A MODIFIED SIGNAL PHASE UNWRAPPING ALGORITHM FOR RANGE ESTIMATION

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ABSTRACT

The Chinese Remainder Theorem (CRT) enables phase unwrapping for measurement of a distance using multiple wrapped phases of sinusoidal transmissions, such as in radar, sonar, or wireless geolocation. In the presence of measurement noise, the existing CRT algorithm assumes that the correct number of wrapping wavelengths can be obtained after a round off operation by ignoring the received signal noise. In this paper, additional hypotheses for the output of the round off process are formulated and a modified CRT algorithm for range estimation is proposed. We show that an improved distance reconstruction rate over the existing CRT algorithm is achieved from both analytical calculations and numerical simulations.

Index Terms— CRT, Wrapped phase, Phase ambiguity

1. INTRODUCTION

The problem of integer ambiguity appears in many engineering fields, such as sensor localization [1], computing, cryptography, and coding theory [2], where the Chinese Remainder Theorem (CRT), invented in the 2nd century [3], plays an important role for solutions. It is well known that the CRT provides a unique integer solution for a given set of known remainders, or correlated unknown remainders under mild conditions. Nevertheless, if the measured remainders are contaminated by noise, the CRT method cannot be applied directly as it is sensitive to noise with standard deviation comparable to the wavelength of underlying signal. In the fundamental work [4], an efficient CRT algorithm was proposed to address the ambiguity integer problem in noise. Later, an enhanced version of the CRT algorithm was presented and analyzed in [5–8]. The problem has also been approached using lattice-based methods in [9], where a lattice algorithm, with identical performance to the CRT algorithm in [6], is proposed. In addition, generalised lattice based algorithms and analysis are discussed in [10–14].

The ambiguity also arises in sensor localisation when a distance is measured using wrapped signal phase measurements, such as RIPS [1] and RFID [15]. It is crucial to reconstruct the correct integers from wrapped phase measurements in the first place. The range estimation can be far away from the ground truth if an incorrect integer solution is chosen. In this paper, we present an improved CRT algorithm with slightly more computational demand compared to the CRT algorithm in [4]. Additional hypotheses on the outcomes of round-off process are proposed and we show that theses hypotheses can be evaluated thanks to the nature of algorithm structure. The proposed algorithm is shown to achieve a higher probability in estimating the correct integer. The performance of the improved algorithm is analysed.

Following this introductory section, the conventional CRTs are reviewed in Section 2 along with some useful propositions and a lemma. In Section 3, an improved CRT for range estimation is presented and its performance is analysed briefly. The simulation is given in Section 4 and Section 5 concludes the paper.

2. PROBLEM DESCRIPTION AND CRT SOLUTION

Multiple sinusoidal signals of different wavelengths are assumed to illuminate an object away from the transmitter. The range estimation problem is to compute the distance between the transmitter and the object using wrapped and noisy phase measurements received by the object.

Let $r \in \mathbb{Z}$ be the unknown range to be estimated using the wrapped phases of the sinusoidal wave signal with wavelengths λ_i, $i = 1, \cdots, k$. The measurement model is described in [9]

$$y_i = y_{i,true} + \omega_i = r \bmod \lambda_i + \omega_i, \quad i = 1, \cdots, k \tag{1}$$

where y_i is the noisy measurement, $y_{i,true}$ is the noise-free measurement, ω_i is the noise, assumed to be normally distributed with mean 0 and variance $\delta^2\lambda_i^2$, i.e. $\omega_i \sim \mathcal{N}(0, \delta^2\lambda_i^2)$, where δ^2 is a small number.

The problem can be solved by the CRT algorithm [6]. The Theorem 1 are given without proof. A useful defintion is given before Theorem 1.

Definition 1. *A modular inverse of integer a is an integer b with respect to modular m satisfying $1 \equiv a\ b \mod m$.*

Theorem 1 (Conventional Chinese Remainder Theorem[4])**.** *Consider following congruence system*

$$\begin{cases} y_{1,true} = r \mod \lambda_1 \\ \vdots \\ y_{i,true} = r \mod \lambda_i \end{cases} \tag{2}$$

where $y_{i,true} \in \mathbb{Z}, i = 1, \cdots, k$ are remainders of $r \in \mathbb{Z}$ modulo $\lambda_i \in \mathbb{Z}$. Assume that GCD$(\lambda_i, \lambda_j) = 1, \forall\ i, j = 1, \cdots, k,\ i \neq j$ and $r <$ LCM$(\lambda_1, \cdots, \lambda_k)$, where GCD$(\cdot)$ is the greatest common divisor and LCM$(\cdot)$ is the least common multiple. Then a unique solution for r exists and is given by $r = \sum_{i=1}^{k} \xi_i y_{i,true}$, where $\xi_i = \frac{\prod_{j=1}^{k}\lambda_j}{\lambda_i} b_i$ and b_i is the modular inverse of $\frac{\prod_{j=1}^{k}\lambda_j}{\lambda_i}$ modulo λ_i.

Theorem 1 cannot be used directly in practice since it is sensitive to noise. Another form is described in Theorem 2 and this can be used to estimate the distance using noise measurements. The following proposition and lemma are useful in proving Theorem 2.

The following Proposition 1 is straightforward.

Proposition 1. *Let $q \in \mathbb{Z}^-$ and $\lambda \in \mathbb{Z}^+$, then $q \mod \lambda$ could be calculated by $q \mod \lambda = \lambda - (|q| \mod \lambda)$.*

Proposition 2. *Let $\{\lambda_1, \cdots, \lambda_k\} \in \mathbb{Z}^+$, and GCD$(\bar{\lambda}_i, \bar{\lambda}_j) = 1, \quad \forall\ i, j = 1, \cdots, k-1,\ i \neq j$, where $\bar{\lambda}_i = \frac{\lambda_i}{g_{i,k}}$ and $g_{i,k} \triangleq$ GCD(λ_i, λ_k). Then LCM$(\lambda_1, \cdots, \lambda_k) = \lambda_k \prod_{i=1}^{k-1} \bar{\lambda}_i$.*

Proof. Let $k = 2$. Then it is easily to see that LCM$(\lambda_1, \lambda_2) = \frac{\lambda_1\lambda_2}{g_{1,2}} = \bar{\lambda}_1\lambda_2$. For $k = 3$, we have

$$\begin{aligned} \text{LCM}(\lambda_1, \lambda_2, \lambda_3) &= \text{LCM}(\text{LCM}(\lambda_1, \lambda_3), \lambda_2) \\ &= \text{LCM}(\bar{\lambda}_1\lambda_3, \lambda_2) \\ &= g_{2,3}\bar{\lambda}_1 \frac{\lambda_3}{g_{2,3}} \frac{\lambda_2}{g_{2,3}} = \bar{\lambda}_1\bar{\lambda}_2\lambda_3 \end{aligned}$$

Repeating the same steps, we have LCM$(\lambda_1, \cdots, \lambda_k) = \lambda_k \prod_{i=1}^{k-1} \bar{\lambda}_i$ □

Lemma 1 is crucial in proving Theorem 2. A slightly different lemma is given in [4]. We will describe the difference between these two lemmas in the followed remark.

Lemma 1. *Let $\{\lambda_1, \lambda_2, x, y\} \in \mathbb{Z}^+$ and $q \in \mathbb{Z}$, where λ_1, λ_2, q are known and x, y are unknown, furthermore, suppose GCD$(\lambda_1, \lambda_2) = 1$. Consider following equations,*

$$\lambda_1 x - \lambda_2 y = q. \tag{3}$$

Then the integral solution for $\{x, y\}$ is, $n \in \mathbb{Z}$,

$$\begin{cases} x = (bq) \mod \lambda_2 + \lambda_2 n \\ y = \frac{((bq) \mod \lambda_2 + \lambda_2 n)\lambda_1 - q}{\lambda_2} \end{cases} \tag{4}$$

where $b \in \mathbb{Z}^+$ is the modular inverse of λ_1 modulo λ_2.

Proof. From Bézout's lemma[16], we know that there always exists a modular inverse of λ_1 modulo λ_2 if GCD$(\lambda_1, \lambda_2) = 1$, i.e.

$$\lambda_1 b - \lambda_2 \frac{\lambda_1 b - 1}{\lambda_2} = 1 \Rightarrow \lambda_1 bq - \lambda_2 \frac{\lambda_1 b - 1}{\lambda_2} q = q$$

where $\frac{\lambda_1 b - 1}{\lambda_2} \in \mathbb{Z}$.

If $bq \geq 0$, then $x = bq$ is a specific solution of (3). $x = (bq) \mod \lambda_2$ is also a solution of (3) since $\exists N \in \mathbb{Z}$ such that

$$y = \frac{\lambda_1((bq) \mod \lambda_2) - q}{\lambda_2} = \frac{\lambda_1(bq - N\lambda_2) - q}{\lambda_2} \in \mathbb{Z}$$

We have a similar result if $bq < 0$, using Proposition 1. Consequently, $(bq) \mod \lambda_2$ is a specific solution of (3) and the general solution is (4).

Next, we aim to show that all the solutions of (3) are included in (4). Suppose that there exits an integral solution $\{x_0, y_0\}$ for (3) which does not satisfy (4).Then, for specific $\{x_0, y_0\}$, one can find $\alpha \in \mathbb{Z}$ and $n_0 \in \mathbb{Z}$ such that

$$\begin{cases} x_0 = (bq) \mod \lambda_2 + \lambda_2 n_0 + \alpha \\ y_0 = \frac{((bq) \mod \lambda_2 + \lambda_2 n_0 + \alpha)\lambda_1 - q}{\lambda_2} \end{cases} \tag{5}$$

where $\alpha \neq 0$ and $\alpha \nmid \lambda_2$, where $a \nmid b$ means that a can not divided by b. Obviously,

$$y_0 = \frac{((bq) \mod \lambda_2 + \lambda_2 n_0)\lambda_1 - q}{\lambda_2} + \frac{\alpha\lambda_1}{\lambda_2} \triangleq C + \frac{\alpha\lambda_1}{\lambda_2}$$

where $C \in \mathbb{Z}$ based on our previous discussion. Since $y_0 \in \mathbb{Z}$, α is either 0 or $\alpha \mid \lambda_2$ which contradicts the assumption. Therefore, all solutions are included in (4). □

Remarks 1. *Another solution with a slightly different form for (3) is given in [4](see Lemma 1, [4]) as:*

$$\begin{cases} x = bq + \lambda_2 nq \\ y = \frac{(bq + \lambda_2 nq)\lambda_1 - q}{\lambda_2} \end{cases} \tag{6}$$

where b is the modular inverse of λ_1 modulo λ_2. However, (6) dose not give all solutions of (3). For example, consider an equation

$$7x - 9y = 2$$

with specific solution $\{x = 17, y = 13\}$. Then we can calculate that $b = 4$ is the modular inverse of 7 modulo 9. From (6), the solution of $7x - 9y = 2$ is given by $x = 4\cdot 2 + 9\cdot 2\cdot n \neq 17$, $\forall n \in \mathbb{Z}$. This is because the term nq in (6) belongs to

$$\{z | z \in \mathbb{Z} \text{ and } z \mod q \equiv 0\}$$

rather than $\mathbb{Z}$.

Based on Theorem 1 and Lemma 1, another form of the CRT algorithm [5], denoted by **Algorithm2**, is given in Theorem 2.

Theorem 2. *Consider the congruence system shown in (2). Assume that* $r < LCM(\lambda_1, \cdots, \lambda_k)$ *and* $GCD(\bar{\lambda}_i, \bar{\lambda}_j) = 1$, $\forall\ i, j = 1, \cdots, k-1,\ i \neq j$ *where* $\bar{\lambda}_i = \frac{\lambda_i}{g_{i,k}}$ *and* $g_{i,k} \triangleq GCD(\lambda_i, \lambda_k)$, $\forall i = 1, \cdots, k-1$. *Then r can be solved uniquely.*

Proof. The system (2) can be rewritten as

$$n_i \lambda_i + y_{i,true} = r, \ \ i = 1, \cdots, k \tag{7}$$

where $n_1, \cdots, n_k$ are unknown integers.

Subtracting the last equation in (7) from the first $k-1$ equations, we have following new system of congruence equations, where $\bar{\lambda}_i = \frac{\lambda_i}{g_{i,k}} \in \mathbb{Z}$ and $\bar{q}_{i,true} = \frac{y_{i,true} - y_{k,true}}{g_{i,k}} \in \mathbb{Z}$.

$$n_k \bar{\lambda}_{i,k} - n_i \bar{\lambda}_i = \bar{q}_{i,true}, \ \ i = 1, \cdots, k-1 \tag{8}$$

From Lemma 1 and assumption GCD$(\bar{\lambda}_i, \bar{\lambda}_j) = 1, \ \ \forall\ i$, $j = 1, \cdots, k,\ i \neq j$ we know that the solutions for the unknown integer n_k in (8) are

$$n_k = \bar{y}_{i,true} + n_i \bar{\lambda}_i, \ \ i = 1, \cdots, k-1 \tag{9}$$

where $\bar{y}_{i,true}$ can be determined using (8) and Lemma 1, and $0 \leq \bar{y}_{i,true} < \bar{\lambda}_i, i = 1, \cdots, k-1$. System (9) is a congruence system with co-prime moduli $\{\lambda_1, \cdots, \lambda_{k-1}\}$ and remainders $\{y_{1,true}, \cdots, y_{k-1,true}\}$. From Theorem 9, we know that the solution of n_k is

$$n_k = \sum_{i=1}^{k-1} \bar{\xi}_i \bar{y}_i \tag{10}$$

where $\bar{\xi}_i = \frac{\prod_{j=1}^{k} \bar{\lambda}_j}{\bar{\lambda}_i} \bar{b}_i$ and $\bar{b}_i \bar{\xi}_i \mod \bar{\lambda}_i = 1$. Then we have $r = n_k \lambda_k + y_k$. From (2), we know that the solution of (9) satisfies $n_k < \prod_{i=1}^{k-1} \bar{\lambda}_i$. Moreover, from the CRT, the solution of (7) satisfies $n_k <$ LCM$(\bar{\lambda}_1, \cdots, \bar{\lambda}_{k-1}) = \prod_{i=1}^{k-1} \bar{\lambda}_i$. Therefore, (10) will be the unique solution of n_k for system (7). In consequence, we can find a unique solution for r. □

Clearly, in practice, estimating $\{\bar{q}_{1,true}, \cdots, \bar{q}_{k-1,true}\}$ in (8) is a key step in Theorem 2 for estimating r via the noisy measurements $\{y_1, \cdots, y_k\}$ which are not integers. Define, for $i = 1, \cdots, k-1$,

$$\bar{q}_i \triangleq \frac{(y_i - y_k)}{g_{i,k}}, \ \text{ and } \ \bar{q}_{i,true} \triangleq \frac{(y_{i,true} - y_{k,true})}{g_{i,k}}. \tag{11}$$

As suggested in [5], the estimate of $\bar{q}_{i,ture}$ is given by a rounding operation $\hat{\bar{q}}_i = \lceil \bar{q}_i \rfloor$. Theorem 2 is then applicable to obtain the estimation of n_k, denoted by $\hat{n}_k$.

3. PROPOSED CRT SOLUTION

Suppose that a set of signal wavelengths $\{\lambda_1, \cdots, \lambda_k\}$ satisfy the conditions of Theorem 2 and we have $\lambda_1 > \lambda_2 > \cdots > \lambda_k$. The smallest wavelength λ_k is selected as the "reference" in the measurement operation (8) because ω_k has the minimum variance.

After obtainign $\hat{n}_k$ via noisy measurements $\{y_1, \cdots, y_k\}$ by Theorem 2, the estimation of r, $\hat{r}$, can be done using a maximum likelihood method [9], i.e

$$\hat{r} = W \textstyle\sum_{i=1}^{k} (\hat{n}_i \lambda_i + y_i) W_i \tag{12}$$

where $W_i = 1/\lambda_i^2, W = 1/\sum_{i=1}^{k} W_i$ and $\hat{n}_i = \left\lceil \frac{\hat{n}_k \lambda_k + y_k - y_i}{\lambda_i} \right\rfloor$.

In this estimation, the calculation of $\hat{\bar{q}}_i$ is crucial and the algorithm will return the correct answer if

$$-1/2 < \frac{\omega_i - \omega_k}{g_{i,k}} < 1/2, \ \ \forall i = 1, \cdots, k-1$$

otherwise, the algorithm will select an incorrect value for $\hat{n}_k$. Furthermore,

$$\left[\frac{\omega_1 - \omega_k}{g_{1,k}}, \cdots, \frac{\omega_{k-1} - \omega_k}{g_{k-1,k}} \right] \sim \mathcal{N}\left(\mathbf{0}, \delta^2 \mathbf{\Sigma}\right)$$

where $\mathbf{0}$ is the zero vector and $\delta^2 \mathbf{\Sigma}$ is the covariance matrix. Since $\left[\frac{\omega_1 - \omega_k}{g_{1,k}}, \cdots, \frac{\omega_{k-1} - \omega_k}{g_{k-1,k}} \right]$ can be written as $[\omega_1, \cdots, \omega_k]\, \mathbf{M}^T$ and $[\omega_1, \cdots, \omega_k]$ is normally distributed with mean 0 and covariance $\delta^2\, \mathrm{diag}(\lambda_1^2, \cdots, \lambda_k^2)$, where

$$\mathbf{M} = \begin{bmatrix} \frac{1}{g_{1,k}} & 0 & \cdots & 0 & -\frac{1}{g_{1,k}} \\ 0 & \frac{1}{g_{2,k}} & \cdots & 0 & -\frac{1}{g_{2,k}} \\ 0 & 0 & \cdots & \frac{1}{g_{k-1,k}} & \frac{1}{g_{k-1,k}} \end{bmatrix},$$

we have $\mathbf{\Sigma} = \mathbf{M}\, \mathrm{diag}(\lambda_1^2, \cdots, \lambda_k^2) \mathbf{M}^T$.

The CRT solution described above indicates that for $\hat{\bar{\mathbf{q}}} = [\hat{\bar{q}}_1, \cdots, \hat{\bar{q}}_{k-1}]$, an integer set $\mathbf{Z}_0 = [z_1, \cdots, z_{k-1}]$ exists such that $\hat{\bar{q}}_i + z_i = \bar{q}_{i,true}, \ \ \forall i = 1, \cdots, k-1$ for a given set of measurement $\{y_1, \cdots, y_k\}$. The algorithm implemented under Theorem 2 will return the correct $\hat{n}_k$, i.e. $\hat{n}_k = n_k$, if $\hat{\bar{\mathbf{q}}} + \mathbf{Z}_0$ is used. However, in practice, the elements contained in $\mathbf{Z}_0$ are unknown, and so we need to estimate the integer set $\mathbf{Z}_0$. It should be noticed that $\mathbf{Z}_0$ can be the zero vector.

Let $\mathbf{Z}_j = [z_{j,1}, \cdots, z_{j,k-1}], \ j = 1, 2, \cdots, \ z_{j,i} \in \mathbb{Z}$ contain all possible values of $\mathbf{Z}_0$. In practice, since δ is small and $\lambda_k < \lambda_i, \forall i = 1, \cdots, k-1$, the variance of $(\omega_i - \omega_k)$ is typically not large. We may reasonably assume that $z_{j,i} \in \{-1, 0, 1\}$[1]. It is easy to find the max value of j is $j_{max} = 3^{k-1}$. Let $\mathcal{Z} = [\mathbf{Z}_1^T, \cdots, \mathbf{Z}_{j_{max}}^T]^T$ be a $j_{max} \times (k-1)$ dimensional matrix containing all possible value of $\mathbf{Z}_0$. The probability that $\mathbf{Z}_0 \in \mathcal{Z}$ can be obtained by

$$Pr\left(\mathbf{Z}_0 \in \mathcal{Z}\right) = \int_{\mathbf{\Omega}} \frac{1}{(2\pi\delta^2 |\mathbf{\Sigma}|)^{\frac{k}{2}}} \exp\left\{ -\frac{1}{2} \mathbf{x}^T (\delta^2 \mathbf{\Sigma})^{-1} \mathbf{x} \right\} d\mathbf{x}$$

[1]In the standard CRT algorithm, it is assume that $z_{j,i} = 0$

where $\mathbf{\Omega}$ is the integral of the area of a rectangular parallelepiped with lower and upper limits $-\frac{3}{2}$ and $\frac{3}{2}$ respectively.

Now the main goal is to find the estimator of $\mathbf{Z}_0$. An efficient algorithm is as follows:

Algorithm 3: (13)

$$\hat{j}_0 = \arg \min_{j \in \{1,\cdots,3^{k-1}\}} \frac{1}{k} \sum_{i=1}^{k} \hat{\hat{\omega}}_{j,i}^2 \quad (14)$$

$$s.t. \quad \hat{n}_{j,k} = \textbf{Algorithm2}\left(\hat{\hat{\mathbf{q}}} + \mathbf{Z}_j\right), \quad \mathbf{Z}_j \in \mathcal{Z} \quad (15)$$

$$\hat{n}_{j,i} = \left\lceil \frac{\hat{n}_{j,k}\lambda_k + y_k - y_i}{\lambda_i} \right\rfloor, \forall i = 1, \cdots, k-1 \quad (16)$$

$$\hat{\hat{\omega}}_{j,i} = (\lfloor \hat{r}_j \rceil - (\hat{n}_{j,i}\lambda_i + y_i))/\lambda_i \quad (17)$$

$$\hat{r}_j = \mathbf{E}[r|\hat{n}_{j,1}, \cdots, \hat{n}_{j,k}] \quad (18)$$

where $\mathbf{E}[r|\hat{n}_{j,1}, \cdots, \hat{n}_{j,k}]$ is the estimation of r given $\hat{n}_{j,1}, \cdots, \hat{n}_{j,k}$ using (12), and **Algorithm2** is the CRT algorithm described in Theorem 2. Thus, the estimator of $\mathbf{Z}_0$ is the $\hat{j}_0$-th row of $\mathcal{Z}$ and so the new estimation of n_k is $\hat{n}_{\hat{j}_0,k}$, i.e. $\hat{n}_{\hat{j}_0,k} = \textbf{Algorithm2}(\hat{\hat{\mathbf{q}}} + \mathbf{Z}_{\hat{j}_0})$.

Because space limitation, we only briefly investigate the performance of the estimator (13) and provide the conclusion (19) without proof. In (13), (15) and (16) provide the estimate of the $[\hat{n}_{j,1}, \cdots, \hat{n}_{j,k}]$ according to a given integer vector $\mathbf{Z}_j$ and fixed $\hat{\hat{\mathbf{q}}}$, and (17) estimates the measurement noise based on $[\hat{n}_{j,1}, \cdots, \hat{n}_{j,k}]$. If $\mathbf{Z}_0$ is in $\mathcal{Z}$, then the $\hat{r}_j$ is close to the true r and the corresponding noise estimation $\left[\hat{\hat{\omega}}_{j,1}, \cdots, \hat{\hat{\omega}}_{j,k}\right]$ is approximately normally distributed with mean 0 and variance δ^2. If the frequencies used are from same band, then their values will be close to each other and we have following conclusion: the reconstruction probability given δ^2, $Pr(\hat{n}_{\hat{j}_0,k} = n_k|\delta^2)$, of the proposed algorithm (14) is approximately

$$\prod_{i=1}^{k} Pr\left(-1/\tilde{\lambda} + 2|\tilde{\omega}_i| < 0\right) \quad (19)$$

where $\tilde{\lambda} = \frac{1}{k}\sum_{i=1}^{k} \lambda_i$ and $\tilde{\omega} \sim \mathcal{N}(0, \delta^2)$.

4. SIMULATION VERIFICATION

The proposed algorithm, i.e. **Algorithm 3**, in Section 3 is evaluated via simulation using two sets of wavelengths: $\Lambda_1 = [46\ 47,\ 49,\ 51,\ 59,\ 61]$ and $\Lambda_2 = [21\ 19\ 17\ 13]$. The first set of wavelengths is from the Wifi 5Ghz band and the latter from $1 \sim 26$MHz which includes all possible frequency differences of US UHF band $902 \sim 928$MHz[10]. These two bands are widely used in localization via range estimation. The distance r is randomly selected between 0 and the LCM of all used wavelengths. The parameter δ in the measurement noise variance $\delta^2\lambda_i^2$ is chosen as $-20\log_{10}\delta = 46 : 2 : 66$, which provides an indication for both noise level and signal-to-noise ratio in the simulation. All simulation results illustrated are averaged over 5000 Monte Carlo runs. The algorithm performance is evaluated in terms of the reconstruction probability $Pr(n_k = \hat{n}_k)$ versus the phase measurement noise level.

The simulation results of reconstruction probability both from Monte Carlo simulation and theoretic computation via (19) are given in Fig. 1. As can be seen, the simulation results agree with the theoretic result. By way of comparison, the performance of the existing algorithm **Algorithm2** is plotted along with that of the proposed algorithm in Fig.1. Clearly, using both wavelength sets the proposed algorithm outperforms **Algorithm2** in terms of reconstruction probability of the underlying integer set when signal-to-noise ratio is low. Finally, we point out that additional computational overhead

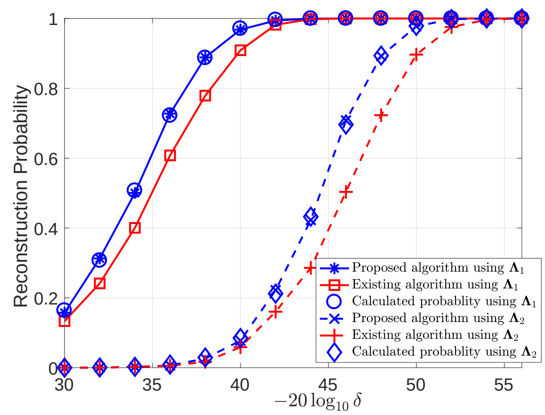

Fig. 1: Reconstruction probability versus error level

of the proposed algorithm over the existing CRT algorithm in [4] is fractional.

5. CONCLUSIONS

A modified CRT algorithm for range estimation is proposed. Compared with existing approaches, the proposed algorithm takes a set of more probable integer outcomes after the rounding off operation is taken into account, by evaluating additional hypotheses based on measurements and known conditions. The probability of correct integer set reconstruction is accordingly derived. Simulation results are presented showing the performance of the proposed algorithm.

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