

# EVOLUTIONARY SPECTRA BASED ON THE MULTITAPER METHOD WITH APPLICATION TO STATIONARITY TEST

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## ABSTRACT

In this work, we propose a new inference procedure for understanding non-stationary processes, under the framework of evolutionary spectra developed by Priestley. Among various frameworks of modeling non-stationary processes, the distinguishing feature of the evolutionary spectra is its focus on the physical meaning of frequency. The classical estimate of the evolutionary spectral density is based on a double-window technique consisting of a short-Fourier transform and a smoothing. However, smoothing is known to suffer from the so-called bias leakage problem. By incorporating Thomson's multitaper method that was originally designed for stationary processes, we propose an improved estimate of the evolutionary spectral density, and analyze its bias/variance/resolution tradeoff. As an application of the new estimate, we further propose a non-parametric rank-based stationarity test, and provide various experimental studies.

**Index Terms**— Non-stationary Processes, Evolutionary Spectra, Spectral Analysis, Multitaper Method, Stationarity Test.

## 1. INTRODUCTION

Nonstationary processes are common across a variety of areas and serve as a natural generalization of the classical wide-sense stationary processes. Because of their wide range of applications, they have been an active research area in many different areas including statistics, neuroscience, and economics.

However, the intrinsic complexity of the non-stationarity precludes a unique way of modeling the non-stationary processes. Various frameworks have been developed over the past few decades: instantaneous power spectra [1], evolutionary spectra [2], Wigner-Ville spectral analysis [3], locally stationary processes [4], and local cosine basis [5] among others. In this work, we adopt the *evolutionary spectra* framework developed by Priestley [2], [6] and his colleagues [7], [8], which takes a spectral analysis approach and is one of the

first frameworks on modeling non-stationary processes. The appealing aspect of this framework is its emphasize on the physical meaning of frequency, while generalizing the spectral representation of the stationary processes to that of the non-stationary processes [9]. Detailed discussions on the different frameworks can be found in [9] and [10] and the references therein.

The estimation procedure of the evolutionary spectra is based on a so-called double-window technique, consisting of a short Fourier transform and smoothing. However, the smoothing step is known nowadays to suffer from bias leakage. To overcome this problem, tapering methods have been developed and the *multitaper method* by Thomson [11] is one of the most widely methods. In a recent work by Abreu and Romero [12], for stationary processes, the mean squared error (MSE) of the spectral estimate based on the multitaper method is characterized. In this work, the analysis is extended to the evolutionary spectra framework, which shows that the non-stationarity calls for additional considerations of the bias/variance/resolution tradeoff. As an application of the estimate, a non-parametric rank-based stationarity test is proposed and compared with the stationarity test investigated by Priestley and Subba Rao in [7] but using the multitaper method instead of smoothing.

Due to the space limit, all the proofs and a few simulations are omitted in this paper. The full version of this work can be found in [13].

## 2. PRIESTLEY'S EVOLUTIONARY SPECTRA FRAMEWORK

### 2.1. Brief Review of the framework

In [2], the main focus is the continuous time setting and the discrete time setting follows immediately. In this work, we will focus on the discrete time setting. In the following, we first briefly review the evolutionary spectra framework. Consider a class of non-stationary processes  $\{X(t)\}$ , with  $E[X(t)] = 0$  and  $E[X^2(t)] < \infty$  for  $t \in \mathbb{Z}$ , such that

$$X(t) = \int_{-\pi}^{\pi} \phi_t(w) dZ(w), t \in \mathbb{Z}, \quad (1)$$

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for some family  $\mathcal{F}$  of functions  $\phi_t(w)$  (defined on  $[-\pi, \pi]$  indexed by  $t$ ) and a measure  $\mu(w)$ , where  $Z(w)$  is an orthogonal process with  $E |dZ(w)|^2 = d\mu(w)$ . If there exists a family  $\mathcal{F}$  such that

$$X(t) = \int_{-\pi}^{\pi} e^{iwt} A_t(w) dZ(w), t \in \mathbb{Z}, \quad (2)$$

where, for fixed  $w$ , the (generalized) Fourier transform  $H_w(v)$  of  $h_w(t) := A_t(w)$  (viewed as a function of  $t$ ) has an absolute maximum at the origin, then  $\{X(t)\}$  is called an *oscillatory process*, and the evolutionary spectrum at time  $t$  with respect to  $\mathcal{F}$  is  $dF_t(w) = |A_t(w)|^2 d\mu(w)$ .

Throughout this paper, assume that  $\mu(w)$  is absolutely continuous with respect to Lebesgue measure. Thus the *evolutionary spectral density at time  $t$*  is  $f_t(w) = |A_t(w)|^2 \frac{d\mu(w)}{dw}$ . Normalize  $A_t(w)$  so that for all  $w$ ,  $A_0(w) = 1$ , which implies that  $d\mu(w)$  represents the evolutionary spectrum at  $t = 0$  and  $|A_t(w)|^2$  represents the change relative to  $t = 0$ . Let  $B_{\mathcal{F}}(w) := \int_{-\pi}^{\pi} |v| |H_w(v)| dv$ , and each family  $\mathcal{F}$  is called *semi-stationary* if  $B_{\mathcal{F}}(w)$  is bounded for all  $w$ . Then  $B_{\mathcal{F}} = (\sup_w B_{\mathcal{F}}(w))^{-1}$  is called the *characteristic width of  $\mathcal{F}$* . A *semi-stationary process*  $\{X(t)\}$  is defined as one that can be represented as (1) with respect to a semi-stationary family  $\mathcal{F}$ . Let  $\mathcal{C}$  denote the class of semi-stationary families such that  $\{X(t)\}$  can be represented as (2). Then  $B_X = \sup_{\mathcal{F} \in \mathcal{C}} B_{\mathcal{F}}$  is called the *characteristic width of  $\{X(t)\}$* .

In the double-window technique, the first window is for the short-window Fourier transform and the second window is for smoothing. In this work, however, the second window will be replaced by the multitaper method as smoothing is known to suffer from the bias leakage problem (see Section 4 for details). The width of the first window  $\{g(u), u \in \mathbb{R}\}$  is defined as  $B_g := \sum_{u=-\infty}^{\infty} |u| |g(u)|$ . Let  $G(w)$  denote the Fourier transform of  $g(u)$ . Assume that  $g(u)$  is square integrable and normalized.

### 3. UNBIASED ESTIMATE OF THE EVOLUTIONARY SPECTRA

In this section, we first revisit Theorem 8.1 [2] and provide an alternative and simplified proof of it as in Proposition 2 in this section. The main difference is that the *pseudo  $\delta$ -function* argument (see Definition 1) is directly applied to the time domain instead of the frequency domain as in [2]. This alternative approach facilitates the analysis of the variance in Section 4.2 and highlight the relationship between the window choice  $g(u)$  and the estimate  $|J_t(w)|^2$  of  $f_t(w)$ . First introduce  $J_t(w)$  as follows, for fixed  $t \in \mathbb{Z}$  and  $w \in [-\pi, \pi]$ ,

$$\begin{aligned} J_t(w) &= \sum_{u=-\infty}^{\infty} g(u-t) X(u) e^{-i w u} \\ &= \int_{-\pi}^{\pi} e^{-i(w-\lambda)t} \sum_{v=-\infty}^{\infty} g(v) A_{v+t}(\lambda) e^{-i(w-\lambda)v} dZ(\lambda). \end{aligned}$$

For two nonnegative functions  $f(x)$  and  $g(x)$ ,  $f(x) = O(g(x))$  means there exists some constant  $0 < C < \infty$  such that  $f(x) \leq Cg(x)$  for sufficiently large  $x$ . The analysis of the spectra estimate of  $f_t(w)$  in [2] depends on an approximation called *pseudo  $\delta$ -function* and the discrete counterpart can be defined as below.

**Definition 1** Consider two functions  $a(\cdot) : \mathbb{Z} \rightarrow \mathbb{R}$  and  $b(\cdot) : \mathbb{Z} \rightarrow \mathbb{R}$ . Then  $a(u)$  is a *pseudo  $\delta$ -function of order  $\epsilon$*  with respect to  $b(u)$  if, for any  $t \in \mathbb{Z}$ , there exists  $\epsilon$  not depending on  $k$  s.t.  $|\sum_{u=-\infty}^{\infty} a(u)b(u+t) - b(t) \sum_{u=-\infty}^{\infty} a(u)| < \epsilon$ .

**Lemma 1** For family  $\mathcal{F}^*$ ,  $a(u) := g(u)e^{-i w u}$  is a *pseudo  $\delta$ -function of  $b(u) := A_u(w)$  with order  $O(B_g/B_X)$* .

**Proposition 1**

$$\begin{aligned} J_t(w) &= \int_{-\pi}^{\pi} A_t(\lambda) G(w-\lambda) e^{-i(w-\lambda)t} dZ(\lambda) \\ &\quad + O(B_g/B_X) \int_{-\pi}^{\pi} e^{-i(w-\lambda)t} dZ(\lambda), \end{aligned}$$

where  $G(w) = \sum_{v=-\infty}^{+\infty} g(v) e^{-i w v}$ .

**Proposition 2**

$$E[|J_t(w)|^2] = \int_{-\pi}^{\pi} |G(w-\lambda)|^2 f_t(\lambda) d\lambda + O(B_g/B_X).$$

Given a sample record  $\{X(0), X(1), \dots, X(T-1)\}$  of length  $T$ , for  $0 \leq t \leq T-1$ , let  $U_t(w) = \sum_{u=0}^{T-1} g(u-t) X(u) e^{-i w u}$ . Assume that  $B_g \ll B_X \ll T$ , then for  $t \gg 0$ ,  $U_t(w)$  becomes almost identical to  $J_t(w)$  and the end effects are negligible. This can be made precise by further assuming that  $g(u)$  is time-limited, i.e.,  $g(u) = 0$  for  $|u| > N$  for some  $N$ . Thus we have for  $t > N/2$ ,  $E[|U_t(w)|^2] = \int_{-\pi}^{\pi} |G(w-\lambda)|^2 f_t(\lambda) d\lambda + O(B_g/B_X)$ .

For stationary processes, one aims to design tapers so that the spectra is concentrated between  $[-W, W]$  with  $2\pi/T < W < \pi$ . We will refer to  $W$  as the resolution of the estimate. For the evolutionary spectra, however, additional constraint  $O(B_g/B_X)$  is crucial to the performance of the estimate.

### 4. ESTIMATE BASED ON THE MULTITAPER METHOD

#### 4.1. Thomson's Multitaper Method

Consider  $N$  sample records  $\{X(0), \dots, X(N-1)\}$ . Assume that the sampling frequency is 1, then for a sequence of length  $N$ , the fundamental frequency is  $2\pi/N$  and the Nyquist frequency is  $\pi$ . For  $2\pi/N < W < \pi$ , one wishes to find sequences with spectral densities concentrated over  $[-W, W]$ . The solution turns out to be a set of sequences  $v_k(N, W; u)$ ,  $0 \leq u \leq N-1$ ,  $0 \leq k \leq N-1$ , which satisfy a certain eigenvalue equation. These  $N$  eigenvectors  $v_k(N, W; u)$

are called the discrete prolate spheroidal and they are ordered by their eigenvalues  $1 > \lambda_0(N, W) > \lambda_1(N, W) > \dots > \lambda_{N-1}(N, W) > 0$ . Only the first  $K = \lfloor 2NW \rfloor$  eigenvalues are close to 1.

The discrete prolate spheroidal wave functions are denoted by  $V_k(N, W; \lambda)$ . For simplicity of notation, we suppress  $N$  and  $W$  and write  $v_k(u) := v_k(N, K; u)$ ,  $V_k(\lambda) := V_k(N, W; \lambda)$ , and  $\lambda_k := \lambda_k(N, K)$ . These  $K$  functions satisfy two types of orthogonality over  $[-W, W]$  and  $[-\pi, \pi]$ , respectively. Consider the average of the  $K$  tapered estimates,  $\frac{1}{K}\rho_K(\lambda) := \frac{1}{K} \sum_{k=0}^{K-1} |V_k(\lambda)|^2$ .

It has been observed numerically that  $(1/K)\rho_K(\lambda)$  is close to  $(1/2W)\mathbf{1}_{[-W, W]}(\cdot)$  by Thomson [11], which is justified recently by Abreu and Romero [12] as given below.

**Theorem 1 ([12])** Let  $N \geq 2$  denote the length of the sequence,  $2\pi/N < W < \pi$  and set  $K := \lfloor 2NW \rfloor$ . Then

$$\left\| \frac{1}{K}\rho_K(\cdot) - \frac{1}{2W}\mathbf{1}_{[-W, W]}(\cdot) \right\|_1 = O\left(\frac{\log N}{K}\right).$$

In the following section, we apply this result to analyze the performance of the multitaper method for semi-stationary processes.

## 4.2. Estimate of the Evolutionary Spectra based on the Multitaper Method

For stationary processes, the bias and variance of the multitaper spectral estimate has been investigated [14], [15], [16]. In this section, we investigate its performance for semi-stationary processes. Let  $g(u)$  be a time-limited function, i.e.,  $|g(u)| = 0$ , for  $|u| > (N-1)/2$ , where  $N$  is assumed to be odd. Apply the multitaper method on  $\{X(t), 0 \leq t \leq T-1\}$  with  $g_k(u) := v_k(u + (N-1)/2)$  for  $0 \leq k \leq K-1$ , then for  $t > (N-1)/2$  we have

$$\begin{aligned} U_t^{(k)}(w) &= \sum_{u=0}^{T-1} g_k(u-t)X(u)e^{-i w u} \\ &= \sum_{u=t-(N-1)/2}^{t+(N-1)/2} g_k(u-t)X(u)e^{-i w u}. \end{aligned}$$

From Proposition 2,

$$E[|U_t^{(k)}(w)|^2] = \int_{-\pi}^{\pi} |G_k(w-\lambda)|^2 f_t(\lambda) d\lambda + O(B_{g_k}/B_X).$$

The estimate of  $f_t(w)$  is the average of  $|U_t^{(k)}(w)|^2$ ,  $\hat{f}_t^K(w) := \frac{1}{K} \sum_{k=0}^{K-1} |U_t^{(k)}(w)|^2$  and the mean of the estimate is shown below.

**Theorem 2** For a semi-stationary process  $\{X(t)\}$ , the expectation of its evolutionary spectral density estimate using the multitaper method is given by,

$$E[\hat{f}_t^K(w)]$$

$$\begin{aligned} &= \frac{1}{K} \int_{-\pi}^{\pi} \rho_K(w-\lambda) f_t(\lambda) d\lambda + O(B_g^{(K)}/B_X) \\ &= \frac{1}{K} \int_{-\pi}^{\pi} \rho_K(\lambda) f_t(w-\lambda) d\lambda + O(B_g^{(K)}/B_X), \end{aligned}$$

where  $\rho_K(\lambda) := \sum_{k=0}^{K-1} |G_k(\lambda)|^2$ ,  $B_g^{(K)} := \max_k B_{g_k}$ .

There is a bias/variance/resolution tradeoff for the estimate  $\hat{f}_t^K(w)$ . The bias can be bounded by invoking Theorem 1 and assuming that  $\|f_t(w)\|_{\infty}$  is bounded for all  $t$ .

**Theorem 3**  $|Bias(\hat{f}_t^K(w))| = O(\log N/K + W^2 + B_g^{(K)}/B_X)$ .

When  $X(t)$  is a normal process, the variance of  $\hat{f}_t^K(w)$  can be characterized as follows.

**Theorem 4**  $Var(\hat{f}_t^K(w)) = O(1/K + B_g^{(K)}/B_X)$ .

From Theorem 3 and 4, the mean squared error (MSE) of  $\hat{f}_t^K(w)$  is given by the following.

**Corollary 1**  $MSE(\hat{f}_t^K(w)) = O((\log N/K)^2 + W^4 + 1/K + B_g^{(K)}/B_X)$ .

## 5. STATIONARITY TEST

The evolutionary spectral density estimate suggests a natural statistical test for the stationarity of a process, as first discussed in Priestley's paper [2] and later investigated by Priestley and Subba Rao (PSR test) in [7]. In a recent package developed by Constantine and Percival [17], smoothing is replaced by the multitaper method. This modified PSR test has been served as a baseline to when compared with other stationarity tests, e.g., in [18]. In this section, a non-parametric version of the stationarity test is proposed, which is based on the Friedman test [19, 20] and is robust to the underlying distribution. It serves as a complementary test to the existing stationarity tests.

### 5.1. PSR Stationarity Test with the Multitaper method

It is a common practice to take the logarithm of the estimate, which stabilizes its variance. Let  $f_t(w)$  denote the evolutionary spectral density of a semi-stationary process  $\{X(t) : 0 \leq t \leq T-1\}$ . Consider the estimate  $\hat{f}_t^K(w)$  based on the multitaper method as in Section 4. Take the logarithm transform of the estimates, which stabilize the variances of the estimates and let  $Y_{ij} := \log \hat{f}_{t_i}^K(w_j)$ . Moreover, to apply the two-way analysis of variance (ANOVA) test, it has to be assumed that the distribution of  $\log \hat{f}_t^K(w)$  is approximately normal [7]. It can be shown that  $W_{ij} := Y_{ij} - \psi(K)$  is approximately distributed according to normal distribution with mean 0 and variance  $\sigma^2 = \psi'(K)$  for  $K \geq 5$  (see [21] and [22] for details).

The approximately independence in time is by choosing non-overlapping short windows of length  $N$  and the approximate independence in frequency is by choosing frequencies that are  $2\pi(K + 1)/(N + 1)$  apart. Now the problem reduces to a two-way ANOVA test for  $W_{ij}$  for  $i \in [1 : I]$  and  $j \in [1 : J]$ , where  $I = \lfloor T/N \rfloor$  and  $J$  is the number of frequencies chosen  $2\pi(K + 1)/(N + 1)$  apart. Details of the test can be found in [7].

### 5.2. A Non-parametric Stationarity Test

There are two main assumptions of the two-way ANOVA test: (1) the samples are uncorrelated and (2) the residuals are normally distributed. There has been extensive research on the robustness of the assumptions of ANOVA test. In the PSR test, the test results are more reliable when the degrees of freedom of time and frequency are large. On the other hand, non-parametric test, e.g., the rank-based Friedman test [19], [20], has an edge when the number of test samples is relatively small.

We now describe the non-parametric test, which will be referred to as *rank-based stationarity test* or *RS test* in short. Take  $\{W_{ij}\}$  introduced in the previous section. In the time-frequency table filled by  $\{W_{ij}\}$ , rank the elements in each column in an increasing order (i.e., 1 corresponds to the smallest element) to form a table of ranks:  $\{R_{ij}\}$ . Whenever there is a tie among  $k$  elements in the same column, assign the mean rank of the  $k$  elements. Similar to the two-way ANOVA test, let  $R_{.}$  denote the mean rank of all ranks, denote  $R_{i.}$  the mean rank of row  $i$ . The sum of square of ranks  $SS_R$  is  $SS_R = J \sum_i (R_{i.} - R_{.})^2$ . The test statistics  $t_R = SS_R / \text{const}$ , where  $\text{const} = I(I + 1)/12$ . It is known that  $t_R$  is (approximately) distributed according to  $\chi^2_{I-1}$  [19], [20].

### 5.3. Simulations

Due to the space limit, we only present synthetic data simulations in this paper. Read data simulations can be found in [13]. The performance of a test is evaluated based on its empirical *size* and *power* values. Generate  $M = 1000$  sample paths/realizations each with length  $T = 512$  and let the nominal size of the test be 0.05. The null hypothesis  $H_0$  is that the process is stationary and the alternative hypothesis  $H_1$  is that the process is not stationarity. The implementation details can be found in [13].

For the size comparison, we generate sample paths from various stationary processes and count the number of rejections of the null hypothesis. Consider the following set of stationary autoregressive and moving-average (ARMA) models used in [23] The noise term  $Z(t)$  is distributed according to  $\mathcal{N}(0, 1)$ .

- (a) i.i.d. standard normal
- (b) AR(1):  $X(t) = 0.9X(t - 1) + Z(t)$ .

- (c) AR(1):  $X(t) = -0.9X(t - 1) + Z(t)$ .
- (d) MA(1):  $X(t) = Z(t) + 0.8Z(t - 1)$ .
- (e) MA(1):  $X(t) = Z(t) - 0.8Z(t - 1)$ .
- (f) ARMA(1,2):  $X(t) = -0.4X(t) + Z(t) - 0.8Z(t - 1)$ .
- (g) AR(2):  $X(t) = \alpha_1 X(t - 1) + \alpha_2 X(t - 2) + Z(t)$  with  $\alpha_1 = 1.385929$  and  $\alpha_2 = -0.9604$  (from [24]).

**Table 1.** Empirical size comparison (%)

models	PSR	RS
(a)	11.1	1.9
(b)	17.7	2.9
(c)	11.2	2.7
(d)	12.7	2.5
(e)	14.8	2.9
(f)	15.3	2.6
(g)	78.7	6.1

For the power comparison, we generate sample paths from semi-stationary processes and count the number of acceptances of the null hypothesis. We focus on the uniformly modulated processes as in [2, 7]. The following model is from [7],

$$X(t) = e^{(n-T/2)^2/2a^2} Y(t), \quad (3)$$

where  $a = 200$  and  $Y_t = 0.8Y_{t-1} - 0.4Y_{t-2} + Z_t$  with  $Z_t \sim \mathcal{N}(0, 100^2)$ . For all the models from Table 1, generate uniformly modulated processes by multiplying each of them with  $e^{(n-T/2)^2/2a^2}$ . To make the numbering consistent with Table 1, these models are also numbered from (a) to (g) and model (3) will be numbered as (h) in the table below.

**Table 2.** Empirical power comparison (%)

models	PSR	RS
(a)	96.7	88.4
(b)	96.8	82.5
(c)	97.3	88.9
(d)	96.4	88.1
(e)	97.3	87.1
(f)	97.1	86.2
(g)	98.3	76.7
(h)	96.4	88.4

Since the empirical size of RS is smaller than that of PSR but the empirical power is larger, RS is a more conservative test compared with PSR.

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