SEMI-CLOSED FORM SOLUTION FOR SUM RATE MAXIMIZATION IN DOWNLINK MULTIUSER MIMO VIA LARGE-SYSTEM ANALYSIS

Hei Victor Cheng, Emil Björnson, and Erik G. Larsson

Department of Electrical Engineering (ISY), Linköping University, Sweden
Email: {hei.cheng, emil.bjornson, erik.g.larsson}@liu.se

ABSTRACT

This work introduces a new approach to solve the joint precoding and power allocation for sum rate maximization problem in the downlink multiuser MIMO by a combination of random matrix theory and optimization theory. The new approach results in a simplified problem that, though non-convex, obeys a simple separable structure. The sum rate maximization problem is decomposed into different single-variable optimization problems that can be solved in parallel. A water-filling-like solution is found, which can be applied under some mild conditions on the SNRs of the users. The proposed scheme provides large gains over heuristic solutions when the number of users in the cell is large, which suggests the applicability in massive MIMO systems.

Index Terms— Beamforming, Multiuser MIMO, Massive MIMO, Optimization, Deterministic Equivalence

1. INTRODUCTION

Multi-antenna techniques are key for achieving high spectral efficiency in the next generation cellular networks (5G). Employing multiple antennas at the base station (BS) to serve multiple users in the same time and frequency resource can lead to a linear increase in the sum channel capacity with respect to the number of users. To achieve this, precoding, also known as digital beamforming, has to be used at the BS. The capacity-achieving precoding for the broadcast channels is the dirty-paper coding scheme [1]. However, it is a non-linear technique which requires high computational complexity at the BS which prohibits it from practical implementation.

In practice, linear beamforming techniques are preferred. The design of optimal linear beamformers is of interest particular as they perform well at low computational complexity. The problem of finding optimal transmit beamforming had received great attentions [2–8]. These work focused on minimizing the transmit power while satisfying certain quality-of-service (QOS) targets at the users such as the rate requirement. For general systems, maximizing different utilities based on the QOS is preferable as the SINR targets might not be known. Among different utilities, sum rate is the one of main interest as it characterizes the total throughput of the system.

1.1. Related Work

The algorithmic solution for the power minimization problem with QOS targets is found in [2–4]. Recently, large-system analysis was applied to investigate the optimal structure of the algorithmic solution [5–8]. Considering the sum rate maximization problem, there are less analytic results available. In general, the sum rate maximization problem was shown in [9] to be NP-hard except in some special cases [10]. Most previous works focus on finding local optimal solutions and have no guarantee for global optimality [11–13]. Another line of work is applying global optimization techniques, but this suffers from high complexity and therefore only act as benchmarks for small-scale systems [14].

1.2. Contributions of this Paper

1. We propose a novel beamforming design that makes use of recent results from large-system analysis. This greatly reduces the complexity of the sum rate problem, while providing global optimality in the large-system limit.
2. The proposed optimization procedure only needs to be performed when the large-scale fading parameters change, and only some scalar parameters need to be calculated. The resulting optimization has almost the same complexity as zero-forcing (ZF) applied together with the water-filling algorithm, which is feasible for practical use.
3. The spirit of the proposed method, which exploits a combination of random matrix theory and optimization theory, sheds light on how large optimization problems can be easier to solve than small problems.

2. SYSTEM MODEL

Consider a single-cell multi-user MIMO system with M antennas at the BS serving $K \leq M$ single-antenna users in the downlink. The BS performs multiuser beamforming to serve the K users in the same time and frequency resource block. Denote by $\boldsymbol{g}_k \sim \mathcal{CN}(\boldsymbol{0}, \beta_k \boldsymbol{I}_M)$ the channel realization between the BS and user k, where β_k represents the large-scale fading. The downlink system model for the transmission to user k can be written as

$$y_k = \boldsymbol{g}_k^H \boldsymbol{W} \boldsymbol{s} + n_k, \quad k = 1, \ldots, K, \tag{1}$$

where $\boldsymbol{s} = [s_1, \ldots, s_K]^T \in \mathbb{C}^{K\times 1}$, s_k denotes the information symbol intended for user k, $n_k \sim \mathcal{CN}(0, \sigma^2)$ represents i.i.d. additive white Gaussian noise, and $\boldsymbol{W} = [\boldsymbol{w}_1, \ldots, \boldsymbol{w}_K]$ is the beamforming matrix where $\boldsymbol{w}_k \in \mathbb{C}^{M\times 1}$ is the beamformer of user k.

With perfect CSI at the BS and at the users, the instantaneous achievable rate in b/s/Hz for user k can be written as

$$R_k = \log_2(1 + \mathrm{SINR}_k), \quad k = 1, \ldots, K, \tag{2}$$

where SINR_k represents the signal-to-interference-plus-noise-ratio

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(SINR) at user k, given by

$$\mathrm{SINR}_k = \frac{|\boldsymbol{g}_k^H \boldsymbol{w}_k|^2}{\sum_{i=1, i\neq k}^{K} |\boldsymbol{g}_k^H \boldsymbol{w}_i|^2 + \sigma^2}. \tag{3}$$

From a system design perspective, the beamformer matrix $\boldsymbol{W}$ is optimized to maximize a certain utility $U(\mathrm{SINR}_1, \ldots, \mathrm{SINR}_K)$ characterized by the SINR values. In this work, we choose to maximize the sum rate in the cell. This corresponds to choosing

$$U(\mathrm{SINR}_1, \ldots, \mathrm{SINR}_K) = \sum_{k=1}^{K} \log_2(1 + \mathrm{SINR}_k). \tag{4}$$

The optimization problem that we are interested in solving is

$$\begin{aligned} \underset{\{\boldsymbol{w}_k\}, \{\gamma_k \geq 0\}}{\text{maximize}} \quad & \sum_{k=1}^{K} \log_2(1 + \gamma_k) \\ \text{subject to} \quad & \sum_{k=1}^{K} \|\boldsymbol{w}_k\|^2 \leq Q \\ & \mathrm{SINR}_k \geq \gamma_k, \forall k, \end{aligned} \tag{5}$$

where Q is the maximum transmit power.

3. OPTIMAL LINEAR BEAMFORMING

In this section, we present the general form of the optimal linear beamforming vectors and how they converge in the asymptotic regime where $K, M \to \infty$ with a fixed $c = \frac{K}{M} \in (0, 1]$. We start by reviewing the solution to the power minimization problem and then apply it to the problem in (5).

3.1. Optimal Beamforming for Power Minimization

We first review results for a closely related problem, namely the power minimization problem with SINR targets, which was the focus of many previous works (see [15] and references therein).

The power minimization problem is formulated to find the minimum power P_o satisfying SINR constraints:

$$\begin{aligned} P_o \triangleq \min_{\{\boldsymbol{w}_k\}} \quad & \sum_{k=1}^{K} \|\boldsymbol{w}_k\|^2 \\ \text{subject to} \quad & \frac{|\boldsymbol{g}_k^H \boldsymbol{w}_k|^2}{\sum_{i=1, i\neq k}^{K} |\boldsymbol{g}_k^H \boldsymbol{w}_i|^2 + \sigma^2} \geq \gamma_k, \quad \forall k. \end{aligned} \tag{6}$$

The structure of the optimal linear precoder matrix $\boldsymbol{W}_o = [\boldsymbol{w}_1, \ldots, \boldsymbol{w}_K]$ is given by

$$\boldsymbol{W}_o = \left(\sum_{k=1}^{K} \lambda_k \boldsymbol{g}_k \boldsymbol{g}_k^H + M \boldsymbol{I}_M \right)^{-1} \boldsymbol{G} \boldsymbol{P}^{\frac{1}{2}}, \tag{7}$$

where $\boldsymbol{G} = [\boldsymbol{g}_1, \ldots, \boldsymbol{g}_K]$ denotes the channel matrix from the BS to all the K users and $\lambda_1, \ldots, \lambda_K$ are the optimal Lagrange multipliers given by the positive unique fixed-points of the equations:

$$\left(1 + \frac{1}{\gamma_k}\right) \lambda_k = \frac{1}{\boldsymbol{g}_k^H \left(\sum_{i=1}^{K} \lambda_i \boldsymbol{g}_i \boldsymbol{g}_i^H + M \boldsymbol{I}_M \right)^{-1} \boldsymbol{g}_k}. \tag{8}$$

The optimal power allocation is a diagonal matrix $\boldsymbol{P} = \mathrm{diag}(p_1, \ldots, p_K)$ given by the vector $\boldsymbol{p} = [p_1, \ldots, p_K]^T$

$$\boldsymbol{p} = \sigma^2 \boldsymbol{A}^{-1} \boldsymbol{1} \tag{9}$$

with the $(k, i)^{th}$ element of $\boldsymbol{A}$ being

$$A_{i,j} = \begin{cases} \frac{1}{\gamma_k} |\boldsymbol{g}_k^H \boldsymbol{c}_k|^2, & k = i \\ -|\boldsymbol{g}_k^H \boldsymbol{c}_i|^2, & k \neq i \end{cases} \tag{10}$$

where $\boldsymbol{c}_k$ is the k^{th} column of

$$\boldsymbol{C} = \left(\sum_{k=1}^{K} \lambda_k \boldsymbol{g}_k \boldsymbol{g}_k^H + M \boldsymbol{I}_M \right)^{-1} \boldsymbol{G}, \tag{11}$$

which can be interpreted as the beamforming direction to user k, $\boldsymbol{p}$ is chosen such that all the SINR constraints are satisfied with equality, and $\boldsymbol{1}$ is a vector with all entries being 1. We see that for any finite M and K, the optimal beamforming vectors are parameterized by $\lambda_1, \ldots, \lambda_K$ and $p_1, \ldots, p_K$ for which the optimal values are obtained through solving equations (8) and (9). Particularly, the optimal $\{\lambda_k\}$ are found through solving fixed-point equations, therefore no insights into the solution to (5). Nevertheless, in the large-system regime where $K,\ M \to \infty$ with $K/M = c \in (0, 1]$, recent results from random matrix theory can be used to obtain the asymptotic values of $\lambda_1, \ldots, \lambda_K$ and $p_1, \ldots, p_K$ [7]. These are as follows:

$$\max_k |\lambda_k - \bar{\lambda}_k| \xrightarrow{a.s.} 0, \quad \max_k |p_k - \bar{p}_k| \xrightarrow{a.s.} 0 \tag{12}$$

where $\xrightarrow{a.s.}$ denotes the almost sure convergence, and $\bar{\lambda}_k$ and $\bar{p}_k$ are "deterministic equivalents" for λ_k and p_k given by

$$\bar{\lambda}_k = \frac{\gamma_k}{\beta_k \eta}, \quad \bar{p}_k = \frac{\gamma_k}{\beta_k \eta^2} \left(P_o + \frac{\sigma^2}{\beta_k} (1 + \gamma_k)^2 \right) \tag{13}$$

and

$$\eta = 1 - \frac{c}{K} \sum_{k=1}^{K} \frac{\gamma_k}{1 + \gamma_k}, \quad P_o = \frac{c\sigma^2}{\eta K} \sum_{k=1}^{K} \frac{\gamma_k}{\beta_k}. \tag{14}$$

3.2. Optimal Beamformer for Sum Rate Maximization

We have seen that, for any given SINR targets, the minimal power is asymptotically given by P_o in (14). Moreover, at the optimal point, all SINR targets are satisfied with equality. For any given set of γ_k, P_o is the minimum power that can achieve these SINR targets. As a result, P_o can be written as a function of $\gamma_1, \ldots, \gamma_K$ and we can transform problem (5) into the following form:

$$\begin{aligned} \underset{\{\gamma_k \geq 0\}}{\text{maximize}} \quad & \sum_{k=1}^{K} \log_2(1 + \gamma_k) \\ \text{subject to} \quad & P_o = \frac{c\sigma^2}{\eta K} \sum_{k=1}^{K} \frac{\gamma_k}{\beta_k} \leq Q. \end{aligned} \tag{15}$$

Problem (15) is asymptotically equivalent to (5), but we will now use it as a large-scale approximation for finite M and K. We observe that P_o depends on $\{\gamma_k\}$ in a complicated way. To simplify the problem we rewrite the constraint by first observing that η is always positive since $K \leq M$:

$$\sum_{k=1}^{K} \frac{c\sigma^2}{K\beta_k} \gamma_k \leq Q - \frac{cQ}{K} \sum_{k=1}^{K} \frac{\gamma_k}{1 + \gamma_k}. \tag{16}$$

Then we divide both sides of (16) with cQ/K to get

$$\sum_{k=1}^{K} a_k \gamma_k \leq M - \sum_{k=1}^{K} \frac{\gamma_k}{1+\gamma_k} \tag{17}$$

where we have defined

$$a_k = \frac{\sigma^2}{\beta_k Q}, \quad k = 1, \ldots, K, \tag{18}$$

which can be interpreted as the inverse signal-to-noise-ratio (SNR) at the users. Finally, we have rewritten (15) as

$$\begin{aligned} \underset{\{\gamma_k \geq 0\}}{\text{maximize}} \quad & \sum_{k=1}^{K} \log_2(1+\gamma_k) \\ \text{subject to} \quad & \sum_{k=1}^{K} \left(a_k \gamma_k + \frac{\gamma_k}{1+\gamma_k} \right) \leq M. \end{aligned} \tag{19}$$

We see that with the use of deterministic equivalents, we obtain the large-system approximation (19) of problem (5) and the approximation is tight when M and K are large. Moreover, (19) has a clear interpretation: we are allocating SINR values to parallel channels with a total cost constraint where the cost of allocating a particular SINR value γ_k to user k is $a_k\gamma_k + \frac{\gamma_k}{1+\gamma_k}$.

4. SUFFICIENT OPTIMALITY CONDITIONS

In this section, we characterize the optimality conditions for problem (19), based on which we develop a water-filling-like algorithm to find the optimal solution.

4.1. Sufficient Condition for Global Optimality

The optimal solution does not change if we change the base of $\log_2$ to $\ln$ in the objective function and this makes the calculations simpler. Define the Lagrange multiplier $\mu \geq 0$, then the Lagrangian function of (19) is

$$\begin{aligned} & L(\gamma_1, \ldots, \gamma_K, \mu) \\ &= \sum_{k=1}^{K} \ln(1+\gamma_k) - \mu \left(\sum_{k=1}^{K} \left(a_k \gamma_k + \frac{\gamma_k}{1+\gamma_k} \right) - M \right). \end{aligned} \tag{20}$$

We are ready to present the optimality conditions:

Lemma 1. *The vector $\boldsymbol{\gamma}^* = [\gamma_1^*, \ldots, \gamma_K^*]^T$ is the optimal solution to* (19) *if $\boldsymbol{\gamma}^*$ is feasible and there exists $\mu^* \geq 0$ such that the following conditions are satisfied:*

$$\boldsymbol{\gamma}^* = \arg \max_{\boldsymbol{\gamma} \geq \mathbf{0}} L(\boldsymbol{\gamma}, \mu^*) \tag{21}$$

$$\mu^* \left(\sum_{k=1}^{K} \left(a_k \gamma_k^* + \frac{\gamma_k^*}{1+\gamma_k^*} \right) - M \right) = 0. \tag{22}$$

Proof. For any feasible $\boldsymbol{\gamma}$, we have the following chain of inequalities

$$\begin{aligned} \sum_{k=1}^{K} \ln(1+\gamma_k^*) &= L(\boldsymbol{\gamma}^*, \mu^*) \geq L(\boldsymbol{\gamma}, \mu^*) \\ &\geq \sum_{k=1}^{K} \ln(1+\gamma_k). \end{aligned} \tag{23}$$

The equality in (23) is due to the condition in (22). The first inequality holds as $\boldsymbol{\gamma}^*$ is the maximizer of the Lagrangian function. The last inequality holds as the Lagrangian is always an upper bound on the original objective function of a maximization problem for any feasible $\boldsymbol{\gamma}$. Optimality follows since $\boldsymbol{\gamma}^*$ is feasible and the resulting objective function is larger than or equal to any other feasible point. □

From Lemma 1 and the fact that the objective function is separable in the K optimization variables, we have

$$\begin{aligned} & \max_{\boldsymbol{\gamma} \geq \mathbf{0}} L(\boldsymbol{\gamma}, \mu) = \\ & \sum_{k=1}^{K} \max_{\gamma_k \geq 0} \left(\ln(1+\gamma_k) - \mu \left(a_k \gamma_k + \frac{\gamma_k}{1+\gamma_k} \right) \right) + \mu M. \end{aligned} \tag{24}$$

As μM is a constant that does not depend on $\boldsymbol{\gamma}$, the optimization problem (19) can be decomposed into K single-variable subproblems in the following form:

$$\max_{\gamma_k \geq 0} \left(\ln(1+\gamma_k) - \mu \left(a_k \gamma_k + \frac{\gamma_k}{1+\gamma_k} \right) \right). \tag{25}$$

The solution to (25) can be found for a given μ as given by the following theorem.

Theorem 1. *The optimal γ_k for* (25) *for a given μ is*

$$\gamma_k(\mu) = \begin{cases} 0, & \mu > 1/(1+a_k) \\ \frac{1-2\mu a_k + \sqrt{1-4\mu^2 a_k}}{2\mu a_k}, & \mu \leq 1/(1+a_k) \end{cases} \tag{26}$$

when $a_k \geq 1$ and

$$\gamma_k(\mu) = \begin{cases} 0, & \mu > \alpha(a_k) \\ \frac{1-2\mu a_k + \sqrt{1-4\mu^2 a_k}}{2\mu a_k}, & \mu \leq \alpha(a_k) \end{cases} \tag{27}$$

when $a_k < 1$, where $\alpha(a_k)$ is the solution of the following equation in μ:

$$\ln \left(1 + \frac{1-2\mu a_k + \sqrt{1-4\mu^2 a_k}}{2\mu a_k} \right) = \mu(1-a_k) + \sqrt{1-4\mu^2 a_k}. \tag{28}$$

Proof Sketch. The solution is obtained by solving for the stationary points in (25), identifying the right one corresponding to the maximum, and comparing to the boundary points. The detailed proof is omitted here due to limited space, but will be provided in the journal version of this work. □

After applying Theorem 1, what remains is to find the μ such that equality is met in the following equation:

$$\sum_{k=1}^{K} \left(a_k \gamma_k(\mu) + \frac{\gamma_k(\mu)}{1+\gamma_k(\mu)} \right) = M. \tag{29}$$

This can be done via the bisection method as $\gamma_k(\mu)$ is a monotonically decreasing function in μ. This leads to a water-filling-like algorithm of finding the "water level" μ which can be implemented efficiently and can be computed in parallel.

4.2. Sufficient Conditions for Strong Duality

The main drawback of our new approach to solve the sum rate maximization problem is that the optimality condition is only sufficient

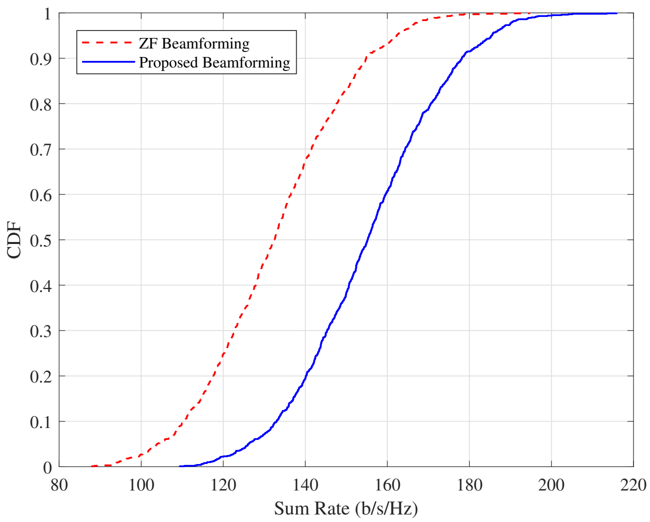

Fig. 1. CDF of the sum rate with $M = 100$, $K = 60$, $R = 500$ m for ZF beamforming and the proposed beamforming.

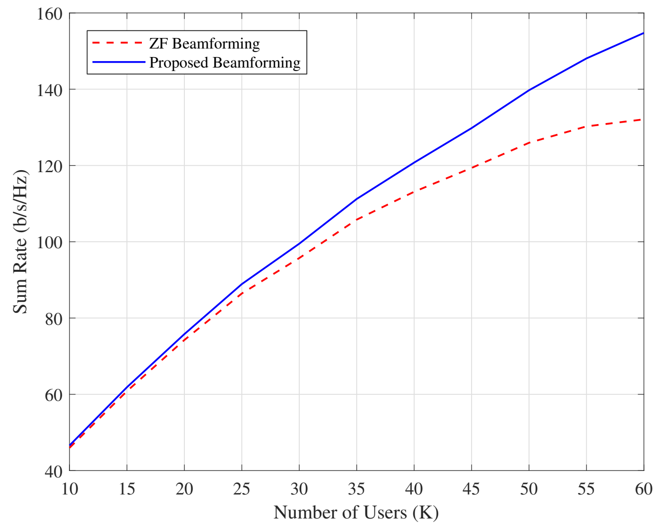

Fig. 2. Average sum rate with $M = 100$, $R = 500$ m and varying K from 10 to 60 for ZF beamforming and the proposed beamforming.

but not necessary. There exist cases where μ satisfying (29) does not exist, since $\gamma_k(\mu)$ is not continuous when $a_k < 1$. When such μ exist, strong duality holds and it is called the *geometric multiplier* [16]. In the following we provide an analytic sufficient condition for which strong duality holds.

Proposition 1. *Denote the number of users with $a_k < 1$ as K_h, then strong duality holds when $K_h \leq M/3 + 1$.*

Proof Sketch. The proof consists of three main steps. First we show that for $a_i \leq a_k$ we always have $\gamma_i(\mu) \geq \gamma_k(\mu)$. Second, we prove that $\alpha(a_k) > 1/(1 + a_k)$. In the last step, we denote $a_{\max}$ as the maximum a_k such that $a_k < 1$ and assume $\mu > 1/(1 + a_{\max})$, we then find an upper bound of the total cost as

$$\sum_{k=1}^{K} \left(a_k \gamma_k(\mu) + \frac{\gamma_k(\mu)}{1 + \gamma_k(\mu)} \right) \leq 3(K_h - 1). \tag{30}$$

Then under condition $K_h \leq M/3 + 1$, the existence of a geometric multiplier μ is guaranteed as we are ensured to be in the continuous regime of $\gamma_k(\mu)$ for all k for (29) to hold. $\square$

This sufficient condition generally holds in cellular networks as most users have rather low SNRs, which implies large a_ks. Moreover, in practice, there is interference from other cells, so the resulting received SINR will be lower, which increase $a_1, \ldots, a_K$ and makes the condition easier to be satisfied.

5. SIMULATION RESULTS

In this section, we present simulation results to demonstrate the benefits of our proposed algorithms and compare the performance with the ZF beamforming, which is generally considered to be close to optimal in massive MIMO. Consider a scenario with $M = 100$ antennas. The power optimization of the ZF scheme is performed by water-filling to optimize the sum rate. The users are assumed to be uniformly and randomly distributed in a cell with radius $R = 500$ m and no user is closer to the BS than 100 m. The path-loss model is chosen as $\beta_k = z_k / r_k^{3.8}$ where r_k is the distance of user k from the BS and z_k is log-normal distributed with a standard deviation of 8 dB and represents the independent shadowing effect. We choose $Q = 10^{-0.5} \cdot R^{3.8}$ such that the median SNR at the cell edge is -5 dB. The Monte-Carlo simulation is run for 1000 realizations, where the user locations and channels are random in each realization.

In Fig. 1, we plot the CDF of the sum rate for different random user locations the proposed scheme and ZF with $K = 60$. From the figure we observe that there is a significant gap between the proposed beamformer and ZF beamforming. For example, at the 90 percentile, there is a gain of about 20 b/s/Hz with the proposed scheme.

In Fig. 2, we plot the average sum rate for different K. From the figure we see that when $c = K/M$ is small, ZF is good enough and performs close to the proposed scheme. However as c increases, the gap between the proposed scheme and ZF increases significantly.

6. CONCLUSION

We introduced a new approach to solve the sum rate maximization problem in the downlink multiuser MIMO by exploiting a large-system approximation from random matrix theory and optimization theory. A water-filling-like semi-closed form solution is found. We draw the following conclusions:

1. The sum rate maximization problem can be simplified to a separable programming problem which, though non-convex, *can* be solved efficiently thanks to the specific structure.

2. The performance comparison with ZF beamforming showed that the proposed scheme is beneficial when the number of users in the cell is large, which suggests the applicability in massive MIMO systems.

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