IMPROVED STEADY STATE ANALYSIS OF THE RECURSIVE LEAST SQUARES ALGORITHM

Muhammad Moinuddin[1,2] *Tareq Y. Al-Naffouri*[3] *Khaled A. Al-Hujaili*[4]

[1]Electrical and Computer Engineering Department, King Abdulaziz University (KAU), Kingdom of Saudi Arabia (KSA), Email: mmsansari@kau.edu.sa
[2]Center of Excellence in Intelligent Engineering Systems (CEIES), KAU, KSA, Email: mmsansari@kau.edu.sa
[3] Electrical Engineering Department, King Abdullah University of Science and Technology, Kingdom of Saudi Arabia, Email:tareq.alnaffouri@kaust.edu.sa
[4] Department of Electrical Engineering, Taibah University, KSA, Email:khujaili@taibah.edu.sa

ABSTRACT

This paper presents a new approach for studying the steady state performance of the Recursive Least Square (RLS) adaptive filter for a circularly correlated Gaussian input. Earlier methods have two major drawbacks: (1) The energy relation developed for the RLS is approximate (as we show later) and (2) The evaluation of the moment of the random variable $\|\mathbf{u}_i\|^2_{\mathbf{P_i}}$, where $\mathbf{u}_i$ is input to the RLS filter and $\mathbf{P}_i$ is the estimate of the inverse of input covariance matrix by assuming that $\mathbf{u}_i$ and $\mathbf{P}_i$ are independent (which is not true). These assumptions could result in negative value of the stead-state Excess Mean Square Error (EMSE). To overcome these issues, we modify the energy relation without imposing any approximation. Based on modified energy relation, we derive the steady-state EMSE and two upper bounds on the EMSE. For that, we derive closed from expression for the aforementioned moment which is based on finding the cumulative distribution function (CDF) of the random variable of the form $\frac{1}{\gamma+||\mathbf{u}||^2_{\mathbf{D}}}$, where $\mathbf{u}$ is correlated circular Gaussian input and $\mathbf{D}$ is a diagonal matrix. Simulation results corroborate our analytical findings.

Index Terms— Adaptive Filters, RLS, Steady-state analysis, Mean square analysis, Excess Mean-Squares-Error

1. INTRODUCTION

The RLS is one of the important algorithm from the adaptive filter's family. Motivation for the RLS adaptive filters relies on the fact that it provide a solution to the least square error minimization problem. The RLS algorithms are more costlier than the basic families such as the Leat mean squares family (LMS) but with much faster convergence speed. Analyzing the performance of the RLS algorithm is not an easy task due to the presence of the input covariance matrix and its inverse which depend on current and past input regressors. As such, only a few works considered the performance of the RLS and its variants [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. The simplest approach is based on the energy relation [9, 13] which (with the aid of separation principle [9]) can be used to state that the EMSE is a function of the moment $E[||\mathbf{u}_i||^2_{\mathbf{P}_i}]$, where $\mathbf{u}_i$ is the input regressor and $\mathbf{P}_i$ is the estimate of the inverse of input covariance matrix. Another separation principle is then used to write $E[||\mathbf{u}_i||^2_{\mathbf{P}_i}] = Tr(E[\mathbf{u}_i^*\mathbf{u}_i]\mathbf{P}_i)$. But this approach is not rigorous as $\mathbf{u}_i$ and $\mathbf{P}_i$ are dependent. Other approaches use the idea of random matrix to study the performance of the RLS [10, 11]. In addition to requiring a much more sophisticated machinery, these approaches are valid for filters relatively larger sizes.

In this work, we perform steady-state analysis of the RLS algorithm for correlated circular Gaussian inputs. The approach is based on evaluating the CDF and the moments of random variable of the form $z = \frac{1}{\gamma+||\mathbf{u}||^2_{\mathbf{D}}}$ where $\mathbf{u}$ is a correlated circular Gaussian vector and $\mathbf{D}$ is a diagonal matrix[1]. To evaluate the CDF, we replace the step function with its equivalent Fourier transform and employ complex integration. Using this CDF, we evaluate in closed form the required moment that appear in the steady-state mean-square analysis of RLS algorithm.

2. THE EXPONENTIALLY WEIGHTED RLS ALGORITHM

The exponentially weighted RLS attempts to solve the following problem

$$\min_{\mathbf{w}}\left[\gamma^{(N+1)}\mathbf{w}^*\Pi\mathbf{w} + (\mathbf{y}_N - \mathbf{H}_N\mathbf{w})^*\mathbf{\Gamma}_N(\mathbf{y}_N - \mathbf{H}_N\mathbf{w})\right] \tag{1}$$

in a recursive manner. Here $\mathbf{y}_N = [d(0),\ d(1),\cdots,d(N)]^T$ is the measurements vector and $\mathbf{H}_N$ is the $(N+1\times M)$

[1]For any matrix $\mathbf{D}$, the quadratic form $||\mathbf{u}||^2_{\mathbf{D}}$ is defined as $||\mathbf{u}||^2_{\mathbf{D}} \triangleq \mathbf{u}^*\mathbf{D}\mathbf{u}$ where the notation $()^*$ denotes conjugate transposition.

data matrix consisting of $(N+1)$ row regressor vectors $\mathbf{u}_k$ and $\mathbf{\Gamma}_N = \text{diag}\{\gamma^N, \gamma^{N-1}, \ldots, \gamma, 1\}$ is a diagonal weighting matrix defined in terms of the forgetting factor γ ($0 \ll \gamma \leq 1$). It can be shown that the solution of (1) is $\mathbf{w}_N = \mathbf{P}_N \mathbf{H}_N^* \mathbf{\Gamma}_N \mathbf{y}_N$. The RLS allows us to obtain the solution of (1) in a recursive manner. Specifically, we have

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mathbf{P}_i \mathbf{u}_i^* [d(i) - \mathbf{u}_i \mathbf{w}_{i-1}] \tag{2}$$

where

$$\mathbf{P}_i = \gamma^{-1} \left[\mathbf{P}_{i-1} - \frac{\mathbf{P}_{i-1} \mathbf{u}_i^* \mathbf{u}_i \mathbf{P}_{i-1}}{\gamma + \mathbf{u}_i \mathbf{P}_{i-1} \mathbf{u}_i^*} \right] \tag{3}$$

which is initialized with $\mathbf{w}_{-1} = 0$ and $\mathbf{P}_{-1} = \mathbf{\Pi}$.

3. THE CONVENTIONAL ENERGY RELATION FOR THE RLS

In the system identification model, the measurement $d(i)$ takes the form $d(i) = \mathbf{u}_i^* \mathbf{w}^o + v(i)$ where $v(i)$ is an additive noise and $\mathbf{w}^o$ is the system coefficients. We can thus define the weight error vector as $\tilde{\mathbf{w}}_i = \mathbf{w}^o - \mathbf{w}_i$ and write (2) as

$$\tilde{\mathbf{w}}_i = \tilde{\mathbf{w}}_{i-1} - \mathbf{P}_i \mathbf{u}_i^* e(i) \tag{4}$$

where $e(i) = d(i) - \mathbf{u}_i \mathbf{w}_{i-1}$ which is called the estimation error. Two other error measures are the a priori and a posteriori estimation errors defined by respectively $e_a(i) = \mathbf{u}_i \tilde{\mathbf{w}}_{i-1}$, $e_p(i) = \mathbf{u}_i \tilde{\mathbf{w}}_i$. If we multiply both sides of (4) from the left by $\mathbf{u}_i$ we obtain

$$e_p(i) = e_a(i) - \|\mathbf{u}_i\|^2_{\mathbf{P}_i} e(i) \tag{5}$$

Next, by evaluating the energies of both sides of (4) with the aid of (5) will lead to the well known *Energy Conservation Relation* [9] given by

$$E\left[\frac{|e_a(i)|^2}{\|\mathbf{u}_i\|^2_{\mathbf{P}_i}}\right] = E\left[\frac{|e_p(i)|^2}{\|\mathbf{u}_i\|^2_{\mathbf{P}_i}}\right] \tag{6}$$

Finally, by using (5) and the fact that $e(i) = e_a(i) + v(i)$, we can show that

$$\sigma_v^2 E\|\mathbf{u}_i\|^2_{\mathbf{P}_i} + E\big(\|\mathbf{u}_i\|^2_{\mathbf{P}_i} \cdot |e_a(i)|^2\big) = 2E|e_a(i)|^2 \tag{7}$$

This relation can be used to evaluate the steady-state EMSE which is defined as $\zeta \stackrel{\triangle}{=} \lim_{i\to\infty} E|e_a(i)|^2$.

3.1. Approach of [9] and its Drawbacks

The approach of [9] hss following drawbacks:

1. In the development of the energy relation (7), it assumes $\lim_{i\to\infty} E[\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{P}_i^{-1}}] = \lim_{i\to\infty} E[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\mathbf{P}_i^{-1}}]$ which is not true because the norms $\|\tilde{\mathbf{w}}_i\|^2$ and $\|\tilde{\mathbf{w}}_{i-1}\|^2$ are not weighted by their respective $\mathbf{P}$ matrices.
2. It assumes that $\mathbf{u}_i$ and $\mathbf{P}_i$ are independent which is not true as can be seen from (3) (which shows that $\mathbf{P}_i$ is a function of $\mathbf{u}_i$ and $\mathbf{P}_{i-1}$). As a result, the moment $E\|\mathbf{u}_i\|^2_{\mathbf{P}_i}$ is approximated as

$$\lim_{i\to\infty} E\|\mathbf{u}_i\|^2_{\mathbf{P}_i} = E\|\mathbf{u}_i\|^2_{\lim_{i\to\infty} E[\mathbf{P}_i]} = Tr(\mathbf{RP}) \tag{8}$$

   where Tr represents the trace operator, $\mathbf{P} = \lim_{i\to\infty} E[\mathbf{P}_i]$ and $\mathbf{R} = E[\mathbf{u}_i^* \mathbf{u}_i]$

3. It assumes that the expectation of the inverse of $\mathbf{P}_i$ is equal to the inverse of expectation of $\mathbf{P}_i$, i.e.,

$$\mathbf{P} \approx \left(\lim_{i\to\infty} E[\mathbf{P}_i^{-1}]\right)^{-1} = (1-\gamma)\mathbf{R}^{-1} \tag{9}$$

   which is also not true in general.

By employing (8) and (9), the required moment is approximated as $E\|\mathbf{u}_i\|^2_{\mathbf{P}_i} \approx \text{Trace}(\mathbf{RP}) \approx (1-\gamma)M$. Thus, the EMSE proposed by [9] is found to be

$$\zeta \approx \frac{\sigma_v^2 (1-\gamma) M}{2 - (1-\gamma)M} \tag{10}$$

The problem with the above result is that it will give negative values of EMSE for the scenario of $(1-\gamma)M > 2$ and a value of infinity at $(1-\gamma)M = 2$ which are not realizable. In order to deal with the above limitations, we propose to modify the energy relation which is presented next.

4. OUR APPROACH: MODIFIED ENERGY RELATION FOR THE RLS

In the proposed approach, we modify the energy relation by rewriting $\|\tilde{\mathbf{w}}_{i-1}\|^2_{\mathbf{P}_i^{-1}}$ in terms of $\mathbf{P}_{i-1}^{-1}$. This is done by expressing the $\mathbf{P}_i^{-1}$ in terms of $\mathbf{P}_{i-1}^{-1}$ using the relation $\mathbf{P}_i^{-1} = \gamma \mathbf{P}_{i-1}^{-1} + \mathbf{u}_i^* \mathbf{u}_i$ which modifies the term $\|\tilde{\mathbf{w}}_{i-1}\|^2_{\mathbf{P}_i^{-1}}$ to

$$\|\tilde{\mathbf{w}}_{i-1}\|^2_{\mathbf{P}_i^{-1}} = \gamma \|\tilde{\mathbf{w}}_{i-1}\|^2_{\mathbf{P}_{i-1}^{-1}} + |e_a(i)|^2 \tag{11}$$

By using the above, the modified energy relation takes the form

$$\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{P}_i^{-1}} + \frac{|e_a(i)|^2}{\|\mathbf{u}_i\|^2_{\mathbf{P}_i}} = \gamma \|\tilde{\mathbf{w}}_{i-1}\|^2_{\mathbf{P}_{i-1}^{-1}} + |e_a(i)|^2 + \frac{|e_p(i)|^2}{\|\mathbf{u}_i\|^2_{\mathbf{P}_i}} \tag{12}$$

Now, by replacing $|e_p(i)|^2$ by its equivalent expression in (5) and by assuming that $\lim_{i\to\infty} \|\tilde{\mathbf{w}}_i\|^2_{\mathbf{P}_i^{-1}} = \lim_{i\to\infty} \|\tilde{\mathbf{w}}_{i-1}\|^2_{\mathbf{P}_{i-1}^{-1}}$ yields

$$|v|^2 \|\mathbf{u}_i\|^2_{\mathbf{P}_i} + \|\mathbf{u}_i\|^2_{\mathbf{P}_i} |e_a(i)|^2 = |e_a(i)|^2 + (1-\gamma)\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{P}_i^{-1}} \tag{13}$$

Next, we evaluate the term $\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{P}_i^{-1}}$ using the relation $\mathbf{P}_i^{-1} = \gamma\mathbf{P}_{i-1}^{-1} + \mathbf{u}_i^*\mathbf{u}_i$ which results in

$$\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{P}_i^{-1}} = \gamma\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{P}_{i-1}^{-1}} + \|\tilde{\mathbf{w}}_i\|^2_{\mathbf{u}_i^*\mathbf{u}_i} \tag{14}$$

But $\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{u}_i^*\mathbf{u}_i}$ is nothing but $|e_p(i)|^2$. As a result, the energy relation in (13) can be rewritten as

$$\begin{aligned}|v|^2\|\mathbf{u}_i\|^2_{\mathbf{P}_i} + \|\mathbf{u}_i\|^2_{\mathbf{P}_i}\,|e_a(i)|^2 = |e_a(i)|^2 + (1-\gamma)\gamma\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{P}_{i-1}^{-1}}\\ +(1-\gamma)|e_p(i)|^2 \end{aligned}\tag{15}$$

In the ensuing section, we derive the steady-state EMSE and its upper bounds using the modified energy relation (15).

5. PROPOSED STEADY-STATE EMSE

In this section, with the aid of separation principle, we evaluate the expression for the steady-state EMSE of the RLS by using the modified energy relation derived in (15). We start by taking expectation on both sides of (15) to arrive at

$$\begin{aligned}\sigma_v^2E[\|\mathbf{u}_i\|^2_{\mathbf{P}_i}] + E[\|\mathbf{u}_i\|^2_{\mathbf{P}_i}\,|e_a(i)|^2] = E[|e_a(i)|^2]\\ +(1-\gamma)\gamma E[\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{P}_{i-1}^{-1}}] + (1-\gamma)E[|e_p(i)|^2]\end{aligned} \tag{16}$$

Since $\mathbf{P}_{i-1}^{-1}$ is a function of $\{\mathbf{u}_j\}$ for all $j < i$ and the independence assumption dictates us that $\tilde{\mathbf{w}}_i$ is independent of $\mathbf{u}_j$ for all $j < i$, we conclude that $\tilde{\mathbf{w}}_i$ is independent of $\mathbf{P}_{i-1}^{-1}$. Hence, the term $E[\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{P}_{i-1}^{-1}}]$ can be simplified to

$$E[\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{P}_{i-1}^{-1}}] = E[\|\tilde{\mathbf{w}}_i\|^2_{E[\mathbf{P}_{i-1}^{-1}]}] = \frac{E[\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{R}}]}{(1-\gamma)} = \frac{E[|e_a(i)|^2]}{(1-\gamma)} \tag{17}$$

where we have used the facts that $E[\mathbf{P}_{i-1}^{-1}] = \frac{\mathbf{R}}{(1-\gamma)}$ and $E[\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{R}}] = E[|e_a(i)|^2]$. Thus, the relation in (16) can be set up as

$$\begin{aligned}\sigma_v^2E[\|\mathbf{u}_i\|^2_{\mathbf{P}_i}] + E[\|\mathbf{u}_i\|^2_{\mathbf{P}_i}\,|e_a(i)|^2] = (1+\gamma)E[|e_a(i)|^2]\\ +(1-\gamma)E[|e_p(i)|^2]\end{aligned} \tag{18}$$

To proceed further, we replace $\|\mathbf{u}_i\|^2_{\mathbf{P}_i}$ in (18) by its equivalent representation[2]

$$\|\mathbf{u}_i\|^2_{\mathbf{P}_i} = \frac{\|\mathbf{u}_i\|^2_{\mathbf{P}_{i-1}}}{\gamma + \|\mathbf{u}_i\|^2_{\mathbf{P}_{i-1}}} = 1 - \frac{\gamma}{\gamma + \|\mathbf{u}_i\|^2_{\mathbf{P}_{i-1}}} \tag{19}$$

At this stage, we define the following random variable

$$Z \triangleq \frac{1}{\gamma + \|\mathbf{u}_i\|^2_{\mathbf{P}_{i-1}}} \tag{20}$$

[2]This equivalent representation is obtained by pre and post multiplying (3) by $\mathbf{u}_i$ and $\mathbf{u}_i^*$, respectively.

Thus, with the aid of the above definition and the relation given in (5), the relation (18) results in the following expression for the steady-state EMSE

$$\zeta = \frac{\sigma_v^2\left(1 + E[Z] - 2\gamma E[Z] - \gamma(1-\gamma)E[Z^2]\right)}{1 + E[Z] + \gamma(1-\gamma)E[Z^2]} \tag{21}$$

where we have employed the separation principle for the terms $E\left[\frac{|e_a(i)|^2}{\gamma+\|\mathbf{u}_i\|^2_{\mathbf{P}_{i-1}}}\right]$ and $E\left[\frac{|e_a(i)|^2}{(\gamma+\|\mathbf{u}_i\|^2_{\mathbf{P}_{i-1}})^2}\right]$. It can be seen from (21) that the calculation of EMSE requires the evaluation of $E[Z]$ and $E[Z^2]$ which are provided in Appendix.

6. UPPER BOUNDS ON THE EMSE

In this section, we aim to derive upper bounds on the EMSE of the RLS without invoking the separation principlefor two extreme scenarios: large values of γ (as $\gamma \to 1$) and small vaues of γ (as $\gamma \to 0$).

6.1. Upper Bound for Large Values of γ

Starting with (15) and by taking expectation with the aid of the result in (17 and the alternate expression for $\|\mathbf{u}_i\|^2_{\mathbf{P}_i}$, we can show that

$$\begin{aligned}\sigma_v^2E[\|\mathbf{u}_i\|^2_{\mathbf{P}_{i-1}}] = (\gamma+\gamma^2)\zeta + \gamma(1-\gamma)E[\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{P}_{i-1}^{-1}}\|\mathbf{u}_i\|^2_{\mathbf{P}_{i-1}}]\\ +(1-\gamma)E[|e_p(i)|^2(\gamma + \|\mathbf{u}_i\|^2_{\mathbf{P}_{i-1}})]\end{aligned} \tag{22}$$

To obtain the upper bound on the EMSE, we drop the last two terms on the right hand side of the above having the term $(1-\gamma)$ which is also a good approximation of the EMSE for larger values of γ as $(1-\gamma)$ approaches to zero as γ approaches to unity. As a result, we arrive to the following upper bound on the EMSE

$$\zeta \leq \frac{\sigma_v^2E\left[\|\mathbf{u}_i\|^2_{\mathbf{P}_{i-1}}\right]}{\gamma+\gamma^2} = \frac{\sigma_v^2Tr(\mathbf{RP})}{\gamma+\gamma^2} \tag{23}$$

where we have used the fact that $\lim_{i\to\infty} E\left[\|\mathbf{u}_i\|^2_{\mathbf{P}_{i-1}}\right] = Tr(\mathbf{RP})$ because $\mathbf{P}_{i-1}$ is independent of $\mathbf{u}_i$.

6.2. Upper Bound for Small Values of γ

We again start with (15) to develop the upper bound for smaller values of γ. By dropping the term $(1-\gamma)\gamma\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{P}_{i-1}^{-1}}$ and approximating the term $(1-\gamma)$ as unity (which is true for small values of γ), we can have the following inequality

$$|v|^2\|\mathbf{u}_i\|^2_{\mathbf{P}_i} + \|\mathbf{u}_i\|^2_{\mathbf{P}_i}|e_a(i)|^2 \geq |e_a(i)|^2 + |e_p(i)|^2 \tag{24}$$

Now, using (5) and alternative representation of $\|\mathbf{u}_i\|^2_{\mathbf{P}_i}$ given in (19), the above relation results in

$$|v|^2\|\mathbf{u}_i\|^2_{\mathbf{P}_{i-1}} \geq 2\gamma|e_a(i)|^2 + |e_a(i)|^2\|\mathbf{u}_i\|^2_{\mathbf{P}_{i-1}} \tag{25}$$

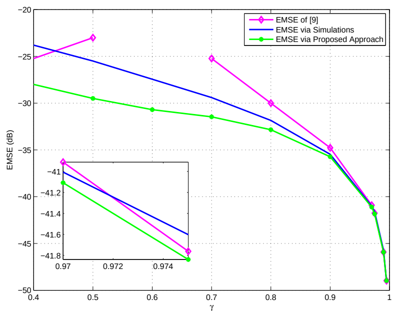

Fig. 1. EMSE of the RLS versus forgetting factor γ.

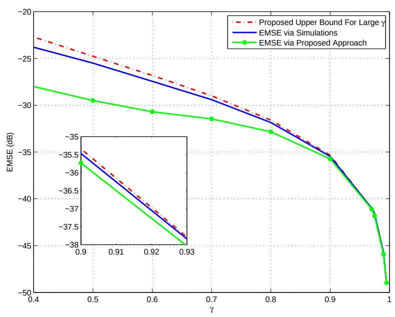

Fig. 2. Upper Bound for Large γ.

Next, by taking expectation on both sides of the above gives us

$$\sigma_v^2 E[\|\mathbf{u}_i\|^2_{\mathbf{P}_{i-1}}] \geq 2\gamma E[|e_a(i)|^2] + E[|e_a(i)|^2\|\mathbf{u}_i\|^2_{\mathbf{P}_{i-1}}] \quad (26)$$

Finally, by dropping the second term from the right hand side of (26) and using the fact that $\mathbf{P}_{i-1}$ is independent of $\mathbf{u}_i$, we get the following upper bound on the EMSE

$$\zeta \leq \frac{\sigma_v^2 Tr(\mathbf{RP})}{2\gamma} \quad (27)$$

7. SIMULATION RESULTS

In simulations, the steady-state performance of the RLS algorithm is investigated for an unknown system identification with $w^o = [0.13484\,,\, 0.26968\,,\, 0.40452\,,\, 0.53936\,,\, 0.67420]^T$. The noise is zero mean i.i.d with variance $\sigma_v^2 = 0.001$. Input to the adaptive filter and to the unknown system is correlated circular complex Gaussian having correlation $\mathbf{R}(i,j) = \alpha_c^{|i-j|}$ $(0 < \alpha_c < 1)$. We first compare our derived EMSE (given in (21)) with the one obtained via simulations and the one proposed in [9] in Fig. 2. It can be depicted from the figure that the proposed EMSE result has a good match with the simulation for larger values of γ (say $\gamma > 0.8$) but it gives a poor estimate for smaller values of γ. This deviation is because of employing the separation principle which is not valid for smaller values of γ. On the other hand, the EMSE using [9] gives positive values only for larger forgetting factor $i.e.$, $\gamma \geq 9$. This is because of the fact that the EMSE expression given in [9] becomes unrealistic (negative or infinity) for $(1-\gamma)M \geq 2$. In contrast, our approach is valid for all values of γ and M. Next, in Fig. 2, the upper bound for large γ (given in (23)) is compared with the EMSE via simulation and using (21). It can be depicted from the result and its zoomed view that the proposed upper bound is tight and very close to the simulation result particularly for large values of γ. Finally, in Fig. 3, the upper bound for small γ (given in (27)) is plotted which also shows that this upper bound is also close to the EMSE via simulations near small values of γ.

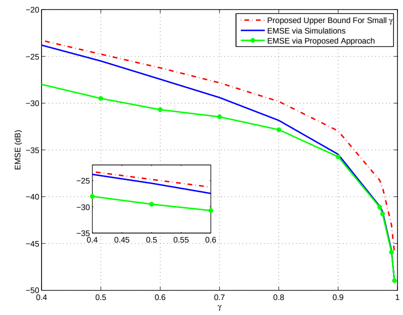

Fig. 3. Upper Bound for small γ.

8. CONCLUSION

In this work, we analyze the RLS algorithm at steady state for correlated complex Gaussian input and we evaluate its steady-state EMSE. The novelty of the work resides in three aspects: first in the modification of energy relation, second in the evaluation of the moment $E[||\mathbf{u}_i||^2_{\mathbf{P}_i}]$ which is based on the derivation of a closed form expression for the CDF and moment of random variable of the form $\frac{1}{\gamma+||\mathbf{u}||^2_{\mathbf{D}}}$, and third in derivation of upper bounds on the EMSE for both large and small values of γ. Simulation results show that, unlike the previous work, EMSE via our approach is valid for a vide range of γ. Also, the derived upper bounds are very close to the simulations in their respective ranges.

APPENDIX: Moments of the Random Variable Z

We evaluate the required moments of Z by first evaluating its CDF using the approach of [14, 15] and are found to be

$$E[Z] = \frac{1}{\gamma} - \sum_{m=1}^{M}\left[\frac{\mathbb{E}_2(\frac{\gamma}{f_m})e^{\frac{\gamma}{f_m}}}{\gamma\prod_{\substack{i=1\\ \neq m}}^{M}[1-\frac{f_i}{f_m}]}\right] \quad (28)$$

and

$$E[Z^2] = \frac{1}{\gamma^2} - \sum_{m=1}^{M}\left[\frac{e^{\frac{\gamma}{f_m}}\left(\mathbb{E}_2(\frac{\gamma}{f_m}) + \mathbb{E}_3(\frac{\gamma}{f_m})\right)}{\gamma^2\prod_{\substack{i=1\\ \neq m}}^{M}[1-\frac{f_i}{f_m}]}\right] \quad (29)$$

9. REFERENCES

[1] E. Eleftheriou and D. Falconer, "Tracking properties and steady-state performance of RLS adaptive filter algorithms," *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 34, no. 5, pp. 1097–1110, 1986.

[2] L. L. L. Guo and P. Priouret, "Performance analysis of the forgetting factor rls algorithm," *Int. J. Adaptive Contr. Signal Processing*, vol. 7, pp. 525–537, 1993.

[3] P. S. R. Diniz and M. G. Siqueira, "Analysis of the qr-rls algorithm for colored-input signals," *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, 1995.

[4] G. V. Moustakides, "Study of the transient phase of the forgetting factor rls," *IEEE Transactions on Signal Processing*, vol. 45, no. 10, pp. 2468–2476, 1997.

[5] S. S. Haykin, *Adaptive filter theory*. Upper Saddle River, N.J.: Prentice Hall, 2002.

[6] S. M. Y. Engel and R. Meir, "The kernel recursive least-squares algorithm," *IEEE Transactions on Signal Processing*, vol. 52, no. 8, pp. 2275–2285, Aug. 2004.

[7] M. L. Chebolu and S. K. Jayaweera, "Energy efficiency analysis of an rls-based adaptive signal processing algorithm for energy aware sensor networks," *Proceedings of 2005 International Conference on Intelligent Sensing and Information Processing*, Jan. 2005.

[8] X.-D. Z. Zi-Zhe Ding and Y.-T. Su, "Study of the transient phase of the forgetting factor rls," *IEEE Signal Processing Letters*, vol. 14, pp. 1–4, 2006.

[9] A. H. Sayed, *Adaptive Filters*. Wiley-IEEE Press, 1 ed., April 2008.

[10] A. Vakili and B. Hassibi, "A stieltjes transform approach for analyzing the RLS adaptive filter," in *2008 46th Annual Allerton Conference on Communication, Control, and Computing*, pp. 432–437, 2008.

[11] M. Pajovic and J. C. Preisig, "Performance analysis of the least squares based lti channel identification algorithm using random matrix methods," *49th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pp. 516–523, Monticello, 2011.

[12] G. Mateos and G. B. Giannakis, "Distributed recursive least-squares: Stability and performance analysis," *IEEE Transactions on Signal Processing*, vol. 60, pp. 3740–3754, April 2012.

[13] T. Y. Al-Naffouri and A. H. Sayed, "Transient analysis of data normalized adaptive filters," *IEEE Transactions on Signal Processing*, vol. 51, no. 3, pp. 639–652, Mar. 2003.

[14] T. Al-Naffouri and B. Hassibi, "On the distribution of indefinite quadratic forms in gaussian random variables," in *IEEE International Symposium on Information Theory, 2009. ISIT 2009*, pp. 1744–1748, 2009.

[15] T. Y. Al-Naffouri, M. Moinuddin, N. Ajeeb, B. Hassibi, and A. L. Moustakas, "On the Distribution of Indefinite Quadratic Forms in Gaussian Random Variables," *IEEE Transactions on Communications*, vol. 64, no. 1, pp. 153–165, 2016.