HYPERCOMPLEX TENSOR COMPLETION WITH CAYLEY-DICKSON SINGULAR VALUE DECOMPOSITION

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ABSTRACT

Expressing multidimensional information as a value in hypercomplex number systems (e.g., quaternion, octonion, etc.) has great potential, in signal processing applications, to enjoy their nontrivial algebraic benefits which are not available in standard real or complex vector systems. Strategic utilizations of such benefits would include, e.g., hypercomplex singular value decomposition (SVD) and low rank approximation of matrices. In this paper, as powerful mathematical tools for wider signal processing applications, we first propose novel definitions of SVD and best low rank approximation of matrices based on algebraic translations of Cayley-Dickson (C-D) number systems. We then derive an algorithmic solution to hypercomplex tensor completion problem based on a convex optimization technique. Numerical experiments in a scenario of color tensor completion problem show that the proposed algorithm recovers much more faithfully the original color information, masked randomly by noise, than a part-wise real tensor completion algorithm.

Index Terms— Hypercomplex number, Cayley-Dickson construction, singular value decomposition, tensor completion, convex optimization

1. INTRODUCTION

Multidimensional information arises naturally in many areas of engineering and science since almost all observations have many attributes. Utilizing hypercomplex number system for representing such multidimensional information is one of effective ways since some physical operations such as rotation can be represented simpler than ordinary real-valued multidimensional vectors. Therefore, it has been used in many areas such as computer graphics [1] and robotics [2, 3] wind forecasting [4, 5, 6] and noise reduction in acoustic systems [7]. In the statistical signal processing field, effective utilization of the m-dimensional Cayley-Dickson number system (C-D number system) [8, 9], which is a standard class of hypercomplex number systems [10], including, e.g., real $\mathbb{R}$, complex $\mathbb{C}$, quaternion $\mathbb{H}$, octonion $\mathbb{O}$ and sedenion $\mathbb{S}$ etc., have been investigated.

Hypercomplex tensors, whose entries are represented by hypercomplex numbers, can play important roles in modeling real world object. For example, in 3D object modeling, each point in 3-dimensional space can have multiple attribute such as color, material, intensity, and so on, and each attribute may has correlation with other attributes. It can be modeled as three dimensional hypercomplex tensor since the correlation of each attribute may be realized as the nontrivial algebraic structure of hypercomplex number systems. Moreover recovering hypercomplex tensor from incomplete observation will be more and more important by the popularization of 3D printer [11], virtual reality, medical imaging etc.

In real or complex domain, one of the standard approximations is the low rank approximation, which can be realized by the truncated singular value decomposition (SVD). However, in general C-D domain, the SVD has not yet been well-established since eigenvalue problems are known to be hard problems [12, 13]. In C-D domain, left and right scalar multiplications are distinct since commutativity of product does not hold in general. Therefore, we have to treat two kinds of eigenvalues, left and right eigenvalues separately. In several previous works, special procedures have been used for calculating eigenvalues in a certain C-D domain. For example, quaternion right eigenvalues are computed by reducing them to the equivalent complex eigenvalues with quaternion-complex matrix translations (complex adjoints) [12, 14, 15]. However, this procedure cannot to be applied to the left eigenvalue problems and hard to be generalized for higher dimensional C-D domain. This situation could be a burden not only to establish low rank approximation frameworks but also to design further advanced algorithms, e.g., tensor completion which utilize eigenvalues and SVD in C-D domain.

In this paper, to establish a low rank hypercomplex tensor completion framework, first, we propose a computation framework for Cayley-Dickson singular value decomposition. To achieve it, we introduce a new notion $\mathbb{R}$-*eigenvalue* for clarifying the relation between the eigenvalues of C-D matrices and real ones. The $\mathbb{R}$-eigenvalue is defined based on the algebraic real translation of C-D linear systems proposed in [16] and can be calculated for general C-D matrices. We also clarify the relation between the $\mathbb{R}$-eigenvalues and existing well-defined quaternion right eigenvalues. Then, we propose a definition of hypercomplex singular value decomposition (SVD) based on the calculation of $\mathbb{R}$-eigenvalues. We also clarify the relation between the proposed SVD, ranks and the known results [17] in well-studied quaternion case. Moreover, we show that the proposed SVD can be utilized for hypercomplex low rank approximation techniques. Utilizing proposed frameworks, we next propose hypercomplex low N-rank tensor completion algorithm based on a convex optimization technique known as *Douglas-Rachford splitting* [18]. The proposed algorithm can be derived straightforwardly with replacing SVD procedure in a tensor completion algorithm [19] and can be applied to general C-D domains.

Numerical experiments including a scenario of color tensor completion problem in quaternion domain show that the proposed algorithm successfully utilizes the correlations of each color space to recover much more faithfully the original color information, masked randomly by noise, than a part-wise real tensor completion algorithm.

2. PRELIMINARIES

2.1. Hypercomplex Number System

Let $\mathbb{N}$ and $\mathbb{R}$ be respectively the set of all non-negative integers and the set of all real numbers. Define an m-dimensional *hypercomplex* number $\mathbb{A}_m$ ($m \in \mathbb{N} \setminus \{0\}$) expanded on the real vector space [8]

$$a := a_1\mathbf{i}_1 + a_2\mathbf{i}_2 + \cdots + a_m\mathbf{i}_m \in \mathbb{A}_m,\ a_1, \ldots, a_m \in \mathbb{R} \quad (1)$$

based on imaginary units $\mathbf{i}_1, \ldots, \mathbf{i}_m$, where $\mathbf{i}_1 = 1$ represents the vector identity element. Any hypercomplex number is expressed uniquely in the form of (1). A *multiplication table* defines the products of any imaginary unit with each other or with itself (e.g., $\mathbf{i}_1^2 = 1, \mathbf{i}_2^2 = -1$ and $\mathbf{i}_1\mathbf{i}_2 = \mathbf{i}_2\mathbf{i}_1 = \mathbf{i}_2$ for $\mathbb{A}_2(=: \mathbb{C})$). We also define the *conjugate* of hypercomplex number a as

$$a^* := a_1\mathbf{i}_1 - a_2\mathbf{i}_2 - \cdots - a_m\mathbf{i}_m. \quad (2)$$

In this paper, we consider the hypercomplex number systems which are constructed recursively by the *Cayley-Dickson construction* (*C-D construction* or *C-D (doubling) procedure*) [8]. The C-D construction is a standard method for extending a number system. This method has been used in extending $\mathbb{R}$ to $\mathbb{C}$, $\mathbb{C}$ to $\mathbb{H}$ and $\mathbb{H}$ to $\mathbb{O}$. By using the C-D construction, an m-dimensional hypercomplex number $\mathbb{A}_m$ is extended to $\mathbb{A}_{2m}$ [8, 9] as

$$z := x + y\mathbf{i}_{m+1} \in \mathbb{A}_{2m}, \quad x, y \in \mathbb{A}_m,$$

where $i_{m+1} \notin \mathbb{A}_m$ is the additional imaginary unit for doubling the dimension of $\mathbb{A}_m$ satisfying $\mathbf{i}_{m+1}^2 = -1$, $\mathbf{i}_1\mathbf{i}_{m+1} = \mathbf{i}_{m+1}\mathbf{i}_1 = \mathbf{i}_{m+1}$ and $\mathbf{i}_v\mathbf{i}_{m+1} = -\mathbf{i}_{m+1}\mathbf{i}_v =: \mathbf{i}_{m+v}$ for all $v = 2, \ldots, m$. For example, the real number system ($\mathbb{A}_1 :=$) $\mathbb{R}$ is extended into complex number system $\mathbb{C}$ ($= \mathbb{A}_2$) by the C-D construction. Note that the value of m is restricted to the form of 2^n ($n \in \mathbb{N}$). The hypercomplex number systems constructed inductively from the real number by the C-D construction are called *Cayley-Dickson number system* (*C-D number system*). The imaginary units appeared in the C-D number systems have many characteristic properties [16] such as $\mathbf{i}_\alpha^2 = -1$ and $\mathbf{i}_\alpha\mathbf{i}_\beta = -\mathbf{i}_\beta\mathbf{i}_\alpha (\alpha \neq \beta)$ for all $\alpha, \{2, \ldots, m\}$. These properties ensures $aa^* = \sum_{\ell=1}^m a_\ell^2 \geq 0$ for any $a \in \mathbb{A}_m$ in (1) and $a^* \in \mathbb{A}_m$ in (2) and enable us to define the absolute values of C-D number a as $|a| := \sqrt{aa^*}$.

A representative example of hypercomplex number is the *quaternion* $\mathbb{H}$. The quaternion number system is constructed from the complex number system by using the C-D construction. A quaternion number is a 4-dimensional hypercomplex which is defined as

$$q = q_1 + q_2\imath + q_3\jmath + q_4\kappa \in \mathbb{H}, \quad q_1, q_2, q_3, q_4 \in \mathbb{R}$$

with the multiplication table:

$$\imath\jmath = -\jmath\imath = \kappa, \; \jmath\kappa = -\kappa\jmath = \imath, \; \kappa\imath = -\imath\kappa = \jmath, \quad \imath^2 = \jmath^2 = \kappa^2 = -1 \tag{3}$$

by letting $M = 4$, $\mathbf{i}_1 = 1$, $\mathbf{i}_2 = \imath$, $\mathbf{i}_3 = \jmath$ and $\mathbf{i}_4 = \kappa$. From (3), quaternions are not *commutative*, i.e., $pq \neq qp$ for $p, q \in \mathbb{H}$ in general.

The octonion $\mathbb{O}$ can be constructed from the quaternion $\mathbb{H}$ by the C-D construction. Note that the multiplication in $\mathbb{O}$ is neither commutative nor *associative*, i.e., $pq \neq qp$ and $(pq)r \neq p(qr)$ for $p, q, r \in \mathbb{O}$ in general [10]. For the octonion multiplication table, see, e.g., [10].

We also define $\mathbb{A}_m^N := \{[x_1, \ldots, x_N]^\top | x_i \in \mathbb{A}_m \, (i = 1, \ldots, N)\}$ for $\forall N \in \mathbb{N} \setminus \{0\}$, where $(\cdot)^\top$ stands for the transpose. Define $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_{\mathbb{A}_m^N} := \boldsymbol{x}^H \boldsymbol{y} \in \mathbb{A}_m, \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{A}_m^N$ and $\|\boldsymbol{x}\|_{\mathbb{A}_m^N} := \langle \boldsymbol{x}, \boldsymbol{x} \rangle_{\mathbb{A}_m^N}^{1/2}$, $\forall \boldsymbol{x} \in \mathbb{A}_m^N$, where $(\cdot)^H$ denotes the *Hermitian transpose* of vectors or matrices (e.g., $\boldsymbol{x}^H := [x_1^*, \ldots, x_N^*]$ for $\boldsymbol{x} := [x_1, \ldots, x_N]^\top \in \mathbb{A}_m^N$, where $x_1 \ldots, x_N \in \mathbb{A}_m$). We also define the *addition* of two hypercomplex vectors $\boldsymbol{x} + \boldsymbol{y} := [x_1 + y_1, \cdots, x_N + y_N]^\top \in \mathbb{A}_m^N$ for $\boldsymbol{x}, \boldsymbol{y} (:= [y_1, \ldots, y_N]^\top) \in \mathbb{A}_m^N$. Let $\mathcal{S} := \mathbb{R}$, $\mathcal{S} := \mathbb{C}$ or $\mathcal{S} := \mathbb{A}_m$ ($m \geq 4$), and call the element of $\mathcal{S}$ *scalar*. If we define the *left scalar multiplication* as $\alpha\boldsymbol{x} := [\alpha x_1, \ldots, \alpha x_N]^\top \in \mathbb{A}_m^N$ for $\alpha \in \mathcal{S}$ and $\boldsymbol{x} \in \mathbb{A}_m^N$, we have $\alpha\boldsymbol{x} + \beta\boldsymbol{y} \in \mathbb{A}_m^N, \forall \alpha, \beta \in \mathcal{S}, \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{A}_m^N$. We can also define the *right scalar multiplication* $\boldsymbol{x}\alpha \in \mathbb{A}_m^N$ in a similar way.

2.2. Hypercomplex Eigenvalue Problems

Similar to the real and complex case, we can formally consider the eigenvalue problems in C-D domain. However, as seen in the quaternion case, the multiplication in C-D number systems is not commutative in general, that is, the left and right scalar multiplications are different, so we need to treat them separately.

In this paper, we respectively call a C-D valued scalar λ^ℓ (λ^r) $\in \mathbb{A}_m$ and a C-D valued vector $\boldsymbol{x}^\ell$ ($\boldsymbol{x}^r$) $\in \mathbb{A}_m^N$ a *left* (*right*) *eigenvalue* and a *left* (*right*) *eigenvector* provided that $\boldsymbol{A}\boldsymbol{x}^\ell = \lambda^\ell \boldsymbol{x}^\ell$ ($\boldsymbol{A}\boldsymbol{x}^r = \boldsymbol{x}^r \lambda^r$). Note that both left eigenvalue and eigenvector are distinct from right ones for a common C-D matrix $\boldsymbol{A} \in \mathbb{A}_m^{N \times N}$ in general.

Unfortunately, there has been known few cases where hypercomplex eigenvalues can be computed systematically. In almost all cases, even the existences of them are still unexplained to the best of our knowledge. In the quaternion domain, it is known that right eigenvalues always exist and indeed a well-defined method is available for computing quaternion right eigenvalues by reducing the quaternion right eigenvalue problem to the equivalent complex one [12, 14, 15]. On the other hand, the method used in the right eigenvalue problem cannot be used in the left one because of the lack of commutativity of multiplication. Wood proved that any $N \times N$ quaternion matrix has at least one left eigenvalue [20] but even for small size the left eigenvalues are still open problem in spite of many previous studies [13, 21, 22]. For example, it was proved that a 2×2 quaternion matrix may have one, two, or an infinite number of left eigenvalues [13] but the proof seems to be difficult to generalize for $N > 2$. For octonion and higher dimensional C-D domain, the general solution is available for very limited cases [23]. However, there are no systematic solutions even for right eigenvalue problem since the method for solving quaternion right eigenvalue problem cannot be generalized for higher dimensional cases because of the lack of associativity of multiplication.

3. C-D SINGULAR VALUE DECOMPOSITION

3.1. $\mathbb{R}$-eigenvalues and Their Properties

In this section, we introduce a new notion $\mathbb{R}$-*eigenvalue* which can be defined for general C-D matrix.

Definition 1 ($\mathbb{R}$-eigenvalues of C-D matrices)**.** For C-D matrix $\boldsymbol{A} \in \mathbb{A}_m^{N \times N}$, we respectively call a complex-valued scalar $\lambda \in \mathbb{C}$ and a complex-valued vector $\boldsymbol{x} \in \mathbb{C}^{mN}$ an $\mathbb{R}$-eigenvalue and $\mathbb{R}$-eigenvector provided that $\widetilde{\boldsymbol{A}}\boldsymbol{x} = \lambda\boldsymbol{x}$, where $\widetilde{\boldsymbol{A}} \in \mathbb{R}^{mN \times mN}$ is the non-trivial mapping of $\boldsymbol{A}$ defined in (5).

As discussed in Section 2.2, the difficulties for computing hypercomplex eigenvalues are mainly from the lack of commutativity and associativity of hypercomplex multiplications. On the other hand, any C-D matrix can be translated to real matrix without loss of any information by the algebraic translation introduced in Appendix. Once the translated real matrix is obtained from a C-D matrix, we are completely freed from any complexity of C-D number system. Moreover, translated real matrix is just real-valued matrix, so any C-D square matrix $\boldsymbol{A} \in \mathbb{A}_m^{N \times N}$ always has mN number of $\mathbb{R}$-eigenvalues in $\mathbb{C}$.

The most well-studied hypercomplex eigenvalue problem is the quaternion right eigenvalue problem. In general, the quaternion square matrix $\boldsymbol{A} \in \mathbb{H}^{N \times N}$ has an infinite number of right eigenvalues [12]. However, it is well-known that any $N \times N$ quaternion matrix $\boldsymbol{A}$ has exactly N right eigenvalues which are complex numbers with non-negative imaginary parts [14, 15]. These eigenvalues are said to be the *standard eigenvalues of* $\boldsymbol{A}$. The standard eigenvalues can be systematically computed by calculating eigenvalues of the *complex adjoint matrix* $\chi_{\boldsymbol{A}} := \begin{bmatrix} \boldsymbol{A}_r & \boldsymbol{A}_i \\ -\boldsymbol{A}_i^* & \boldsymbol{A}_r^* \end{bmatrix} \in \mathbb{C}^{2N \times 2N}$ of $\boldsymbol{A} := \boldsymbol{A}_r + \boldsymbol{A}_i\jmath$ ($\boldsymbol{A}_r, \boldsymbol{A}_i \in \mathbb{C}^{N \times N}$). Note that this calculation is available only for quaternion since the complex adjoints can be defined only for quaternion matrices. By calculating the $\mathbb{R}$-eigenvalues of the complex adjoint matrix, we have the relation between the $\mathbb{R}$-eigenvalues and the standard eigenvalues [14, 15] of quaternion matrices as follows:

Theorem 1 ($\mathbb{R}$-eigenvalue of quaternion matrices)**.** *Suppose that the quaternion square matrix* $\boldsymbol{A} \in \mathbb{H}^{N \times N}$ *has a standard eigenvalue* $\lambda \in \mathbb{C}$. *Then* $\boldsymbol{A}$ *also has* $\mathbb{R}$-*eigenvalues* λ *and* λ^* *with multiplicity 2. If* $\boldsymbol{A}$ *is Hermitian, it has real-valued* $\mathbb{R}$-*eigenvalues* λ *with multiplicity 4.*

3.2. Singular Value Decompositions and Ranks

In this section, we propose propose novel definitions of singular value decomposition (SVD) best low rank approximation of C-D matrices and clarify their properties.

Definition 2 (C-D singular value decomposition and $\mathbb{R}$-rank)**.** For a C-D matrix $\boldsymbol{A} \in \mathbb{A}_m^{M \times N}$, we call $\widetilde{\boldsymbol{A}} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top$ *C-D singular value decomposition*, where $\boldsymbol{U} \in \mathbb{R}^{mM \times mM}$ and $\boldsymbol{V} \in \mathbb{R}^{mN \times mN}$ are orthogonal real matrices and $\boldsymbol{\Sigma} := \operatorname{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0) \in \mathbb{R}^{mM \times mN}$ is a rectangular diagonal matrix with positive singular values $\sigma_1 \geq \cdots \geq \sigma_r (> 0)$ of $\widetilde{\boldsymbol{A}}$ on the diagonal. We denote $r = \operatorname{rank}^{\mathbb{R}}(\boldsymbol{A}) := \operatorname{rank}(\widetilde{\boldsymbol{A}}) \leq \max(mM, mN)$ and call it *$\mathbb{R}$-rank*.

Note that the mapping: $\widetilde{(\cdot)} : \mathbb{A}_m^{M \times N} \to \mathbb{R}^{mM \times nN}$ is defined in Appendix. Similar to the eigenvalue problem, the SVDs have not been well-established for almost C-D number systems. However, in the quaternion case, both the SVD and the rank of quaternion matrix are well-established [12]. Obviously, the $\mathbb{R}$-rank can be defined for general C-D matrices similar to the $\mathbb{R}$-eigenvalues. Moreover, the $\mathbb{R}$-rank has very strong relation to well-established ranks in C-D domain.

Lemma 1 (Relation between the $\mathbb{R}$-rank and original ranks in C-D domain)**.** *For complex ($m = 2$) or quaternion ($m = 4$) cases, it holds that* $\operatorname{rank}^{\mathbb{R}}(\boldsymbol{A}) = m\operatorname{rank}(\boldsymbol{A})$ *for all* $\boldsymbol{A} \in \mathbb{A}_m^{M \times N}$.

Lemma 1 implies that the $\mathbb{R}$-rank is equivalent to the product of the dimension of C-D number and the original rank and thus minimizing $\mathbb{R}$-rank is equivalent to the original rank at least in well-established complex and quaternion case. In this section, by passing through the Schmidt-Eckart-Young theorem [24], we propose a new best low rank approximation technique of C-D matrices.

Lemma 2 (Low $\mathbb{R}$-rank approximation of hypercomplex matrices)**.** *For a C-D matrix $\boldsymbol{A} \in \mathbb{A}_m^{M \times N}$ of $\mathbb{R}$-rank r, a best p $\mathbb{R}$-rank approximation is achieved by*

$$\min_{\substack{\boldsymbol{X} \in \mathbb{R}^{mM \times mN} \\ \operatorname{rank}^{\mathbb{R}}(\boldsymbol{X}) \leq p}} \left\| \widetilde{\boldsymbol{A}} - \boldsymbol{X} \right\|_F = \left\| \widetilde{\boldsymbol{A}} - \boldsymbol{A}_p \right\|_F = \sqrt{\sum_{i=p+1}^{r} \sigma_i^2},$$

where $\|\cdot\|_F$ is the Frobenius norm, $\widetilde{\boldsymbol{A}} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top$, $\boldsymbol{A}_p = \boldsymbol{U}\boldsymbol{\Sigma}_p\boldsymbol{V}^\top$ and $\boldsymbol{\Sigma} = \operatorname{diag}(\sigma_1, \ldots, \sigma_p, 0, \ldots, 0) \in \mathbb{R}^{mM \times mN}$.

By carefully selecting reduced $\mathbb{R}$-rank, the low $\mathbb{R}$-rank approximation also achieves the low rank approximation in the original sense for well-established complex and quaternion domain.

Theorem 2 (Relation to the known low rank approximations)**.** *Consider complex ($m = 2$) or quaternion ($m = 4$) cases. If $\boldsymbol{A}^\star \in \mathbb{R}^{mM \times mN}$ achieves the best low mp $\mathbb{R}$-rank approximation of $\boldsymbol{A} \in \mathbb{A}_m^{M \times N}$, then the reverted C-D matrix $\underset{\sim}{\boldsymbol{A}}^\star \in \mathbb{A}_m^{M \times N}$ (see Appendix) also achieves the best low p rank approximation of $\boldsymbol{A}$ in [17].*

This theorem implies that Lemma 2 is the generalization of the known result in [17].

4. HYPERCOMPLEX TENSOR COMPLETION

4.1. Formulations

We basically adopt the nomenclature of [25]. A tensor is the generalization of a matrix to higher dimension. In this paper we denote it by a calligraphic letter e.g., $\mathcal{X} \in \mathbb{A}_m^{N_1 \times \cdots \times N_n} =: \mathscr{T}^{\mathbb{A}_m}$.The *order* (also called *ways* or *modes*) n of tensor is the number of dimensions. *Fibers* are the higher-order analogue of matrix row and columns. A fiber is defined by fixing every index but one. The mode-k fibers are all vectors $\boldsymbol{x}_{i_1 \ldots i_{k-1} : i_{k+1} \ldots i_n}$ which are obtained by fixing the value of $\{i_1, \ldots, i_n\} \setminus i_k$. The *mode-$k$ unfolding* (also called *matricization* or *flattening*) of a tensor $\mathcal{X} \in \mathscr{T}^{\mathbb{A}_m}$ denoted by $\boldsymbol{X}_{(k)} \in \mathbb{A}_m^{N_k \times I_k}$ ($I_k = \prod_{\ell=1, \ell \neq k}^{n} N_\ell$) is a matrix obtained by concatenating all mode-k fibers along columns. The inner product of two same-sized tensors $\mathcal{X}, \mathcal{Y} \in \mathscr{T}^{\mathbb{A}_m}$ is the sum of the products of their entries, i.e., $\langle \mathcal{X}, \mathcal{Y} \rangle_{\mathscr{T}^{\mathbb{A}_m}} := \sum_{i_1=1}^{N_1} \cdots \sum_{i_n=1}^{N_n} x_{i_1 \ldots i_n} y_{i_1 \ldots i_n}$.

Algorithm 1: $\mathbb{A}_m$-Douglas-Rachford splitting for hypercomplex tensor completion ($\mathbb{A}_m$-DRS)

Input : $\mathcal{L}_\Omega, \Omega, \boldsymbol{b}, t_k, \lambda, c_\lambda, \gamma$
Output: Recovered tensor $\mathcal{X} \in \mathscr{T}^{\mathbb{A}_m}$
Initialize $k \leftarrow 0, \tau \leftarrow \gamma(n+1), \mathcal{X}_i^{(0)} \leftarrow \mathcal{O}$ $(\forall i = 0, \ldots, n)$;
while *not converged* **do**
  $\lambda \leftarrow c_\lambda \lambda$;
  $\overline{\mathcal{X}} \leftarrow \operatorname{mean}(\mathcal{X}_0^{(k)}, \ldots, \mathcal{X}_n^{(k)})$;
  for $i = 1, \ldots, n$ **do**
    $\boldsymbol{X}' \leftarrow \operatorname{shrink}\left(2\widetilde{\overline{\boldsymbol{X}}}_{(i)} - \widetilde{\boldsymbol{X}}_{i,(i)}^{(k)}, \gamma(N+1)\right)$;
    $\operatorname{prox}_{\tau f_i}(2\overline{\mathcal{X}} - \mathcal{X}_i^{(k)}) \leftarrow \operatorname{refold}(\underset{\sim}{\boldsymbol{X}}')$;
    $\mathcal{X}_i^{(k+1)} \leftarrow \mathcal{X}_i^{(k)} + t_k \left[\operatorname{prox}_{\tau f_i}(2\overline{\mathcal{X}} - \mathcal{X}_i^{(k)}) - \overline{\mathcal{X}}\right]$;
  end
  $\left[\operatorname{prox}_{\tau f_0}(2\overline{\mathcal{X}} - \mathcal{X}_0^{(k)})\right]_{\mathbb{i}} \leftarrow \begin{cases} \left[\frac{\tau}{\lambda\tau+1}\{\lambda\mathcal{L}_\Omega^*(\boldsymbol{b}) + \frac{1}{\tau}(2\overline{\mathcal{X}} - \mathcal{X}_0^{(k)})\}\right]_{\mathbb{i}} & \text{if } \mathbb{i} \in \Omega, \\ [2\overline{\mathcal{X}} - \mathcal{X}_0^{(k)}]_{\mathbb{i}} & \text{otherwise} \end{cases}$;
  $\mathcal{X}_0^{(k+1)} \leftarrow \mathcal{X}_0^{(k)} + t_k \left[\operatorname{prox}_{\tau f_0}(2\overline{\mathcal{X}} - \mathcal{X}_0^{(k)}) - \overline{\mathcal{X}}\right]$;
  $k \leftarrow k+1$;
end
$\mathcal{X} \leftarrow \operatorname{mean}(\mathcal{X}_0^{(k)}, \ldots, \mathcal{X}_n^{(k)})$

There are several notions of tensor rank but the *N-rank* is easy to compute. Originally, the N-rank is defined as the tuple of the rank of the mode-k unfoldings. However, the rank is not well-defined for general C-D domain, so we newly define the *N-$\mathbb{R}$-rank* of a n-dimensional hypercomplex tensor $\mathcal{X} \in \mathscr{T}^{\mathbb{A}_m}$ as the tuple of the $\mathbb{R}$-ranks of the mode-k unfoldings, i.e.,

$$N\text{-rank}^{\mathbb{R}}(\mathcal{X}) := \left[\operatorname{rank}^{\mathbb{R}}(\boldsymbol{X}_{(1)}), \ldots, \operatorname{rank}^{\mathbb{R}}(\boldsymbol{X}_{(n)})\right] \in \mathbb{N}^n.$$

In this paper, we will only focus on it as a rank of hypercomplex tensor.

Based on the N-$\mathbb{R}$-rank introduced above, we formulate the hypercomplex low rank tensor completion problem. Given a linear map $\mathcal{L} : \mathscr{T}^{\mathbb{A}_m} \to \mathbb{A}_m^p$ with $p \leq \prod_{i=1}^{N} n_i$ and given $\boldsymbol{b} \in \mathbb{A}_m^p$. The goal of the low N-$\mathbb{R}$-rank tensor completion problem is to find the hypercomplex tensor $\mathcal{X}$ that minimizes a function of N-$\mathbb{R}$-rank fulfilling the linear measurements $\mathcal{L}(\mathcal{X}) = \boldsymbol{b}$. This can be expressed as the following optimization problem:

$$\underset{\mathcal{X} \in \mathscr{T}^{\mathbb{A}_m}}{\text{minimize}} \quad f(N\text{-rank}^{\mathbb{R}}(\mathcal{X})) \quad \text{s.t.} \quad \mathcal{L}(\mathcal{X}) = \boldsymbol{b},$$

where $f : \mathbb{N}^n \to \mathbb{R}$. Following the convex relaxation in [19], we obtain the following unconstrained formulation:

$$\underset{\mathcal{X} \in \mathscr{T}^{\mathbb{A}_m}}{\text{minimize}} \quad \sum_{i=1}^{n} \left\| \widetilde{\boldsymbol{X}}_{(i)} \right\|_* + \frac{\lambda}{2} \left\| \mathcal{L}(\mathcal{X}) - \boldsymbol{b} \right\|_{\mathbb{A}_m}^2, \tag{4}$$

where $\|\cdot\|_*$ is the nuclear norm of matrices.

4.2. Hypercomplex Tensor Completion Algorithm via Convex Optimization

The problem (4) can be efficiently solved by well-established convex optimization algorithms such as the *Douglas-Rachford splitting (DRS)* [18]. In this section we propose a new hypercomplex tensor completion algorithm based on the DRS. In the case of tensor completion, we use the sampling operator $\mathcal{L}_\Omega$ as $\mathcal{L}$. It extracts the entries of the tensor $\mathcal{X}$ into the vector $\boldsymbol{b}$ at positions given by the set of revealed entries denoted by Ω. We can summarize the proposed hypercomplex tensor completion algorithm in Algorithm 1, following the derivation performed in [19]. Here, $(t_k)_{k \geq 0} \subset [0, 2]$ satisfies $\sum_{k \geq 0} t_k(2 - t_k)$, $\gamma \in (0, \infty)$, c_λ controls the increase of the Lagrange multiplier, $\mathcal{O} \in \mathscr{T}^{\mathbb{A}_m}$ is the tensor of all zero and a white letter $\mathbb{i}$ is the tuple of the indices at each dimension of a tensor, i.e., $\mathbb{i} := (i_1, \ldots, i_n)$. Moreover, $\mathcal{L}_\Omega^* : \mathbb{A}_m^p \to \mathscr{T}^{\mathbb{A}_m}$ denotes the ad-

Table 1. Performance comparison of three algorithms

$\mathscr{T}^{\mathbb{H}} = \mathbb{H}^{64\times64\times64}, \rho = 0.6, r = 4$				$\mathscr{T}^{\mathbb{H}} = \mathbb{H}^{10\times10\times10\times10}, \rho = 0.6, r = 2$			
Algorithm	# iter.	error	time [s]	Algorithm	# iter.	error	time [s]
$\mathbb{A}_m$-DRS	126	8.0e-6	391	$\mathbb{A}_m$-DRS	898	1.5e-4	170
DRS-QSVD	126	8.0e-6	5.0e+4	DRS-QSVD	898	1.5e-4	8,407
DRS-PW	210	1.0e-1	252	DRS-PW	1,077	35.6	63

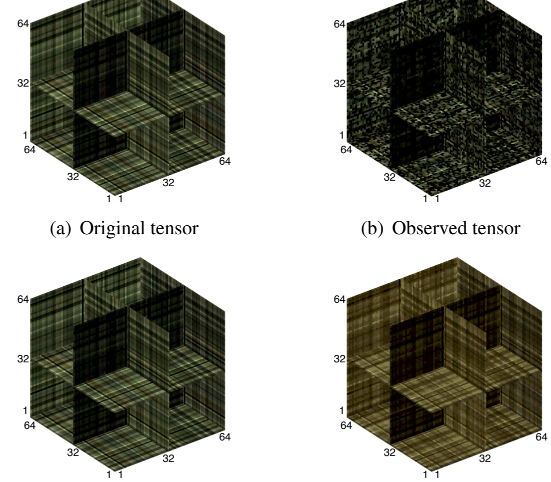
joint operator of $\mathcal{L}_\Omega$ satisfying $\langle \mathcal{L}_\Omega(\mathcal{X}), \boldsymbol{v}\rangle_{\mathbb{A}_m^p} = \langle \mathcal{X}, \mathcal{L}_\Omega^*(\boldsymbol{v})\rangle_{\mathscr{T}^{\mathbb{A}_m}}$ for all $\mathcal{X} \in \mathscr{T}^{\mathbb{A}_m}$ and $\boldsymbol{v} \in \mathbb{A}_m^p$, $\mathrm{refold}(\cdot)$ denotes the refolding of matrix into a tensor and $\mathrm{shrink}(\widetilde{\boldsymbol{A}}, \tau)$ denotes the singular value shrinkage operator given by $\mathrm{shrink}(\widetilde{\boldsymbol{A}}, \tau) = \boldsymbol{U}\boldsymbol{\Sigma}_\tau \boldsymbol{V}^\top$ for the singular value decomposition $\widetilde{\boldsymbol{A}} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top$ with the singular values σ_k $(k = 1, \ldots, r)$ and the shrunk diagonal matrix $\boldsymbol{\Sigma}_\tau := \mathrm{diag}(\max\{\sigma_1 - \tau, 0\}, \ldots, \max\{\sigma_r - \tau, 0\}, 0, \ldots, 0)$. Note that this shrinkage operator is calculated based on Definition 2 and Theorem 2, and all calculation in Algorithm 1 can be done only with real-valued calculation. Moreover, if we set $\mathbb{A}_m = \mathbb{R}$, the the proposed algorithm is reduced to the original DRS for low N-rank tensor completion algorithm in [19]. Similar to the real case, we have the following theorem:

Theorem 3 (Convergence of $\mathbb{A}_m$-DRS)**.** *Let parameters of Algorithm 1 be chosen so that* $\gamma \in (0, \infty)$, $(t_n)_{n\geq 0} \subset [0, 2]$ *satistying* $\sum_{n\geq 0} t_n(2 - t_n) = \infty$ *and let* $c_\lambda = 1$. *Then, the output of Algorithm 1 is a minimizer of (4).*

5. NUMERICAL EXAMPLES

We examine the efficiency of the proposed algorithm in the context of hypercomplex tensor completion problem. We perform experiments in the quaternion domain and compare three algorithms, the proposed $\mathbb{A}_m$-DRS for quaternion case, the DRS extended to quaternion domain using quaternion SVD, which is available in the MATLAB quaternion toolbox [26] (DRS-QSVD), and the part-wise DRS (DRS-PW). The DRS-PW optimizes real and each imaginary part separately with the original real-valued DRS. Note that the proposed $\mathbb{A}_m$-DRS can be applied to general C-D case, and DRS-QSVD is available only for quaternion case but also a new algorithm since the quaternion tensor completion itself is a new problem to the best of our knowledge. These three algorithms are implemented with MATLAB and the SVD in $\mathbb{A}_m$-DRS and DRS-PW are based on the QR-decomposition.

In each experiment we generate low N-rank quaternion tensor $\mathcal{X}_0$ which we used as ground truth. We fix the dimension r of a 'core tensor' $\mathcal{C} \in \mathbb{H}^{r\times\cdots\times r}$. Then we generate matrices $\boldsymbol{\Psi}^{(1)}, \ldots, \boldsymbol{\Psi}^{(n)}$ with $\boldsymbol{\Psi}^{(k)} \in \mathbb{H}^{N_k\times r}$ and set $\mathcal{X}_0 = \mathcal{C} \times_1 \boldsymbol{\Psi}^{(1)} \times_2 \cdots \times_n \boldsymbol{\Psi}^{(n)} \in \mathscr{T}^{\mathbb{H}}$, where $\times_k$ $(k = 1, \ldots, n)$ is the *k-mode product* satisfying $\mathcal{Y} = \mathcal{X} \times_k \boldsymbol{\Psi}^{(k)} \Leftrightarrow \boldsymbol{Y}_{(k)} = \boldsymbol{\Psi}^{(k)}\boldsymbol{X}_{(k)}$. All entries of $\mathcal{C}$ and $\boldsymbol{\Psi}^{(k)}$ are i.i.d. from $\mathcal{N}(0, 1)$. With this construction, the n-rank of $\mathcal{X}_0$ equals to $[r, \ldots, r]$ almost surely. We fix the percentage ρ of the entries to be known and randomly chose the support of the known entries. The value and the locations of the known entries of $\mathcal{X}_0$ are used as inputs for the algorithms. For the parameters, we set $c_\lambda = 1, \lambda = n$ and $t_k = 1$. Table 1 shows that the performance of the $\mathbb{A}_m$-DRS and the DRS-QSVD are the same since these two methods are mathematically equivalent but the $\mathbb{A}_m$-DRS is much faster than DRS-QSVD. This is because DRS-QSVD uses slow external library for computing quaternion SVD and have to use it many times in large-scale tensor completion problems. The DRS-PW does not converge to the optimal solution since it cannot approximate the ground truth. Fig.1 depicts some slices of (a) original, (b) observed tensors and the completion results of the left case in Table 1 by (c) $\mathbb{A}_m$-DRS and (d) DRS-PW. Each pixel in tensors is represented by the three imaginary parts of a quaternion as the RGB color space. It shows that the proposed method indeed recovers well both color information and low rank structure of the original tensor while DRS-PW cannot recover the color information of the original tensor since it cannot utilize the correlation of each color space.



(a) Original tensor (b) Observed tensor (c) Completed by $\mathbb{A}_m$-DRS (d) Completed by DRS-PW

Fig. 1. Completion results of a 3-dimensional quaternion tensor

6. CONCLUSIONS

In this paper, we have proposed an algorithmic solution to hypercomplex tensor completion problem based on a convex optimization technique. This solution utilizes a new definition of SVD and best low rank approximation of matrices based on algebraic translations of C-D number systems. Numerical experiments show that the proposed algorithm recovers much more faithfully the original color information than existing algorithm.

APPENDIX

Algebraic Real Translations of C-D Linear Systems

We briefly review the algebraic translation of C-D valued vectors and matrices proposed in [16]. A trivial correspondence (mapping) of hypercomplex vectors or matrices to real ones is

$$\widehat{(\cdot)} : \mathbb{A}_m^{M\times N} \to \mathbb{R}^{mM\times N} : \boldsymbol{A} \mapsto \widehat{\boldsymbol{A}} := \left[\boldsymbol{A}_1^\top, \ldots, \boldsymbol{A}_m^\top\right]^\top.$$

This correspondence is just concatenating a real and all imaginary parts in the hypercomplex vectors or matrices. Obviously, this mapping is invertible and thus we can also define $\overset{\smile}{(\cdot)} : \mathbb{R}^{mM\times N} \to \mathbb{A}_m^{M\times N} : \widehat{\boldsymbol{A}} \mapsto \boldsymbol{A}$. Only in terms of the mappings $\widehat{(\cdot)}$ and $\overset{\smile}{(\cdot)}$, it is hard to obtain the correspondence of matrix-vector product $\boldsymbol{A}\boldsymbol{x}$, so we also introduce the following non-trivial mapping:

$$\widetilde{(\cdot)} : \mathbb{A}_m^{M\times N} \to \mathbb{R}^{mM\times mN} :$$
$$\boldsymbol{A} \mapsto \widetilde{\boldsymbol{A}} := \left[\boldsymbol{L}_M^{(1)\top}\widehat{\boldsymbol{A}}, \boldsymbol{L}_M^{(2)\top}\widehat{\boldsymbol{A}}, \ldots, \boldsymbol{L}_M^{(m)\top}\widehat{\boldsymbol{A}}\right], \quad (5)$$

where the matrix $\boldsymbol{L}_M^{(\ell)} \in \mathbb{R}^{mM\times mM}$ $(\ell = 1, \ldots, m)$ is defined for the m-dimensional hypercomplex number $\mathbb{A}_m$ as

$$\boldsymbol{L}_M^{(\ell)} := \begin{bmatrix} \delta_{1,1}^{(\ell)}\boldsymbol{I}_M & \cdots & \delta_{1,m}^{(\ell)}\boldsymbol{I}_M \\ -\delta_{2,1}^{(\ell)}\boldsymbol{I}_M & \cdots & -\delta_{2,m}^{(\ell)}\boldsymbol{I}_M \\ \vdots & \ddots & \vdots \\ -\delta_{m,1}^{(\ell)}\boldsymbol{I}_M & \cdots & -\delta_{m,m}^{(\ell)}\boldsymbol{I}_M \end{bmatrix}, \delta_{\alpha,\beta}^{(\gamma)} := \begin{cases} 1 & (\text{if } \mathbf{i}_\alpha\mathbf{i}_\beta = \mathbf{i}_\gamma), \\ -1 & (\text{if } \mathbf{i}_\alpha\mathbf{i}_\beta = -\mathbf{i}_\gamma), \\ 0 & (\text{otherwise}), \end{cases}$$

and $\boldsymbol{I}_M$ is the M-dimensional identity matrix. Similar to the trivial mapping, $\widetilde{(\cdot)}$ is also invertible and thus we define $\underset{\smile}{(\cdot)} : \mathbb{R}^{mM\times mN} \to \mathbb{A}_m^{M\times N} : \widetilde{\boldsymbol{A}} \mapsto \boldsymbol{A}$. These translations have many useful algebraic properties. For detail, see [16].

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