OPTIMAL TONE RESERVATION FOR PEAK TO AVERAGE POWER CONTROL OF CDMA SYSTEMS

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ABSTRACT

In this paper we study the tone reservation technique for the reduction of the peak to average power ratio (PAPR) in code division multiple access (CDMA) systems that employ the Walsh functions. In the tone reservation method, the available carriers are partitioned into two sets, the information set, which carries the information, and the compensation set, which is used to reduce the PAPR. Central questions are: What is the best possible reduction of the PAPR? What is the optimal information set that achieves this reduction, and how can it be found? What is the general structure of the information set? So far, the answers were unknown. In this paper we completely solve these questions for CDMA systems that employ the Walsh functions. Interestingly, using the first N Rademacher functions is optimal under all sets of size N.

Index Terms— Peak to average power, tone reservation, code division multiple access, Walsh system, optimal constant

1. INTRODUCTION

Code division multiple access (CDMA) is a transmission technique that is used in many systems, for example in 3G and UMTS, GPS, and Galileo [1]. Moreover, multiple extensions such as multicarrier CDMA [2] exist.

The control of the peak to average power ratio (PAPR) is an important task in any orthogonal transmission scheme, and thus also for CDMA systems that employ orthogonal functions [3–6]. Large PAPR values are undesired, because they can overload amplifiers, distort the signals, and lead to out-of-band radiation. For a further discussion of these concepts and problems, we would like to refer to [6].

In order to reduce the PAPR, several methods have been proposed [7, 8], among them the popular tone reservation method [9–11], which we consider in this paper. In this method, the set of available carriers is partitioned into two sets: the information set $\mathcal{K}$, which is used to carry the information, and the compensation set $\mathcal{K}^{\complement}$, which is used to reduce the PAPR. A significant advantage of the tone reservation method is that no additional information exchange is needed between the transmitter and receiver. The set $\mathcal{K}$ is fixed and known by both the transmitter and receiver. The receiver simply has to select the carriers corresponding to the set $\mathcal{K}$. In Section 3 we will explain the tone reservation method in more detail.

Tone reservation is an elegant procedure and easy to define. The practical implementation, however, is difficult, because there exist few explicit algorithms for the calculation of the compensation set, and their complexity is high in general. Further, most available results are based on simulations and little of them are analytic in nature.

In this paper we analytically treat the PAPR reduction problem via tone reservation for CDMA systems that are based on the Walsh functions. Important questions about the tone reservation method in those systems are: 1. What is the best possible reduction of the PAPR? 2. What is the optimal information set that achieves this reduction, and how can it be found? And 3. What is the general structure of the information set? We solve all three question in Section 5. It is surprising that for CDMA the optimal constants and information sets can be derived.

In [12] a general theory for the solvability of the PAPR reduction problem was developed for arbitrary orthogonal transmission schemes. The solvability of the PAPR problem for CDMA systems that are based on Walsh functions was studied in [13, 14]. The results in these publications initiated the research for answering the above questions. It would be interesting to further develop the theory from [12], such that above questions can also be answered for other orthogonal transmission schemes, in particular for OFDM. However, this task seems to be very difficult.

2. NOTATION AND WALSH SYSTEMS

By $L^p[0,1]$, $1 \leq p \leq \infty$, we denote the usual L^p-spaces on the interval $[0,1]$, equipped with the norm $\|\cdot\|_p$. For an index set $\mathcal{I} \subset \mathbb{Z}$, we denote by $\ell^2(\mathcal{I})$ the set of all square summable sequences $c = \{c_k\}_{k\in\mathcal{I}}$ indexed by $\mathcal{I}$. The norm is given by $\|c\|_{\ell^2(\mathcal{I})} = (\sum_{k\in\mathcal{I}}|c_k|^2)^{1/2}$. By $|A|$ we denote the cardinality of a set A.

The Rademacher functions r_n, $n \in \mathbb{N}$, on $[0,1]$ are defined by $r_n(t) = \operatorname{sgn}[\sin(\pi 2^n t)]$, where sgn denotes the signum function with the convention $\operatorname{sgn}(0) = -1$. The Walsh functions w_n, $n \in \mathbb{N}$, on $[0,1]$ are defined by $w_1(t) = 1$ and $w_{2^k+m}(t) = r_{k+1}(t)w_m(t)$ for $k = 0, 1, 2, \dots$ and $m = 1, 2, \dots, 2^k$. Note that we use an indexing of the Walsh functions that starts with 1. The Rademacher system $\{r_n\}_{n\in\mathbb{N}}$ is an orthonormal system (ONS) in $L^2[0,1]$, but not a basis. The Walsh functions $\{w_n\}_{n\in\mathbb{N}}$ form an orthonormal basis for $L^2[0,1]$, and we have $\int_0^1 w_n(t)\ \mathrm{d}t = 0$ for all $n \in \mathbb{N}$, $n \geq 2$. For further details about the Walsh function, see for example [15].

3. PAPR AND TONE RESERVATION

Without loss of generality, we can restrict ourselves to signals defined on the interval $[0,1]$. Signals with other duration can be simply scaled to be concentrated on $[0,1]$. For a signal $s \in L^2[0,1]$, we define

$$\operatorname{PAPR}(s) = \frac{\|s\|_{L^\infty[0,1]}}{\|s\|_{L^2[0,1]}},$$

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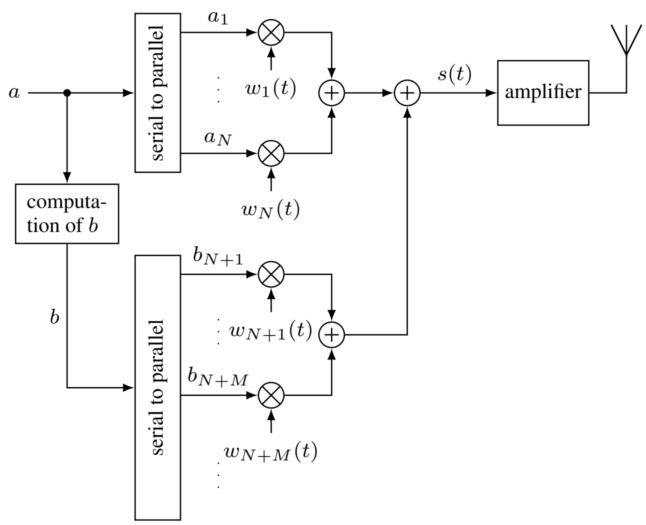

Fig. 1. Block diagram of a CDMA transmission scheme with tone reservation. In this example we have $\mathcal{I} = \mathbb{N}$, $\mathcal{K} = \{1, \ldots, N\}$ and $\mathcal{K}^{\complement} = \mathbb{N} \setminus \mathcal{K}$.

i.e., the PAPR is the ratio between the peak value of the signal and the square root of the power of the signal. Note that the PAPR is usually defined as the square of this value. This however makes, from a mathematical point of view, no difference for the results in this paper. In the case of an orthogonal transmission scheme, using the ONS $\{\phi_k\}_{k\in\mathcal{I}} \subset L^2[0,1]$, the PAPR of the transmit signal

$$s(t) = \sum_{k\in\mathcal{I}} c_k\phi_k(t), \quad t \in [0,1],$$

with coefficients $c = \{c_k\}_{k\in\mathcal{I}}$, is given by

$$\mathrm{PAPR}(s) = \frac{\|\sum_{k\in\mathcal{I}} c_k\phi_k\|_{L^\infty[0,1]}}{\|c\|_{\ell^2(\mathcal{I})}},$$

because $\|s\|_{L^2[0,1]} = \|c\|_{\ell^2(\mathcal{I})}$, due to the fact that $\{\phi_k\}_{k\in\mathcal{I}}$ is an ONS.

For an orthogonal transmission scheme, the peak value of the signal s, and hence the PAPR, can become large, as the following result shows. Given any system $\{\phi_n\}_{n=1}^N$ of N orthonormal functions in $L^2[0,1]$, then there exist a sequence $\{c_n\}_{n=1}^N \subset \mathbb{C}$ of coefficients with $\sum_{n=1}^N |c_n|^2 = 1$, such that $\|\sum_{n=1}^N c_n\phi_n\|_{L^\infty[0,1]} \geq \sqrt{N}$ [16]. This increase of the PAPR with an order of $\sqrt{N}$ is undesired and ways to battle it are needed.

Tone reservation is one approach to reduce the PAPR. Let $\{\phi_k\}_{k\in\mathcal{I}}$ be an ONS in $L^2[0,1]$. We additionally assume that $\|\phi_k\|_\infty < \infty$, $k \in \mathcal{I}$, i.e., we consider the practically relevant case of bounded carriers. In the tone reservation method, the index set $\mathcal{I}$ is partitioned in two disjoint sets $\mathcal{K}$ and $\mathcal{K}^{\complement}$. Note that the set $\mathcal{K}$ can be finite or infinite. For a given sequence $a = \{a_k\}_{k\in\mathcal{K}} \in \ell^2(\mathcal{K})$, the goal is to find a sequence $b = \{b_k\}_{k\in\mathcal{K}^{\complement}} \in \ell^2(\mathcal{K}^{\complement})$ such that the peak value of the signal

$$s(t) = \underbrace{\sum_{k\in\mathcal{K}} a_k\phi_k(t)}_{=:A(t)} + \underbrace{\sum_{k\in\mathcal{K}^{\complement}} b_k\phi_k(t)}_{=:B(t)}, \quad t \in [0,1],$$

is as small as possible. $A(t)$ denotes the signal part which contains the information and $B(t)$ the part which is used to reduce the PAPR.

A block diagram, illustrating the tone reservation method for a CDMA transmission system using the Walsh functions, is given in Fig. 1.

Note that we allow infinitely many carriers to be used for the compensation of the PAPR. This is also of practical interest, since the solvability of the PAPR problem in this setting is a necessary condition for the solvability of the PAPR problem in the setting with finitely many carriers.

4. SOLVABILITY OF THE PAPR PROBLEM

We define the solvability of the PAPR problem next.

Definition 1 (Solvability of the PAPR problem)**.** For an ONS $\{\phi_k\}_{k\in\mathcal{I}}$ and a set $\mathcal{K} \subset \mathcal{I}$, we say that the PAPR problem is solvable with finite extension constant C_{EX}, if for all $a \in \ell^2(\mathcal{K})$ there exists a $b \in \ell^2(\mathcal{K}^{\complement})$ such that

$$\left\| \sum_{k\in\mathcal{K}} a_k\phi_k + \sum_{k\in\mathcal{K}^{\complement}} b_k\phi_k \right\|_{L^\infty[0,1]} \leq C_{\mathrm{EX}}\|a\|_{\ell^2(\mathcal{K})}. \qquad (1)$$

We call the PAPR problem solvable if it is solvable for some finite extension constant C_{EX}.

If the PAPR reduction problem is solvable, condition (1) immediately implies that $\|b\|_{\ell^2(\mathcal{K}^{\complement})} \leq C_{\mathrm{EX}}\|a\|_{\ell^2(\mathcal{K})}$, because

$$\begin{aligned}\left(\sum_{k\in\mathcal{K}^{\complement}} |b_k|^2\right)^{\frac{1}{2}} &\leq \left(\sum_{k\in\mathcal{K}} |a_k|^2 + \sum_{k\in\mathcal{K}^{\complement}} |b_k|^2\right)^{\frac{1}{2}} \\ &= \left(\int_0^1 \left|\sum_{k\in\mathcal{K}} a_k\phi_k(t) + \sum_{k\in\mathcal{K}^{\complement}} b_k\phi_k(t)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}} \\ &\leq \operatorname*{ess\,sup}_{t\in[0,1]} \left|\sum_{k\in\mathcal{K}} a_k\phi_k(t) + \sum_{k\in\mathcal{K}^{\complement}} b_k\phi_k(t)\right|, \qquad (2)\end{aligned}$$

that is, the energy of the compensation signal is also bounded by $(C_{\mathrm{EX}}\|a\|_{\ell^2(\mathcal{K})})^2$. Further, we have $\mathrm{PAPR}(s) \leq C_{\mathrm{EX}}$. It is clear that finding the optimal, i.e., minimal extension constant is an important problem that is relevant for applications.

In [11,13] the following different but equivalent characterization of the solvability of the PAPR problem was given.

Theorem 1. *Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a complete ONS, $\mathcal{K} \subset \mathbb{N}$, and $C_{EX} > 0$. We have*

$$\|f\|_{L^2[0,1]} \leq C_{EX}\|f\|_{L^1[0,1]} \qquad (3)$$

for all $f \in L^1[0,1]$ having the representation

$$f = \sum_{k\in\mathcal{K}} a_k\phi_k$$

for some $\{a_k\}_{k\in\mathcal{K}} \subset \mathbb{C}$, if and only if the PAPR problem is solvable for $\{\phi_n\}_{n\in\mathbb{N}}$ and $\mathcal{K}$ with constant C_{EX}.

Theorem 1 will be useful for our proof. In particular, because it shows that for any set $\mathcal{K}$, the smallest constant C_{EX} for which (3) is true is also the smallest constant C_{EX} for which (1) is true.

5. OPTIMAL PAPR CONTROL AND OPTIMAL SET

As already mentioned in the introduction, in the remainder of this paper we will answer the following three questions for CDMA systems that employ the Walsh system: 1. What is the best possible reduction of the PAPR, i.e, how small is the optimal extension constant C_{EX}? 2. What is the optimal information set $\mathcal{K}$ that achieves this reduction, and how can it be found? And 3. What is the general structure of the information set $\mathcal{K}$?

Let $\mathcal{K} = \{k_1, k_2, \dots, k_N\} \subset \mathbb{N}$ be a set of N arbitrary distinct natural numbers. By $C_{\text{EX}}(\mathcal{K})$ we denote the optimal, i.e., smallest, extension constant for which the PAPR problem is solvable for the Walsh system $\{\phi_n\}_{n\in\mathbb{N}} = \{w_n\}_{n\in\mathbb{N}}$ and the set $\mathcal{K}$. Next, we want to study how small the optimal extension constant can become for different sets $\mathcal{K}$ of cardinality N, i.e., we are interested in

$$\underline{C}_{\text{EX}}(N) := \inf_{\substack{\mathcal{K}\subset\mathbb{N}\\ |\mathcal{K}|=N}} C_{\text{EX}}(\mathcal{K}). \tag{4}$$

We will see in Theorem 3 that for each $N \in \mathbb{N}$ there indeed exists a set $\mathcal{K}^{\text{opt}}(N) \subset \mathbb{N}$ with $|\mathcal{K}^{\text{opt}}(N)| = N$, such that $\underline{C}_{\text{EX}}(N) = C_{\text{EX}}(\{\mathcal{K}^{\text{opt}}(N)\})$. That is, the infimum in (4) is in fact attained and a minimum.

A priori it is not clear how the set $\mathcal{K}^{\text{opt}}(N)$ depends on N. It could be that for different N we obtain completely different sets $\mathcal{K}^{\text{opt}}(N)$. In particular it does not need to hold that $\mathcal{K}^{\text{opt}}(N) \subset \mathcal{K}^{\text{opt}}(N+1)$. However, we will see in Corollary 1 that exactly this is the case.

The next theorem completely describes the smallest possible extension constant $\underline{C}_{\text{EX}}$, and thus answers question 1.

Theorem 2. *We have $\underline{C}_{EX}(1) = 1$ and $\underline{C}_{EX}(N) = \sqrt{2}$ for all $N \geq 2$.*

Question 2 about the optimal information set $\mathcal{K}^{\text{opt}}(N)$ that achieves the best possible PAPR reduction is answered by the next theorem.

Theorem 3. *For $N \in \mathbb{N}$ we have $\underline{C}_{EX}(N) = C_{EX}(\{2^k + 1\}_{k=0}^{N-1})$. That is, $\mathcal{K}^{opt}(N) = \{2^k + 1\}_{k=0}^{N-1}$, showing that the first N Rademacher functions achieve the minimal extension constant $\underline{C}_{EX}(N)$.*

Finally, we study the structure of the optimal information sets, and thus answer question 3. For $N \in \mathbb{N}$ let

$$T_N := \{\mathcal{K} \subset \mathbb{N} \colon |\mathcal{K}| = N, \underline{C}_{\text{EX}}(N) = C_{\text{EX}}(\mathcal{K})\}$$

denote the set of all optimal information sets $\mathcal{K}$, i.e., the set of all $\mathcal{K}$ that attain the smallest possible extension constant $\underline{C}_{\text{EX}}(N)$.

Of course, we cannot conclude for $\mathcal{K} \in T_N$ and $k_l \notin \mathcal{K}$ that $\mathcal{K} \cup \{k_l\} \in T_{N+1}$. It is also not immediately clear whether there exists an infinite set $\mathcal{K} = \{k_1, k_2, \dots\}$, $k_n < k_{n+1}$, $l \in \mathbb{N}$, such that the first N elements $K_N = \{k_1, \dots, k_N\}$ always satisfy $K_N \in T_N$. The following corollary gives an answer, and shows that we have this situation.

Corollary 1. *Let $N \geq 2$ and $\mathcal{K} = \{k_1, \dots, k_N\} \in T_N$. Then we have $\mathcal{K} \setminus \{k_l\} \in T_{N-1}$ for all $1 \leq l \leq N-1$.*

The proofs of all above results will be given in Section 8.

6. ELEMENTARY FACTS ABOUT WALSH FUNCTIONS

Before we can give the proofs, we need some elementary facts about the Walsh functions, which were introduced in Section 2.

Let $\mathcal{K} = \{k_1, k_2, \dots, k_N\} \subset \mathbb{N}$ be a set of N arbitrary distinct natural numbers. Let $W(\mathcal{K})$ denote the largest number C_1 such that

$$C_1 \left(\sum_{l=1}^{N} |\alpha_l|^2\right)^{\frac{1}{2}} \leq \int_0^1 \left|\sum_{l=1}^{N} \alpha_l w_{k_l}(t)\right| \,\mathrm{d}t \tag{5}$$

for all $\alpha_1, \dots, \alpha_N \in \mathbb{C}$. We have

$$\begin{aligned} W(\mathcal{K}) &= \inf_{\substack{\{\alpha_l\}_{l=1}^N \\ \sum_{l=1}^N |\alpha_l|^2 = 1}} \int_0^1 \left|\sum_{l=1}^{N} \alpha_l \omega_{k_l}(t)\right| \,\mathrm{d}t \\ &= \min_{\substack{\{\alpha_l\}_{l=1}^N \\ \sum_{l=1}^N |\alpha_l|^2 = 1}} \int_0^1 \left|\sum_{l=1}^{N} \alpha_l \omega_{k_l}(t)\right| \,\mathrm{d}t. \end{aligned} \tag{6}$$

The minimum in (6) is indeed attained, since the mapping

$$(\alpha_1, \dots, \alpha_N) \mapsto \int_0^1 \left|\sum_{l=1}^{N} \alpha_l \omega_{k_l}(t)\right| \,\mathrm{d}t$$

is continuous, and the minimum is taken over a compact set in $\mathbb{C}^N$. According to the Cauchy–Schwarz inequality we have

$$\begin{aligned} \int_0^1 \left|\sum_{l=1}^{N} \alpha_l w_{k_l}(t)\right| \,\mathrm{d}t &\leq \left(\int_0^1 \left|\sum_{l=1}^{N} \alpha_l w_{k_l}(t)\right|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \\ &= \left(\sum_{l=1}^{N} |\alpha_l|^2\right)^{\frac{1}{2}}, \end{aligned}$$

which implies that

$$W(\mathcal{K}) \leq 1. \tag{7}$$

Further, for arbitrary $l \in \{1, \dots, N\}$, we see from (6) that

$$W(\mathcal{K}) \leq W(\mathcal{K} \setminus \{k_l\}). \tag{8}$$

For $N \in \mathbb{N}$ we set

$$\overline{W}(N) := \sup_{\substack{\mathcal{K}\subset\mathbb{N}\\ |\mathcal{K}|\leq N}} W(\mathcal{K}).$$

Clearly, we have $0 \leq \overline{W}(N) \leq 1$ for all $N \in \mathbb{N}$, according to (7). For $N \geq 2$ and all sets $\{k_1, \dots, k_N\}$, it follows from (8) that

$$W(\{k_1, \dots, k_N\}) \leq \overline{W}(N-1),$$

and consequently

$$\overline{W}(N) \leq \overline{W}(N-1). \tag{9}$$

Hence, we see that $0 \leq \overline{W}(N) \leq 1$ and, further, that $\{\overline{W}(N)\}_{N=1}^{\infty}$ is a monotonically decreasing sequence of real numbers that is bounded from below. Hence the limit $\lim_{N\to\infty} \overline{W}(N)$ exists.

7. AUXILIARY RESULT ABOUT RADEMACHER FUNCTIONS

In this section, we prove an auxiliary result about Rademacher functions, that will be needed for the proof of our main results.

For $N \in \mathbb{N}$, let $R(N)$ denote the largest number C_2 such that

$$C_2 \left(\sum_{l=1}^{N} |\alpha_l|^2 \right)^{\frac{1}{2}} \leq \int_0^1 \left| \sum_{l=1}^{N} \alpha_l r_l(t) \right| \, \mathrm{d}t \tag{10}$$

for all $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$. By the same reasoning as in Section 6, there exists such a number.

Lemma 1. *We have $R(1) = 1$ and $R(N) = 1/\sqrt{2}$ for all $N \geq 2$.*

Proof. For $N = 1$, eq. (10) becomes $C_2|\alpha_1| \leq |\alpha_1|$, which shows that $R(1) = 1$.

Next, we treat the case $N \geq 2$. According to Khinchin's inequality [17], we have for all $N \in \mathbb{N}$ and all $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$ that

$$\frac{1}{\sqrt{2}} \left(\sum_{l=1}^{N} |\alpha_l|^2 \right)^{\frac{1}{2}} \leq \int_0^1 \left| \sum_{l=1}^{N} \alpha_l r_l(t) \right| \, \mathrm{d}t. \tag{11}$$

The constant $1/\sqrt{2}$ in (11) is the best, i.e., largest possible constant that holds for all $N \in N$ [17, 18]. For fixed $N \in \mathbb{N}$, $N \geq 2$, inequality (11) implies that

$$R(N) \geq \frac{1}{\sqrt{2}}. \tag{12}$$

A simple calculation shows that $\int_0^1 |r_1(t) + r_2(t)| \, \mathrm{d}t = 1$. Hence, for $N = 2$ and $\alpha_1 = \alpha_1 = 1$, inequality (10) becomes $C_2\sqrt{2} \leq 1$, and and we see that $R(2) \leq 1/\sqrt{2}$. Due to (12), it follows that

$$R(2) = \frac{1}{\sqrt{2}} \tag{13}$$

Further, for $N \geq 3$ we have $1/\sqrt{2} \leq R(N) \leq R(2) \leq 1/\sqrt{2}$, where the first inequality follows from (12), the second inequality from the same arguments that led to (9), and the third inequality from (13). Hence, for $N \geq 2$, we have $R(N) = 1/\sqrt{2}$. □

8. PROOFS

In this section we prove Theorems 2 and 3 and Corollary 1.

Based on Lemma 1 we can prove Lemma 2, which is ultimately needed for the proof of Theorem 2.

Lemma 2. *We have $\overline{W}(1) = 1$ and $\overline{W}(N) = 1/\sqrt{2}$ for all $N \geq 2$.*

Proof. Since $W(\{k\}) = \int_0^1 |w_k(t)| \, \mathrm{d}t = 1$, for all $k \in \mathbb{N}$, it immediately follows that $\overline{W}(1) = 1$. According to the definition of $\overline{W}$, we have $\overline{W}(N) \geq R(N)$ for all $N \in \mathbb{N}$. Hence, it follows from Lemma 1 that

$$\overline{W}(N) \geq \frac{1}{\sqrt{2}} \tag{14}$$

for all $N \geq 2$. Let $k_1 < k_2$ be two arbitrary natural numbers. We have

$$\begin{aligned} \int_0^1 |w_{k_1}(t) + w_{k_2}(t)| \, \mathrm{d}t &= \int_0^1 |w_{k_1}(t)(w_{k_1}(t) + w_{k_2}(t))| \, \mathrm{d}t \\ &= \int_0^1 |1 + w_{k_1}(t) w_{k_2}(t)| \, \mathrm{d}t = \int_0^1 |1 + w_{k'}(t)| \, \mathrm{d}t \\ &= \int_0^1 1 + w_{k'}(t) \, \mathrm{d}t = 1, \end{aligned} \tag{15}$$

and it follows that

$$W(\{k_1, k_2\}) \leq \int_0^1 \left| \frac{1}{\sqrt{2}} w_{k_1}(t) + \frac{1}{\sqrt{2}} w_{k_2}(t) \right| \, \mathrm{d}t = \frac{1}{\sqrt{2}},$$

where we used (6) in the inequality and (15) in the equality. Hence, we see that

$$\overline{W}(2) \leq \frac{1}{\sqrt{2}}. \tag{16}$$

Since $\overline{W}(N) \leq \overline{W}(2)$ for all natural numbers $N \geq 2$ according to (9), it follows that

$$\frac{1}{\sqrt{2}} \leq \overline{W}(N) \leq \overline{W}(2) \leq \frac{1}{\sqrt{2}},$$

where we used (14) in the first and (16) in the last inequality. Consequently, we have $\overline{W}(N) = 1/\sqrt{2}$ for all $N \geq 2$. □

Now we are in the position to prove Theorems 2 and 3, as well as Corollary 1.

Proof of Theorem 2. According to the definitions of $\underline{C}_{\text{EX}}(N)$ and $\overline{W}(N)$, and Theorem 1 we see that $\underline{C}_{\text{EX}}(N) = 1/\overline{W}(N)$. Hence, Lemma 2 completes the proof. □

Proof of Theorem 3. In the proof of Lemma 1 we have already seen that, for every $N \in \mathbb{N}$, the first N Rademacher functions $r_1, \ldots, r_N$ give the maximal constant $\overline{W}(N)$ and hence the minimal extension constant $\underline{C}_{\text{EX}}(N)$. The first N Rademacher functions $\{r_n\}_{n=1}^N$ correspond to the Walsh functions $\{w_{2^k+1}\}_{k=0}^{N-1}$. □

Proof of Corollary 1. For $N \geq 3$ we have for $\{k_1, \ldots, k_N\} \in T_N$ that

$$\frac{1}{\sqrt{2}} = W(\{k_1, \ldots, k_N\}) \leq W(\{k_1, \ldots, k_N\} \setminus \{k_l\}) \leq \frac{1}{\sqrt{2}}$$

according to (8) and Lemma 2. It follows that

$$W(\{k_1, \ldots, k_N\} \setminus \{k_l\}) = \frac{1}{\sqrt{2}},$$

which in turn implies that

$$\begin{aligned} C_{\text{EX}}(\{k_1, \ldots, k_N\} \setminus \{k_l\}) &= \frac{1}{W(\{k_1, \ldots, k_N\} \setminus \{k_l\})} \\ &= \sqrt{2} = \underline{C}_{\text{EX}}(N-1), \end{aligned} \tag{17}$$

where the first equality follows from the Definition of W and Theorem 1. From (17) we see that $\{k_1, \ldots, k_N\} \setminus \{k_l\} \in T_{N-1}$. □

9. RELATION TO PRIOR WORK

The control of the PAPR and finding optimal information sets is an important problem. In [14] it was shown for CDMA based Walsh functions that the information sets $\mathcal{K}$ for which the PAPR is solvable need to be sparse, in particular their upper densities needs to be zero. However, no statement about the optimal information set was made. In general, little is known about the answers to questions 1–3, and most of the results are based on simulations and not on analytic considerations. For our proof the optimal constant in Khinchin's inequality [17, 18] was essential. It would be interesting to answer the three questions also for other orthogonal transmission schemes, e.g., OFDM, where the ONS is the system of complex exponentials. However, the present proof technique is tailored to the specific properties of the Walsh functions, and therefore cannot be used for other ONS.

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