \frac{\partial p_k}{\partial x_k} \right\}$$

$$+ 6 \mathbb{E} \left\{ \frac{1}{g_k^2 p_k^2} \frac{\partial g_k}{\partial x_k} \frac{\partial^2 g_k}{\partial x_k^2} \left(\frac{\partial p_k}{\partial x_k} \right)^2 \right\} + \mathbb{E} \left\{ \frac{1}{p_k^2} \frac{\partial^2 p_k}{\partial x_k^2} \frac{\partial^3 p_k}{\partial x_k^3} \right\} + 6 \mathbb{E} \left\{ \frac{1}{g_k^2 p_k^2} \left(\frac{\partial g_k}{\partial x_k} \right)^2 \frac{\partial p_k}{\partial x_k} \frac{\partial^2 p_k}{\partial x_k^2} \right\}, \quad (11e)$$

$$G_{33} = \mathbb{E} \left\{ \frac{1}{g_k^2} \left(\frac{\partial^3 g_k}{\partial x_k^3} \right)^2 \right\} + 6 \mathbb{E} \left\{ \frac{1}{g_k^2 p_k} \frac{\partial^2 g_k}{\partial x_k^2} \frac{\partial^3 g_k}{\partial x_k^3} \frac{\partial p_k}{\partial x_k} \right\} + 9 \mathbb{E} \left\{ \frac{1}{g_k^2 p_k^2} \left(\frac{\partial^2 g_k}{\partial x_k^2} \frac{\partial p_k}{\partial x_k} \right)^2 \right\} + \mathbb{E} \left\{ \frac{1}{p_k^2} \left(\frac{\partial^3 p_k}{\partial x_k^3} \right)^2 \right\} + 6 \mathbb{E} \left\{ \frac{1}{g_k^2 p_k} \frac{\partial g_k}{\partial x_k} \frac{\partial^3 g_k}{\partial x_k^3} \frac{\partial^2 p_k}{\partial x_k^2} \right\} + 9 \mathbb{E} \left\{ \frac{1}{g_k^2 p_k^2} \left(\frac{\partial g_k}{\partial x_k} \frac{\partial^2 p_k}{\partial x_k^2} \right)^2 \right\} + 18 \mathbb{E} \left\{ \frac{1}{g_k^2 p_k^2} \frac{\partial g_k}{\partial x_k} \frac{\partial^2 g_k}{\partial x_k^2} \frac{\partial p_k}{\partial x_k} \frac{\partial^2 p_k}{\partial x_k^2} \right\}. \quad (11f)$$

The expectations involving only higher-order derivatives of g_k are available in closed-form for many models (see the example in this paper). Alternatively, these can be approximated by a simple (non-sequential) Monte Carlo integration approach. On the other hand, the evaluation of higher-order derivatives of the prediction density p_k is more involved. Here, we approximate these quantities using particle filtering, see e.g. [2, 4, 20].

The PF approximates the joint smoothing distribution $p(X_k|Z_k)$ with a set of weighted particles $\{X_k^{(p)}, w_k^{(p)}\}_{p=1}^{N_p}$, yielding a point-mass approximation given as

$$\hat{p}^{N_p}(X_k|Z_k) \triangleq \sum_{p=1}^{N_p} w_k^{(p)} \delta_{X_k^{(p)}}(X_k), \quad (12)$$

where $\delta_x(\cdot)$ denotes the Dirac distribution at point x and $X_k^{(p)}$ is a particle state trajectory with corresponding weight $w_k^{(p)}$. The PF is based on sequential importance sampling method, where particles are generated from a proposal distribution $q(x_k|x_{k-1}, z_k)$ followed by an update step of the particle weights according to

$$w_k^{(p)} \propto w_{k-1}^{(p)} \frac{p(z_k|x_k^{(p)})p(x_k^{(p)}|x_{k-1}^{(p)})}{q(x_k^{(p)}|x_{k-1}^{(p)}, z_k)}. \quad (13)$$

In order to make PF work in practice, a resampling step is performed to reduce the variance in the weights.

By using the approximate density in (12), one can approximate the prediction density $p(x_k|Z_{k-1})$ and the corresponding higher-order derivatives as follows

$$p(x_k|Z_{k-1}) = \int p(x_k|x_{k-1})p(X_{k-1}|Z_{k-1})dX_{k-1} \approx \sum_{p=1}^{N_p} w_{k-1}^{(p)} p(x_k|x_{k-1}^{(p)}), \quad (14a)$$

$$\frac{\partial^i p(x_k|Z_{k-1})}{\partial x_k^i} \approx \sum_{p=1}^{N_p} w_{k-1}^{(p)} \frac{\partial^i p(x_k|x_{k-1}^{(p)})}{\partial x_k^i}. \quad (14b)$$

Using this approximation, any expectation E_{ij} in the form $E_{ij} \triangleq \mathbb{E}_{x_k, Z_{k-1}} \left\{ \frac{1}{p_k^2} \frac{\partial^i p_k}{\partial x_k^i} \frac{\partial^j p_k}{\partial x_k^j} \right\}$ can be approximated as

$$\hat{E}_{ij} = \frac{1}{N_{mc}} \sum_{\ell=1}^{N_{mc}} \frac{1}{p_k^2} \frac{\partial^i p_k}{\partial x_k^i} \frac{\partial^j p_k}{\partial x_k^j} \Big|_{x_k=x_k^{(\ell)}, Z_{k-1}=Z_{k-1}^{(\ell)}}, \quad (15)$$

where $x_k^{(\ell)}, Z_{k-1}^{(\ell)}$ with $\ell = 1, \dots, N_{mc}$ are independent and identically distributed (i.i.d.) random variables such that $(x_k^{(\ell)}, Z_{k-1}^{(\ell)}) \sim p(x_k, Z_{k-1})$ holds.

5. SIMULATIONS

We investigate the dynamical system proposed in [12], where the process model is linear Gaussian with transition pdf

$$p(x_k|x_{k-1}) = \frac{1}{\sqrt{2\pi Q}} \exp \left\{ -\frac{(x_k - x_{k-1})^2}{2Q} \right\} \quad (16)$$

and the measurement model is described by the skewed Gaussian likelihood given as

$$p(z_k|x_k) = \begin{cases} \frac{\sqrt{2}}{\sqrt{\pi}(\sigma_1+\sigma_2)} \exp \left\{ -\frac{(z_k-x_k)^2}{2(\sigma_1)^2} \right\}, & z_k < x_k, \\ \frac{\sqrt{2}}{\sqrt{\pi}(\sigma_1+\sigma_2)} \exp \left\{ -\frac{(z_k-x_k)^2}{2(\sigma_2)^2} \right\}, & \text{otherwise.} \end{cases}$$

In the simulations below, the following parameters are used: $\sigma_1 = 1, \sigma_2 = 3, Q = 10, x_0 \sim \mathcal{N}(0, 1)$. The expressions for

G_{ij} to compute the M-BBLB can be found as follows.

$$G_{11} = \frac{1}{\sigma_1\sigma_2} + E_{11}, \quad (17a)$$

$$G_{12} = \sqrt{\frac{2}{\pi}} \cdot \frac{\sigma_1 - \sigma_2}{(\sigma_1\sigma_2)^2} + E_{12}, \quad (17b)$$

$$G_{22} = 2 \left(\frac{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2}{(\sigma_1\sigma_2)^3} \right) + \frac{4E_{11}}{\sigma_1\sigma_2} + E_{22}, \quad (17c)$$

$$G_{13} = E_{13}, \quad (17d)$$

$$G_{23} = 3\sqrt{\frac{2}{\pi}} \left(\frac{\sigma_1^3 - \sigma_2^3 - \sigma_1^2\sigma_2 + \sigma_2^2\sigma_1}{(\sigma_1\sigma_2)^4} \right) + 6\sqrt{\frac{2}{\pi}} \cdot \frac{\sigma_1 - \sigma_2}{(\sigma_1\sigma_2)^2} \cdot E_{11} + \frac{6}{\sigma_1\sigma_2} \cdot E_{12} + E_{23}, \quad (17e)$$

$$G_{33} = 6 \left(\frac{\sigma_1^4 + \sigma_2^4 - \sigma_1^3\sigma_2 - \sigma_1\sigma_2^3 + (\sigma_1\sigma_2)^2}{(\sigma_1\sigma_2)^5} \right) + 18 \left(\frac{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2}{(\sigma_1\sigma_2)^3} \right) \cdot E_{11} + 18\sqrt{\frac{2}{\pi}} \cdot \frac{\sigma_1 - \sigma_2}{(\sigma_1\sigma_2)^2} \cdot E_{12} + 9 \frac{E_{22}}{\sigma_1\sigma_2} + E_{33}. \quad (17f)$$

Notice that some expectations appearing in G_{ij} were calculated analytically thanks to the structure of the likelihood. The challenge then remains to approximate numerically the expectations E_{ij} . For this purpose, we used particle filters with $N_p = 1000$ particles and $p(x_k|x_{k-1})$ as importance density whose results were averaged over $N_{mc} = 100\,000$ Monte Carlo runs.

We compare the proposed bounds to 1) best linear unbiased estimator (BLUE), i.e., the Kalman filter, 2) Particle filter (PF), 3) joint Bayesian Bhattacharyya lower bound (J-BBLB) of order 2, see [12] and [21] for corrections, 4) marginal Bayesian Cramér-Rao lower bound (M-BCRLB) and 5) joint BCRLB (J-BCRLB). The MSE performances of the estimators are shown along with the bounds in Figure 1. As expected we observe that M-BBLB of order 3 is tighter than M-BBLB of order 2; and M-BBLB of order 2 is tighter than J-BBLB of order 2. It is seen that the gain obtained by marginalization in BBLBs of order 2 is slightly more than that is observed with BCRLBs (i.e., BBLBs with order 1). The increase of the BBLB order (from 2 to 3) seems to provide a significant improvement in tightness. Overall, the proposed bounds predict the estimators' performance much better than BCRLBs.

6. CONCLUSION AND FUTURE WORK

Marginal BBLBs have been proposed as tighter alternatives to BCRLB in bounding discrete-time filtering performance. Expressions for marginal BBLBs of order 2 and 3 have been obtained and a suitable numerical calculation methodology has been outlined.

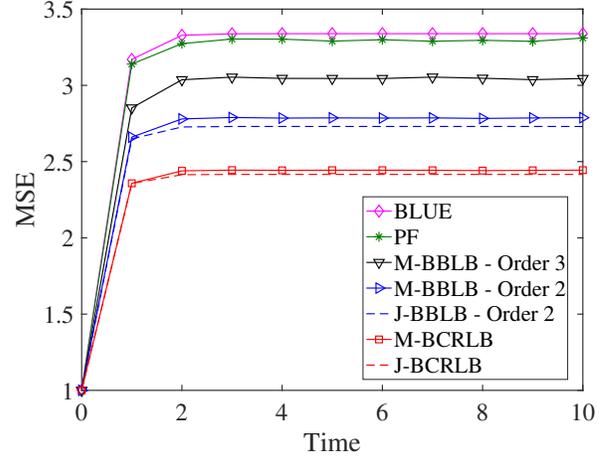


Fig. 1. MSE performance of different filters and bounds.

The simulation results on a scalar nonlinear dynamic system show that the order increase from 2 to 3 yields a significant improvement in tightness. An interesting question would be the behavior of the BBLBs when the order is further increased. As the bounds get closer to the filter performances the improvements are expected to be smaller. A promising future research idea can be to find out the order of BBLB above which the improvements become minor.

7. APPENDIX

For the computation of the M-BBLB of order $r = 2, 3$ we require the following higher-order derivatives

$$\begin{aligned} \frac{1}{\tilde{p}_k} \frac{\partial \tilde{p}_k}{\partial x_k} &= - \left(\frac{x_k - \hat{x}_{k|k}}{P_{k|k}} \right), \\ \frac{1}{\tilde{p}_k} \frac{\partial^2 \tilde{p}_k}{\partial x_k^2} &= \left(\frac{x_k - \hat{x}_{k|k}}{P_{k|k}} \right)^2 - \frac{1}{P_{k|k}}, \\ \frac{1}{\tilde{p}_k} \frac{\partial^3 \tilde{p}_k}{\partial x_k^3} &= - \left[\left(\frac{x_k - \hat{x}_{k|k}}{P_{k|k}} \right)^3 - 3 \left(\frac{x_k - \hat{x}_{k|k}}{P_{k|k}} \right) \frac{1}{P_{k|k}} \right]. \end{aligned}$$

where $\tilde{p}_k = p(x_k|Z_k)$. Straightforward calculations yield

$$\begin{aligned} V_3 G^{-1} V_3^T &= V_3 \left[\text{diag} \left(P_{k|k}^{-1}, 2P_{k|k}^{-2}, 6P_{k|k}^{-3} \right) \right]^{-1} V_3^T \\ &= P_{k|k}, \end{aligned}$$

where we have a diagonal G matrix due to the fact that the odd moments of a Gaussian random variable are zero and the even moments to compute G_{13} cancel each other. $P_{k|k}$ is the error variance of the Kalman filter, which for linear Gaussian systems is also equivalent to the BCRLB. Equivalence of BCRLB and M-BCRLB for linear Gaussian systems was proven in [15].

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