A REFINED ANALYSIS OF THE GAP BETWEEN EXPECTED RATE FOR PARTIAL CSIT AND THE MASSIVE MIMO RATE LIMIT

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ABSTRACT

Optimal BeamFormers (BFs) that maximize the Weighted Sum Rate (WSR) for a Multiple-Input Multiple-Output (MIMO) interference broadcast channel (IBC) remains an important research area. Under practical scenarios, the problem is compounded by the fact that only partial channel state information at the transmitter (CSIT) is available. Hence, a typical choice of the optimization metric is the Expected Weighted Sum Rate (EWSR). However, the presence of the expectation operator makes the optimization a daunting task. On the other hand, for the particular, but significant, special case of massive MIMO (MaMIMO), the EWSR converges to Expected Signal covariance Expected Interference covariance based WSR (ESEI-WSR) and this metric is more amenable to optimization. Recently, [1] considered a multi-user Multiple-Input Single-Output (MISO) scenario and proposed approximating the EWSR by ESEI-WSR. They then derived a constant bound for this approximation. This paper performs a refined analysis of the gap between EWSR and ESEI-WSR criteria for finite antenna dimensions.

Index Terms— Beamforming, partial CSIT, EWSR, ESEI-WSR, MaMIMO

1. INTRODUCTION

Interference is the main limiting factor in wireless transmission. Base stations (BSs) with multiple antennas are able to serve multiple Mobile Terminals (MTs) simultaneously, which is called Spatial Division Multiple Access (SDMA) or Multi-User (MU) MIMO. We are particularly concerned here with maximum Weighted Sum Rate (WSR) designs accounting for finite SNR. Typical approaches for maximizing WSR are based on a link to Weighted Sum MSE (WSMSE) [2] or an approach based on Difference of Convex function programming [3] (which is actually better interpreted as an instance of majorization). However, these approaches rely on perfect channel CSIT, which is not practical. Hence, an alternative approach is to maximize the EWSR for the case of partial CSIT.

Partial CSIT formulations can typically be categorized as either bounded error / worst case (relevant for quantization error in digital feedback) or Gaussian error (relevant for analog feedback, prediction error, second-order statistics information etc.). The Gaussian CSIT formulation with mean and covariance information was first introduced for SDMA (a Direction of Arrival (DoA) based historical precedent of MU MIMO), in which the channel outer product was typically replaced by the transmit side channel correlation matrix,

and worked out in more detail for single user (SU) MIMO, e.g. [4]. The use of covariance CSIT was made in the context of Massive MIMO [5], where a not so rich propagation environment leads to subspaces (slow CSIT) for the channel vectors so that the fast CSIT can be reduced to the smaller dimension of the subspace. Such CSIT (feedback) reduction is especially crucial for Massive MIMO. Due to the difficulty in directly optimizing the EWSR metric, optimization of the expected WSMSE (EWSMSE), which is a lower bound for the EWSR, was proposed in [6]. In fact, exact expressions exist for a number of MISO [7] and MIMO cases [8]. However, those expressions are very hard to interpret and to optimize with respect to BFs. This issue has led to the development of large system analysis to try to get simpler expressions for the expected rate [9], [10]. Recently, though under a single user MIMO setting, the authors [11] used a large system approximation for the optimization of the EWSR metric under partial CSIT to counter the impact of Doppler created Inter Carrier Interference (ICI). On the other hand, for the particular, but significant, special case of MaMIMO where the number of transmit antennas is large compared to the number of receive antennas, the EWSR converges to ESEI-WSR and this metric is more amenable to optimization. In another recent publication, [1] considered a multi-user Multiple-Input Single-Output (MISO) scenario and proposed approximating the EWSR by ESEI-WSR. They then derived a constant bound for this approximation. The approximate metric was then used for optimization of the EWSR. Inspired by this, we perform a refined analysis of the gap between EWSR and ESEI-WSR criteria for finite antenna dimensions to evaluate the usefulness of using the ESEI-WSR metric (that is more mathematically tractable) instead of the EWSR.

The main goal of this paper is to show that the much simpler expressions obtained in the ESEI approximation (MaMIMO limit) in fact exhibit only a finite and even small gap to the exact expected rate. Towards this end, we first show in section 3.1 for a general non-zero mean correlated MIMO scenario that the gap is monotonically increasing as a function of SNR and hence is maximum at infinite SNR. Then, we go about deriving this gap at infinite SNR for specific MISO (section 3.3) and MIMO(section 3.4) scenarios. The swift reduction in the gap with increasing number of antennas is clearly seen for the MISO scenarios. The second order Taylor Series Expansion of EWSR for a general MIMO setting is also derived in section 3.2 and observed to concur with the infinite SNR limits for the gap derived independently. Henceforth, the term gap would refer to the gap between ESEI-WSR and the EWSR. However, we shall also briefly analyze the actual gap in EWSR, between optimal BFs and BFs optimized by ESEI-WSR. In the following text, the notation $|\mathbf{A}|$ refers to the determinant of the matrix $\mathbf{A}$. $\mathcal{CN}(\mu, C)$ refers to a circularly complex Gaussian distribution with mean μ and covariance C. In this paper, Tx may denote transmit/transmitter/transmission and Rx may denote receive/receiver/reception.

2. MIMO IBC SIGNAL MODEL

Consider an IBC with C cells and a total of K users with d_k streams per user. We shall consider a system-wide numbering of the users. User k has N_k antennas and is served by BS b_k. The $N_k \times 1$ received signal at user k in cell b_k is,

$$
\mathbf{y}_k = \underbrace{\mathbf{H}_{k,b_k}\,\mathbf{G}_k\,\mathbf{x}_k}_{\text{signal}} + \underbrace{\sum_{\substack{i \neq k \\ b_i = b_k}} \mathbf{H}_{k,b_k}\,\mathbf{G}_i\,\mathbf{x}_i}_{\text{intracell interf.}} + \underbrace{\sum_{j \neq b_k} \sum_{i:b_i=j} \mathbf{H}_{k,j}\,\mathbf{G}_i\,\mathbf{x}_i}_{\text{intercell interf.}} + \mathbf{v}_k \tag{1}
$$

where $\mathbf{x}_k$ is the intended (white, identity covariance) signal, $\mathbf{H}_{k,b_k}$ is the $N_k \times M_{b_k}$ channel from BS b_k to user k. BS b_k serves $K_{b_k} = \sum_{i:b_i=b_k} 1$ users. We consider a noise whitened signal representation so that we get for the noise $\mathbf{v}_k \sim \mathcal{CN}(0, I_{N_k})$. The $M_{b_k} \times d_k$ spatial Tx filter or beamformer (BF) is $\mathbf{G}_k$.

The scenario of interest is that of partial CSIT available globally with all the BSs. The Gaussian CSIT model for the partial CSIT is

$$
\mathbf{H}_{k,b_k} = \overline{\mathbf{H}}_{k,b_k} + \widetilde{\mathbf{H}}_{k,b_k}\,\mathbf{C}_t^{1/2} \tag{2}
$$

where $\overline{\mathbf{H}}_{k,b_k} = \mathbf{E}\mathbf{H}_{k,b_k}$, and $\mathbf{C}_t^{1/2}$ is the Hermitian square-roots of the Tx side covariance matrices. The elements of $\widetilde{\mathbf{H}}_{k,b_k}$ are i.i.d. $\sim \mathcal{CN}(0,1)$.

$$
\mathbf{E}_{\mathbf{H}_{k,b_k}|\overline{\mathbf{H}}_{k,b_k}} (\mathbf{H}_{k,b_k} - \overline{\mathbf{H}}_{k,b_k})(\mathbf{H}_{k,b_k} - \overline{\mathbf{H}}_{k,b_k})^H = \mathrm{tr}\{\mathbf{C}_t\}\mathbf{I}_{N_k}
$$
$$
\mathbf{E}_{\mathbf{H}_{k,b_k}|\overline{\mathbf{H}}_{k,b_k}} (\mathbf{H}_{k,b_k} - \overline{\mathbf{H}}_{k,b_k})^H(\mathbf{H}_{k,b_k} - \overline{\mathbf{H}}_{k,b_k}) = N_k \mathbf{C}_t \tag{3}
$$

Note that the expectation is done over $\mathbf{H}_{k,b_k}$, for a known $\overline{\mathbf{H}}_{k,b_k}$. This is true for all the expectation operations done in this paper. However, as the parameter over which the expectation is done is clear from the context, henceforth, we just mention the expectation operator $\mathbf{E}$ to reduce notational overhead.

2.1. Expected WSR (EWSR)

Once the CSIT is imperfect, various optimization criteria could be considered, such as outage capacity. Here we shall consider the EWSR for a known channel mean $\overline{\mathbf{H}}$.

$$
\mathrm{EWSR}(\mathbf{G}) = \mathbf{E} \sum_k u_k \ln |\mathbf{I} + \mathbf{G}_k^H\,\mathbf{H}_{k,b_k}^H\,\mathbf{R}_{\overline{k}}^{-1}\,\mathbf{H}_{k,b_k}\,\mathbf{G}_k|
$$
$$
= \mathbf{E} \sum_{k=1}^{K} u_k\,(\ln|\mathbf{R}_k| - \ln|\mathbf{R}_{\overline{k}}|). \tag{4}
$$

Here, $\mathbf{G}$ represents the collection of BFs $\mathbf{G}_k$, u_k are rate weights.

$$
\mathbf{R}_k = \mathbf{H}_{k,b_k}\,\mathbf{Q}_k\,\mathbf{H}_{k,b_k}^H + \mathbf{R}_{\overline{k}},\ \ \mathbf{Q}_i = \mathbf{G}_i\mathbf{G}_i^H,
$$
$$
\mathbf{R}_{\overline{k}} = \sum_{i \neq k} \mathbf{H}_{k,b_i}\,\mathbf{Q}_i\,\mathbf{H}_{k,b_i}^H + I_{N_k}. \tag{5}
$$

The EWSR cost function needs to be augmented with the power constraints $\sum_{k:b_k=j} \mathrm{tr}\{\mathbf{Q}_k\} \leq P_j$.

2.2. MaMIMO limit and ESEI-WSR

If the number of Tx antennas M becomes very large, we get a convergence for any quadratic term of the form

$$
\mathbf{H}\mathbf{Q}\mathbf{H}^H \xrightarrow{M \to \infty} \mathbf{E}\,\mathbf{H}\mathbf{Q}\mathbf{H}^H = \overline{\mathbf{H}}\mathbf{Q}\overline{\mathbf{H}}^H + \mathrm{tr}\{\mathbf{Q}\mathbf{C}_t\}\,\mathbf{I} \tag{6}
$$

and hence we get the following MaMIMO limit matrices

$$
\breve{\mathbf{R}}_k = \breve{\mathbf{R}}_{\overline{k}} + \overline{\mathbf{H}}_{k,b_k}\mathbf{Q}_k\overline{\mathbf{H}}_{k,b_k}^H + \mathrm{tr}\{\mathbf{Q}_k\mathbf{C}_{t,k,b_k}\}\,\mathbf{I}_{N_k}
$$
$$
\breve{\mathbf{R}}_{\overline{k}} = I_{N_k} + \sum_{i \neq k}^{K} \left(\overline{\mathbf{H}}_{k,b_i}\mathbf{Q}_i\overline{\mathbf{H}}_{k,b_i}^H + \mathrm{tr}\{\mathbf{Q}_i\mathbf{C}_{t,k,b_i}\}\,\mathbf{I}_{N_k} \right) \tag{7}
$$

Now, typical approaches to solve the WSR (eg. the DC approach in [3]) can be run to obtain the max EWSR BF. We shall refer to this approach as the ESEI-WSR approach as (channel dependent) signal and interference covariance matrices are replaced by their expected values. In the following sections, we analyze the gap between the EWSR and the ESEI-WSR to suggest an approximation of the first by the latter in the design of the BF. We would like to remark here that the ESEI-WSR may also be interpreted as the WSR that would be obtained if we assume that the received signal and interference are also Gaussian.

3. EWSR TO ESEI-WSR GAP ANALYSIS

We are interested in bounding the difference between ESEI-WSR and the EWSR. At the level of each user k, we stack the channel estimates relevant for each user k.

$$
\mathbf{H}_k = [\mathbf{H}_{k,b_1} \cdots \mathbf{H}_{k,b_{k-1}}\ \mathbf{H}_{k,b_k}\ \mathbf{H}_{k,b_{k+1}} \cdots \mathbf{H}_{k,b_K}]
$$
$$
= \overline{\mathbf{H}}_k + \widetilde{\mathbf{H}}_k \mathbf{C}_{t,k}^{\frac{1}{2}} \tag{8}
$$

where the elements of $\widetilde{\mathbf{H}}_k$ are i.i.d $\sim \mathcal{CN}(0,1)$ and $\overline{\mathbf{H}}_k$ refers to the mean part of $\mathbf{H}_k$. $\mathbf{C}_{t,k}$ is a block diagonal matrix whose i^{th} diagonal block is $\mathbf{C}_{t,k,b_i}$. Let $\mathbf{Q}$ be a block diagonal matrix with i^{th} diagonal block being $\sum_{l:b_l=b_i} \mathbf{Q}_l$. Note that this summation corresponds to contributions from all the intracell precoding vectors. $\mathbf{Q}_{\overline{k}}$ is similar to $\mathbf{Q}$ but with the kth block diagonal set to $\sum_{l \neq k}^{l:b_l=b_k} \mathbf{Q}_l$. Thus, in $\mathbf{Q}_{\overline{k}}$, only the interfering precoders (intracell and intercell) are included. Then,

$$
\mathbf{R}_k = \mathbf{I} + \mathbf{H}_k\mathbf{Q}\mathbf{H}_k^H, \qquad \mathbf{R}_{\overline{k}} = \mathbf{I} + \mathbf{H}_k\mathbf{Q}_{\overline{k}}\mathbf{H}_k^H \tag{9}
$$

$$
\mathrm{EWSR}(\mathbf{G}) = \sum_{k=1}^{K} u_k\,\mathbf{E}_{\mathbf{H}_k}\,(\ln|\mathbf{R}_k| - \ln|\mathbf{R}_{\overline{k}}|)
$$
$$
= \mathbf{E}\sum_{k=1}^{K} u_k\,\left(\ln|\mathbf{I} + \mathbf{H}_k\mathbf{Q}\mathbf{H}_k^H| - \ln|\mathbf{I} + \mathbf{H}_k\mathbf{Q}_{\overline{k}}\mathbf{H}_k^H| \right) \tag{10}
$$

$$
\mathrm{ESEI\text{-}WSR}(\mathbf{G})
$$
$$
= \sum_{k=1}^{K} u_k\,\left(\ln|\mathbf{I} + \mathbf{E}\,\mathbf{H}_k\mathbf{Q}\mathbf{H}_k^H| - \ln|\mathbf{I} + \mathbf{E}\,\mathbf{H}_k\mathbf{Q}_{\overline{k}}\mathbf{H}_k^H| \right) \tag{11}
$$

Thus, the EWSR and ESEI-WSR have been rewritten in a convenient format so that one can focus on the gap between the two by comparing terms of the form $\mathbf{E}\ln|\mathbf{I} + \mathbf{H}_k\mathbf{Q}\mathbf{H}_k^H|$ and $\ln|\mathbf{I} + \mathbf{E}\,\mathbf{H}_k\mathbf{Q}\mathbf{H}_k^H|$.

3.1. Monotonicity of gap with SNR

For an SNR ρ, define
$$
\Gamma_k(\rho) = \ln|\mathbf{I} + \rho\mathbf{E}\mathbf{H}_k'\mathbf{H}_k'^H| - \mathbf{E}\ln|\mathbf{I} + \rho\mathbf{H}_k'\mathbf{H}_k'^H| \tag{12}
$$

where $\mathbf{H}_k' \sim \mathcal{CN}(\overline{\mathbf{H}}_k', \mathbf{C})$, $\overline{\mathbf{H}}_k' = \frac{1}{\sqrt{\rho}}\overline{\mathbf{H}}_k\mathbf{Q}^{\frac{1}{2}}$, and $\mathbf{C} = \frac{1}{\rho}\mathbf{C}_t^{\frac{1}{2}}\mathbf{Q}\mathbf{C}_t^{\frac{1}{2}}$. Then, $\mathbf{I} + \mathbf{H}_k\mathbf{Q}\mathbf{H}_k^H = \mathbf{I} + \rho\mathbf{E}\mathbf{H}_k'\mathbf{H}_k'^H$.

Theorem 1. $\Gamma_k(\rho)$ is monotonically increasing in ρ

Proof. By Jensen's inequality, $\Gamma_k(\rho) \geq 0$ and it can be seen easily that the equality is attained when $\rho = 0$. To show the monotonicity, we show that the derivative with respect to ρ is always non-negative. We omit the subscripts and superscripts on $\mathbf{H}$ for convenience.

$$
\frac{\partial}{\partial \rho}\left(\ln|\mathbf{I} + \rho\mathbf{E}\mathbf{H}\mathbf{H}^H| - \mathbf{E}\ln|\mathbf{I} + \rho\mathbf{H}\mathbf{H}^H| \right) =
$$
$$
\mathrm{tr}\left(\{\mathbf{I} + \rho\mathbf{E}\mathbf{H}\mathbf{H}^H\}^{-1}\mathbf{E}\mathbf{H}\mathbf{H}^H - \mathbf{E}\left(\{\mathbf{I} + \rho\mathbf{H}\mathbf{H}^H\}^{-1}\mathbf{H}\mathbf{H}^H \right) \right) \tag{13}
$$

Noting that, $\{\mathbf{I} + \rho\mathbf{E}\mathbf{H}\mathbf{H}^H\}^{-1}\mathbf{E}\mathbf{H}\mathbf{H}^H$ can be written as $\frac{1}{\rho}\mathbf{I} - \frac{1}{\rho}\{\mathbf{I} + \rho\mathbf{E}\mathbf{H}\mathbf{H}^H\}^{-1}$,

$$
\frac{\partial}{\partial \rho}\left(\ln|\mathbf{I} + \rho\mathbf{E}\mathbf{H}\mathbf{H}^H| - \mathbf{E}\ln|\mathbf{I} + \rho\mathbf{H}\mathbf{H}^H| \right) =
$$
$$
\frac{1}{\rho}\mathrm{tr}\,\mathbf{E}\left(\{\mathbf{I} + \rho\mathbf{H}\mathbf{H}^H\}^{-1} \right) - \mathrm{tr}\,\frac{1}{\rho}\{\mathbf{I} + \rho\mathbf{E}\mathbf{H}\mathbf{H}^H\}^{-1} \geq 0 \tag{14}
$$

where we have applied Jensen's inequality as $\{\mathbf{I} + \rho\mathbf{H}\mathbf{H}^H\}^{-1}$ is a convex function. $\qquad\square$

As a result, the largest value of $\Gamma_k(\rho)$ will be observed at infinite SNR for a general non-zero mean MIMO with channel $\mathbf{H}$ with arbitrary transmit covariance matrix. Now, following the same steps as in [1], we can obtain, for any collection of BFs $\mathbf{G}$,

$$
\begin{aligned}
\text{ESEI-WSR} - \sum_{k=1}^{K} u_k \Gamma_k(\infty) &\leq \text{ESEI-WSR} - \sum_{k=1}^{K} u_k \Gamma_k(\rho) \\
&\leq \text{EWSR} \leq \\
\text{ESEI-WSR} + \sum_{k=1}^{K} u_k \Gamma_{\bar{k}}(\rho) &\leq \text{ESEI-WSR} + \sum_{k=1}^{K} u_k \Gamma_{\bar{k}}(\infty)
\end{aligned}
\tag{15}
$$

In the above, Γ_k and $\Gamma_{\bar{k}}$ are terms corresponding to the first and the second terms of equation (10). Remains now to obtain the $\Gamma_k(\infty)$ for different scenarios. However, we first look at the Taylor series expansion of EWSR to get an alternative expression for the gap.

3.2. Second-Order Taylor Series Expansion of EWSR

Consider the Taylor series expansion for matrices $\mathbf{X}$, $\mathbf{Y}$ of dimension N_k.

$$
\ln|\mathbf{X} + \mathbf{Y}| \approx \ln|\mathbf{X}| + \text{tr}\,\mathbf{X}^{-1}\mathbf{Y} - \frac{1}{2}\text{tr}\,\mathbf{X}^{-1}\mathbf{Y}\mathbf{X}^{-1}\mathbf{Y}
\tag{16}
$$

Consider $\mathbf{X} + \mathbf{Y} = \mathbf{I} + \rho\mathbf{H}\mathbf{H}^H$, $\mathbf{H} = \overline{\mathbf{H}} + \widetilde{\mathbf{H}}\mathbf{C}^{\frac{1}{2}}$, $\widetilde{\mathbf{H}} \sim \mathcal{CN}(0, \mathbf{I})$. For expansion around $\mathbf{I} + \rho\mathbf{E}\,\mathbf{H}\mathbf{H}^H$, choose $\mathbf{X} = \mathbf{I} + \rho\mathbf{E}\,\mathbf{H}\mathbf{H}^H$, $\mathbf{Y} = \rho\left(\mathbf{E}\,\mathbf{H}\mathbf{H}^H - \mathbf{H}\mathbf{H}^H\right)$. Hence, we get,

$$
\begin{aligned}
\mathbf{E}\,\ln|\mathbf{I} + \rho\mathbf{H}\mathbf{H}^H| &\approx \ln|\mathbf{I} + \rho\mathbf{E}\,\mathbf{H}\mathbf{H}^H| - \\
\frac{\rho^2}{2}\mathbf{E}\,\text{tr}&\{\mathbf{X}^{-1}(\mathbf{H}\mathbf{H}^H - \mathbf{E}\,\mathbf{H}\mathbf{H}^H)\mathbf{X}^{-1}(\mathbf{H}\mathbf{H}^H - \mathbf{E}\,\mathbf{H}\mathbf{H}^H)\}
\end{aligned}
\tag{17}
$$

Using 4th order Gaussian moments [12], we get

$$
\begin{aligned}
\mathbf{E}\,\ln|\mathbf{I} + \rho\mathbf{H}\mathbf{H}^H| &\approx \ln|\mathbf{I} + \rho\mathbf{E}\,\mathbf{H}\mathbf{H}^H| - \frac{\rho^2}{2}\text{tr}\Big\{\text{tr}\{\mathbf{X}^{-1}\}^2\mathbf{C}^2 \\
&+ 2\text{tr}\{\mathbf{X}^{-1}\}\overline{\mathbf{H}}^H\mathbf{X}^{-1}\overline{\mathbf{H}}\mathbf{C} - (\overline{\mathbf{H}}^H\mathbf{X}^{-1}\overline{\mathbf{H}})^2\Big\}.
\end{aligned}
\tag{18}
$$

Let us denote this second order approximation by $\Gamma_2(\rho)$. i.e,

$$
\begin{aligned}
\Gamma_2(\rho) = \frac{\rho^2}{2}\text{tr}\Big\{&\text{tr}\{\mathbf{X}^{-1}\}^2\mathbf{C}^2 \\
&+ 2\text{tr}\{\mathbf{X}^{-1}\}\overline{\mathbf{H}}^H\mathbf{X}^{-1}\overline{\mathbf{H}}\mathbf{C} - (\overline{\mathbf{H}}^H\mathbf{X}^{-1}\overline{\mathbf{H}})^2\Big\}.
\end{aligned}
\tag{19}
$$

Consider the mean zero special case, $\overline{\mathbf{H}} = 0$. Then, $\mathbf{E}\,\mathbf{H}\mathbf{H}^H = \text{tr}\{\mathbf{C}\}\mathbf{I}$ and $\mathbf{X} = \mathbf{I}_{N_k} + \rho\text{tr}\{\mathbf{C}\}\mathbf{I}_{N_k}$. Therefore,

$$
\mathbf{E}\,\ln|\mathbf{I} + \rho\mathbf{H}\mathbf{H}^H| \approx \ln(1 + \rho\text{tr}(\mathbf{C})) - \frac{\rho^2 N_k^2}{2}\frac{\text{tr}\{\mathbf{C}^2\}}{(1 + \rho\text{tr}\{\mathbf{C}\})^2}.
\tag{20}
$$

At high SNR, as $\rho \to \infty$,

$$
\mathbf{E}\,\ln|\mathbf{I} + \rho\mathbf{H}\mathbf{H}^H| \approx \ln(1 + \rho\text{tr}(\mathbf{C})) - \frac{N_k^2}{2}\frac{\text{tr}\{\mathbf{C}^2\}}{(\text{tr}\{\mathbf{C}\})^2}.
\tag{21}
$$

Thus, $\Gamma_2(\infty) = \frac{N_k^2}{2}\frac{\text{tr}\{\mathbf{C}^2\}}{(\text{tr}\{\mathbf{C}\})^2}$. Continuing from Theorem 1, we now determine the value of $\Gamma(\infty)$ for different scenarios.

3.3. MISO correlated channel

In the MISO correlated channel, the relevant metric is of the form $\ln(1 + ||\mathbf{h}||^2)$, where $\mathbf{h}$ is a $1 \times M$ MISO channel vector with $\lambda_i \cdots \lambda_p$ being the p non-zero, non-identical eigen values of the correlation matrix $\mathbf{E}\,\mathbf{h}\mathbf{h}^H$.

Theorem 2.

$$
\begin{aligned}
0 &\leq \ln(1 + \rho\sum_{i=1}^{p}\lambda_i) - \mathbf{E}\ln(1 + \rho||\mathbf{h}||^2) \\
&\leq \gamma - \left(\sum_{i=1}^{p}\frac{\ln\lambda_i}{\pi_{l\neq i}(1 - \lambda_l/\lambda_i)} - \ln(\sum_{i=1}^{p}\lambda_i)\right),
\end{aligned}
\tag{22}
$$

where ρ is the SNR, γ is Euler constant.

The proof is given in Appendix of the companion Arxiv paper [13] due to lack of space. Note that for $M = 1$, the bound reduces to that in [1], namely γ. Thus, this bound is a much more refined and tighter bound than what is provided in [1]. Though in general, the correlation matrix would have non-equal eigen values, it is illustrative to consider an extreme case where the eigen values are all identical. In fact, this is identical to a MISO i.i.d channel with just p antennas instead of M.

Theorem 3.

$$
0 \leq \ln(1 + \rho\,M) - \mathbf{E}\ln(1 + \rho||\mathbf{h}||^2) \leq \gamma - \left(\sum_{k=1}^{p}\frac{1}{k} - \ln(p)\right) + \frac{1}{p}
$$

The proof is given in Appendix of the companion Arxiv paper [13] due to lack of space. We further explore the bound using the properties of the harmonic series. Define $\mathcal{H}_p = \sum_{k=1}^{p}\frac{1}{k}$. It is known that,

$$
\mathcal{H}_p = \ln(p) + \gamma + \frac{1}{2p} - \frac{1}{12p^2} + \frac{1}{120p^4}\cdots
\tag{23}
$$

Using this in (3), we get

$$
\gamma - (\mathcal{H}_p - \ln(p)) + \frac{1}{p} = \frac{1}{2p} + \frac{1}{12p^2} - \frac{1}{120p^4}\cdots
\tag{24}
$$

Thus, the second order term for the bound is $\frac{1}{2p}$, which is also in agreement with equation (21), $\frac{1}{2}\frac{\text{tr}\{\mathbf{C}^2\}}{(\text{tr}\{\mathbf{C}\})^2} = \frac{\sum_{i=1}^{p}\lambda^2}{2(\sum_{i=1}^{p}\lambda)^2} = \frac{1}{2p}$

3.4. MIMO zero mean i.i.d channel

In a multi-user scenario, the regime of interest is $M \geq N_k$. To tackle this scenario, we first introduce the LDU (Lower Diagonal Upper triangular factorization) of the channel Gram matrix,

$$
\mathbf{H}\mathbf{H}^H = \mathbf{L}\mathbf{D}\mathbf{L}^H = (\mathbf{L}\mathbf{D}^{\frac{1}{2}})(\mathbf{L}\mathbf{D}^{\frac{1}{2}})^H
\tag{25}
$$

where $\mathbf{L}$ has unit diagonal and $\mathbf{D}$ is a diagonal matrix with diagonal entries $(\mathbf{D}_i)$ greater than zero. The second factorization corresponds to a Cholesky decomposition. The Cholesky factorization of a Wishart matrix (such as $\mathbf{H}\mathbf{H}^H$) leads to,

$$
\mathbf{D}_i \sim \frac{1}{2}\chi_{2(M-i+1)}^2, i \in 1 \cdots N_k \qquad \mathbf{L}_{i,j}\mathbf{D}_i^{\frac{1}{2}} \sim \mathcal{CN}(0,1), i > j
$$

which is also known as Bartlett's decomposition [14]. Note that $|\mathbf{H}\mathbf{H}^H| = |\mathbf{L}\mathbf{D}\mathbf{L}^H| = |\mathbf{D}|$. Hence, $\ln|\mathbf{H}\mathbf{H}^H| = \sum_{i=1}^{N_k}\ln|\mathbf{D}_i|$ and the MIMO case reduces to a sum of MISO scenarios, each having a χ^2 distribution with a reducing number of degrees of freedom. Thus, reusing the results in section 3.3, we get,

$$\Gamma(\infty) = \sum_{i=1}^{N_k} \left(\gamma - \left(\sum_{k=1}^{M-i} \frac{1}{k} - \ln(M) \right) \right) \qquad (26)$$

For illustration, let us also consider $M \gg N_k$. Then using the approximation of the Harmonic series, it can be easily shown that $\Gamma(\infty) \approx \frac{N_k^2}{2M}$, which concurs with the second order Taylor series term in (21). The general case of correlated MIMO channel with non-zero mean is a future work to be addressed. However, we conjecture that in the case of a non-zero mean MIMO, the gap would further reduce based on the rice factor (the ratio of the power in the mean to that of the random part). However, a few comments are in order. Whenever $\Gamma(\infty)$ is closely approximated by $\Gamma_2(\infty)$ then $\Gamma(\rho)$ should be closely approximated by $\Gamma_2(\rho)$ also. We can also observe that whenever the gap $\Gamma(\rho)$ gets small, the second-order term $\Gamma_2(\rho)$ becomes good, in the sense that $\Gamma(\rho) = \Gamma_2(\rho) + O(\Gamma_2^2(\rho))$.

4. ACTUAL EWSR GAP

MISO Covariance CSIT only case:
Note that our real interest is in bounding the difference $|\text{EWSR}(\mathbf{G}^*) - \text{EWSR}(\mathbf{G}^{**})|$, where $\mathbf{G}^*$ are the optimal BFs that maximize the EWSR and $\mathbf{G}^{**}$ are the BFs that result from the optimization of the approximate ESEI-WSR metric. We now provide some insights into this for cases where the Tx has more antennas than the dimension of the total interference covariance subspace. At infinite SNR, the optimal beamformers perform zero forcing (ZF). Thus, there is no interference observed at user k and only the signal part needs to be optimized. Then, the equivalent scalar channel observed at the receiver for a zero mean Gaussian channel $\mathbf{h}_k$ may be written as $\mathbf{h}_k \mathbf{g}_k$, which is clearly complex Gaussian for any choice of the BF vector $\mathbf{g}_k$. Thus, at infinite SNR, $\mathbf{G}^*$ optimizes $\sum_{k=1}^{K} \mathbf{E} \ln(|\mathbf{h}_k \mathbf{g}_k|^2)$ and the ESEI-WSR optimizes $\sum_{k=1}^{K} \ln(\mathbf{E}|\mathbf{h}_k \mathbf{g}_k|^2)$. However, as shown in the proof of Theorem 3, $\ln(\mathbf{E}|\mathbf{h}_k \mathbf{g}_k|^2) = \mathbf{E} \ln(|\mathbf{h}_k \mathbf{g}_k|^2) - \gamma$, and it follows immediately that $\mathbf{G}^* = \mathbf{G}^{**}$.

Result 1. For a zero mean correlated MISO IBC channel allowing covariance CSIT based ZF, at infinite SNR, $|\text{EWSR}(\mathbf{G}^*) - \text{EWSR}(\mathbf{G}^{**})| = 0$.

MISO Channel Estimate Plus Covariance CSIT case:
In this case $\mathbf{h}_k \mathbf{g}_k$ not only has a variance σ^2 but also a non-zero mean m. The ESEI-WSR approach maximized a weighted combination over users of $\mathbf{E}|\mathbf{h}_k \mathbf{g}_k|^2 = |m|^2 + \sigma^2$ whereas the EWSR cost function is also an increasing function of the σ^2 and $|m|^2$ but through a solution dependent weighted sum instead of just the sum. A similar phenomenon occurs in the large MIMO system limit [9], [10], [11].

MIMO case:
In the MIMO case, consider a per stream approach in which now index k will refer to stream k (users possibly getting multiple streams). At the output of a linear Rx $\mathbf{f}_k$, a signal estimate appears

$$\widehat{x}_k = \mathbf{f}_k^H \mathbf{H}_{k,b_k} \mathbf{g}_k x_k + \sum_{i \neq k} \mathbf{f}_k^H \mathbf{H}_{k,b_i} \mathbf{g}_i x_i + \mathbf{f}_k^H \mathbf{v}_k. \qquad (27)$$

The per stream and per user approaches are equivalent in the perfect CSIT case [2], though in the partial CSIT case the relation is less clear. Nevertheless, proceeding from (27), the scalar $\mathbf{f}_k^H \mathbf{H}_{k,b_k} \mathbf{g}_k$ is of similar nature as the $\mathbf{h}_k \mathbf{g}_k$ of the MISO case. Hence the MISO results above also hold for the MIMO per stream approach. We should not though note that esp. in the multi-cell MIMO case, the (E)WSR has typically multiple local optima which at high SNR correspond to different distributions of the ZF roles between Txs and Rxs. Hence the gap analysis above should be understood to apply to corresponding local optima, including the global optimum.

5. NUMERICAL RESULTS

Figure 1 verifies the infinite-SNR bounds for MISO correlated scenario by comparing them against the true values of the gap for different SNRs and different values of M. The true values of the gap are obtained from Matlab simulations by averaging across different channel realizations and channel correlations. As expected, the gap is zero at very low SNR. As the SNR increases, the gap monotonically increases to the infinite SNR limit, as predicted in section 3.1. In addition, the gap reduces rapidly with increasing M. Further,
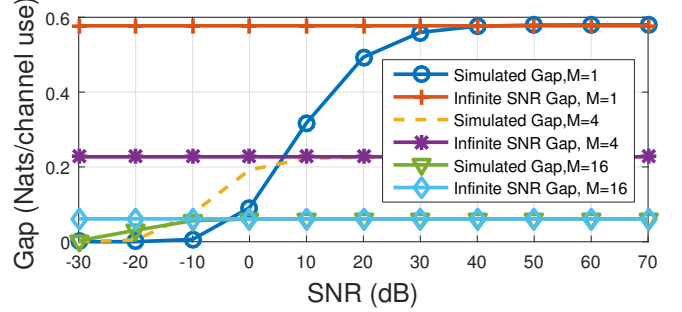

Fig. 1. Gap between ESEI-WSR and EWSR for the MISO correlated scenario for different values of transmit antennas.

to verify the goodness of the second order Taylor series approximation, Figure 2 compares the true gap to the gap approximated from the Taylor series expansion for a zero mean correlated MIMO scenario. This scenario is chosen as we expect gap to be maximum here. The number of receive antennas for each user was chosen as $N_k = 4$. ρ was chosen as 1000. As expected, the Taylor series approximation becomes more accurate with increasing number of Tx antennas. Indeed, even in this MIMO correlated scenario, the gap reduces quickly as the number of Tx antennas increases.
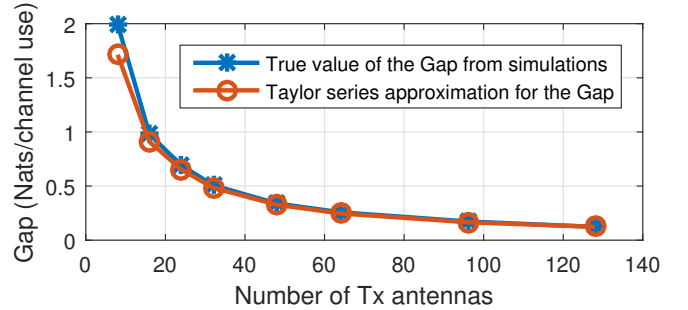

Fig. 2. Gap obtained from the second order Taylor series approximation vs. the true value of the gap for a MIMO correlated scenario. The number of antennas at each receiver, N_k, is taken as 4.

6. CONCLUSION

In this paper, we have motivated the use of the ESEI-WSR metric (or the MaMIMO limit of the EWSR) for utility optimization involving partial CSIT. Towards this end, we presented a refined bound for the gap between EWSR and the ESEI-WSR. We first showed that the gap is maximum at infinite SNR. The results clearly show that the gap reduces with increasing number of transmit antennas, thereby concurring with the well-known result for the MaMIMO limit. We also derived an alternative simple approximate expression for the gap using the second order Taylor series approximation. The general case of correlated MIMO channel with non-zero mean and the actual EWSR gap are subjects of future work.

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