# TONE RESERVATION AND SOLVABILITY CONCEPTS FOR THE PAPR PROBLEM IN GENERAL ORTHONORMAL TRANSMISSION SYSTEMS

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# ABSTRACT

Large peak to average power ratios (PAPRs) are problematic for communication systems. One possible approach to control the PAPR is the tone reservation method. We analyze the tone reservation method for general complete orthonormal systems, and consider two solvability concepts: strong solvability and weak solvability. Strong solvability requires a rather strong control of the peak value of the transmit signal by the energy of the information signal, and thus might be to restrictive for practical applications. Therefore, the concept of weak solvability was introduced, which only requires the boundedness of the transmit signal. In this paper we prove that weak solvability and strong solvability are equivalent for arbitrary complete orthonormal systems.

*Index Terms*— Orthonormal transmission system, peak to average power ratio, tone reservation, OFDM, CDMA

### 1. INTRODUCTION

The control of the peak to average power ratio (PAPR) is an important task in orthogonal frequency division multiplexing (OFDM) and code division multiple access (CDMA) based communication systems [1–4]. Large PAPR values are undesired, because they can overload amplifiers, distort the signal, and lead to out-of-band radiation. For a further discussion of these concepts, we would like to refer to [4]. For future communication systems other, more general waveform transmission schemes are discussed [5,6].

Large PAPR values, however, are not specific to OFDM and CDMA systems, but rather occur for arbitrary bounded orthonormal systems (ONS). It is well known that the PAPR of such signals can be as large as  $\sqrt{N}$ , where N denotes the number of carriers [7].

In order to reduce the PAPR, several methods have been proposed [8,9], among them the popular tone reservation method [10–14], which we consider in this paper. In this method, the set of available carriers is partitioned into two sets, the first of which is used to carry the information (information set), and the second of which to reduce the PAPR (compensation set).

Tone reservation is an elegant procedure and easy to define. The practical implementation however is difficult. Little of the available results are analytic in nature, and there exist few explicit and efficient algorithms for the calculation of the compensation set. Further, optimal constants for the control of the PAPR value are in general unknown, and hence the maximal possible reduction of the PAPR value by tone reservation is unclear.

In Section 3 we will explain the tone reservation method in more detail, and in Section 4 we introduce the concepts of weak and strong

solvability of the PAPR problem. In Section 5 we present our main result, the equivalence of weak and strong solvability for arbitrary bounded complete ONSs.

#### 2. NOTATION

By  $L^p[0, 1]$ ,  $1 \le p \le \infty$ , we denote the usual  $L^p$ -spaces on the interval [0, 1], equipped with the norm  $\|\cdot\|_p$ . For an index set  $\mathcal{I} \subset \mathbb{Z}$ , we denote by  $\ell^2(\mathcal{I})$  the set of all square summable sequences  $c = \{c_k\}_{k\in\mathcal{I}}$  indexed by  $\mathcal{I}$ . The norm is given by  $\|c\|_{\ell^2(\mathcal{I})} = (\sum_{k\in\mathcal{I}} |c_k|^2)^{1/2}$ . By |A| we denote the cardinality of a set A. The Rademacher functions  $r_n$ ,  $n \in \mathbb{N}$ , on [0, 1] are defined by

The Rademacher functions  $r_n$ ,  $n \in \mathbb{N}$ , on [0, 1] are defined by  $r_n(t) = \operatorname{sgn}[\sin(\pi 2^n t)]$ , where sgn denotes the signum function with the convention  $\operatorname{sgn}(0) = -1$ . The Walsh functions  $w_n$ ,  $n \in \mathbb{N}$ , on [0, 1] are defined by  $w_1(t) = 1$  and  $w_{2^k+m}(t) = r_{k+1}(t)w_m(t)$  for  $k = 0, 1, 2, \ldots$  and  $m = 1, 2, \ldots, 2^k$ . Note that we use an indexing of the Walsh functions that starts with 1. The Walsh functions  $\{w_n\}_{n\in\mathbb{N}}$  form an orthonormal basis for  $L^2[0, 1]$ . For further details about the Walsh function, see for example [15].

#### 3. TONE RESERVATION

Without loss of generality, we can restrict ourselves to signals defined on the interval [0, 1]. Signals with other duration can be simply scaled to be concentrated on [0, 1]. For a signal  $s \in L^2[0, 1]$ , we define

PAPR(s) = 
$$\frac{\|s\|_{L^{\infty}[0,1]}}{\|s\|_{L^{2}[0,1]}}$$

i.e., the PAPR is the ratio between the peak value of the signal and the square root of the power of the signal. Note that the PAPR is usually defined as the square of this value. This however, from a mathematical point of view, makes no difference for the results in this paper. In the case of an orthogonal transmission scheme, using the ONS  $\{\phi_k\}_{k \in \mathcal{I}} \subset L^2[0, 1]$ , the PAPR of the transmit signal

$$s(t) = \sum_{k \in \mathcal{I}} c_k \phi_k(t), \quad t \in [0, 1],$$

with coefficients  $c = \{c_k\}_{k \in \mathcal{I}}$ , is given by

$$PAPR(s) = \frac{\left\|\sum_{k \in \mathcal{I}} c_k \phi_k\right\|_{L^{\infty}[0,1]}}{\|c\|_{\ell^2(\mathcal{I})}},$$

because  $\{\phi_k\}_{k\in\mathcal{I}}$  is a ONS, which implies that  $||s||_{L^2[0,1]} = ||c||_{\ell^2(\mathcal{I})}$ .

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For an orthogonal transmission scheme, the peak value of the signal s, and hence the PAPR, can become large, as the following result shows. Given any system  $\{\phi_n\}_{n=1}^N$  of N orthonormal functions



Fig. 1. Block diagram of a general orthogonal waveform transmission scheme with tone reservation. In this example  $\{\phi_n\}_{n\in\mathbb{N}}$  is an ONS, and we have  $\mathcal{I} = \mathbb{N}$ ,  $\mathcal{K} = \{1, \dots, N\}$  and  $\mathcal{K}^{\mathsf{C}} = \mathbb{N} \setminus \mathcal{K}$ .

in  $L^2[0, 1]$ , then there exist a sequence  $\{c_n\}_{n=1}^N \subset \mathbb{C}$  of coefficients with  $\sum_{n=1}^N |c_n|^2 = 1$ , such that  $\|\sum_{n=1}^N c_n \phi_n\|_{L^{\infty}[0,1]} \ge \sqrt{N}$  [7]. For the OFDM case, where the CONS consists of complex ex-ponentials  $\{e^{ik \cdot 2\pi}\}_{k \in \mathbb{Z}}$  and the CDMA case, where the CONS is given by the Walsh functions  $\{w_n\}_{n\in\mathbb{N}}$ , this can be easily seen. For the sequence

$$c_n = \begin{cases} \frac{1}{\sqrt{N}}, & 1 \le n \le N\\ 0, & \text{otherwise,} \end{cases}$$

we clearly have  $\sum_{n=1}^{N} |c_n|^2 = 1$ . In the OFDM case we have

$$\left\|\sum_{n=1}^{N} c_n e^{int2\pi}\right\|_{L^{\infty}[0,1]} = \frac{1}{\sqrt{N}} \max_{t \in [0,1]} \left|\sum_{n=1}^{N} e^{int2\pi}\right| = \sqrt{N},$$

and in the Walsh case

$$\left\|\sum_{n=1}^{N} c_n w_n\right\|_{L^{\infty}[0,1]} = \frac{1}{\sqrt{N}} \left\|\sum_{n=1}^{N} w_n\right\|_{L^{\infty}[0,1]} = \sqrt{N},$$

because  $\sum_{n=1}^{N} w_n(t) = N$  for all  $t \in (0, 1/N)$ . Tone reservation, which is illustrated in Fig. 1, is one approach to reduce the PAPR. Let  $\{\phi_k\}_{k\in\mathcal{I}}$  be an ONS in  $L^2[0,1]$ . We additionally assume that  $\|\phi_k\|_{\infty} < \infty, k \in \mathcal{I}$ , i.e., we consider the practically relevant case of bounded functions. In the tone reservation method, the index set  $\mathcal I$  is partitioned in two disjoint sets  $\mathcal K$  (information set) and  $\mathcal{K}^{\complement}$  (compensation set). The set  $\mathcal{K}$  is used to carry the information and the set  $\mathcal{K}^{\complement}$  to reduce he PAPR. Note that the set  $\mathcal{K}$  can be finite or infinite. For a given sequence  $a = \{a_k\}_{k \in \mathcal{K}} \in \ell^2(\mathcal{K})$ , the goal is to find a sequence  $b = \{a_k\}_{k \in \mathcal{K}^{\complement}} \in \ell^2(\mathcal{K}^{\complement})$  such that the peak value of the signal

$$s(t) = \underbrace{\sum_{k \in \mathcal{K}} a_k \phi_k(t)}_{=:A(t)} + \underbrace{\sum_{k \in \mathcal{K}^{\complement}} b_k \phi_k(t)}_{=:B(t)}, \quad t \in [0, 1],$$

is as small as possible. A(t) denotes the signal part which contains the information and B(t) the part which is used to reduce the PAPR.

Note that we allow infinitely many carriers to be used for the compensation of the PAPR. This is also of practical interest, since the solvability of the PAPR problem in this setting is a necessary condition for the solvability of the PAPR problem in the setting with finitely many carriers.

# 4. STRONG AND WEAK SOLVABILITY

Next we introduce two solvability concepts for the tone reservation method. The difference of the two concepts lies in the way how we control the peak value, i.e. the  $L^{\infty}$  norm, of the transmit signal.

Definition 1 (Strong solvability of the PAPR problem). For an ONS  $\{\phi_k\}_{k\in\mathcal{I}}$  and a set  $\mathcal{K}\subset\mathcal{I}$ , we say that the PAPR problem is strongly solvable with finite constant  $C_{\text{EX}}$ , if for all  $a \in \ell^2(\mathcal{K})$  there exists a  $b \in \ell^2(\mathcal{K}^{\complement})$  such that

$$\left\| \sum_{k \in \mathcal{K}} a_k \phi_k + \sum_{k \in \mathcal{K}^{\complement}} b_k \phi_k \right\|_{L^{\infty}[0,1]} \le C_{\mathrm{EX}} \|a\|_{\ell^2(\mathcal{K})}.$$
(1)

We call the PAPR problem strongly solvable if it is strongly solvable for some finite constant  $C_{\text{EX}}$ .

Formally, this solvability concept was introduced in [13, 16]. If the PAPR reduction problem is strongly solvable, condition (1) immediately implies that  $\|b\|_{\ell^2(\mathcal{K}^{\mathsf{G}})} \leq C_{\mathrm{EX}} \|a\|_{\ell^2(\mathcal{K})}$ , because

$$\left(\sum_{k\in\mathcal{K}^{\complement}}|b_{k}|^{2}\right)^{\frac{1}{2}} \leq \left(\sum_{k\in\mathcal{K}}|a_{k}|^{2}+\sum_{k\in\mathcal{K}^{\complement}}|b_{k}|^{2}\right)^{\frac{1}{2}}$$
$$= \left(\int_{0}^{1}\left|\sum_{k\in\mathcal{K}}a_{k}\phi_{k}(t)+\sum_{k\in\mathcal{K}^{\complement}}b_{k}\phi_{k}(t)\right|^{2}\mathrm{d}t\right)^{\frac{1}{2}}$$
$$\leq \operatorname{ess\,sup}_{t\in[0,1]}\left|\sum_{k\in\mathcal{K}}a_{k}\phi_{k}(t)+\sum_{k\in\mathcal{K}^{\complement}}b_{k}\phi_{k}(t)\right|.$$
(2)

Further, we have  $PAPR(s) \leq C_{EX}$ .

It is easy to show there exist infinite sets  $\mathcal{K} \subset \mathcal{I}$  for which the PAPR is strongly solvable.

Example 1. We will give two examples next.

For the Walsh ONS  $\{w_n\}_{n\in\mathbb{N}}$  (CDMA case) we can use the information set  $\mathcal{K} = \{2^l\}_{l \in \mathbb{N} \cup \{0\}}$ . Then the PAPR problem is strongly solvable, and it can be shown that the optimal extension constant is  $C_{\rm EX} = \sqrt{2} \, [17].$ 

For the Fourier ONS  $\{e^{ik \cdot 2\pi}\}_{k \in \mathbb{Z}}$  (OFDM case), the same information set  $\mathcal{K} = \{2^l\}_{l \in \mathbb{N} \cup \{0\}}$  makes the PAPR problem strongly solvable. However, in this case the optimal extension constant is yet unknown.

For OFDM, using the complex exponentials, and CDMA, using the Walsh functions, the information sets  $\mathcal{K}$  for which the PAPR is strongly solvable need to be sparse, similar to Example 1 where the gaps grow larger and larger [18, 19]. In [13] the following result was proved for OFDM: If  $\mathcal{K} \subset \mathbb{Z}$  is a set such that the PAPR is strongly solvable for  $\mathcal{K}$  with some finite extension constant  $C_{\text{EX}}$  then we have

$$\lim_{N\to\infty}\frac{|\mathcal{K}\cap [-N,N]|}{2N+1}=0,$$

that is the relative density of information bearing carriers in [-N, N]needs to go to zero. A similar result was shown in [16, 20] for the Walsh system: If  $\mathcal{K} \subset \mathbb{N}$  is a set such that the PAPR is strongly solvable for  $\mathcal{K}$  with some finite extension constant  $C_{EX}$  then we have

$$\lim_{N \to \infty} \frac{|\mathcal{K} \cap [1, N]|}{N} = 0.$$

This is true regardless of the specific value of the constant  $C_{\text{EX}}$ .

In view of the discouraging results about the density of the information set, one could ask if it is too restricting to require (1), i.e., the control of the peak value by a constant  $C_{\text{EX}}$  times the norm of *a*. Therefore, in [20] the concept of weak solvability was introduced.

**Definition 2** (Weak solvability of the PAPR problem). For an ONS  $\{\phi_k\}_{k \in \mathcal{I}}$  and a set  $\mathcal{K} \subset \mathcal{I}$ , we say that the PAPR problem is weakly solvable if for all  $a \in \ell^2(\mathcal{K})$  we have

$$\inf_{b\in\ell^2(\mathcal{K}^\complement)}\left\|\sum_{k\in\mathcal{K}}a_k\phi_k+\sum_{k\in\mathcal{K}^\complement}b_k\phi_k\right\|_{L^\infty[0,1]}<\infty.$$

This is a weaker form of solvability compared to the strong solvability that was given in Definition 1. The peak value of the transmit signal is only required to be bounded and not to be controlled by the norm of the sequence  $a = \{a_k\}_{k \in \mathcal{K}}$  as in (1).

Clearly, strong solvability always implies weak solvability. In [20] the question was raised if maybe the converse implication is also true, that is, if maybe both concepts are equivalent. In [18] this equivalence was proved for OFDM by showing that weak solvability implies strong solvability. The question remained whether this is also true for other ONS. The goal of this work is to prove the equivalence of strong solvability and weak solvability for general complete ONS. Since the proof in [18] was tailored to the specific properties of the OFDM ONS, we need to use a completely different approach for the proof here.

# 5. EQUIVALENCE OF SOLVABILITY CONCEPTS

The goal of this section is to show that, for arbitrary complete ONS, weak solvability, as stated in Definition 2, implies string solvability, as stated in Definition 1. Hence, both concepts of stability are equivalent.

To this end, we start with the following simple lemma, which gives a different but equivalent characterization of the weak solvability concept.

**Lemma 1.** Let  $\{\phi_n\}_{n\in\mathbb{N}}$  be an ONS and  $\mathcal{K} \subset \mathbb{N}$ . The PAPR problem is weakly solvable for  $\{\phi_n\}_{n\in\mathbb{N}}$  and  $\mathcal{K}$  if and only if for all  $a \in \ell^2(\mathcal{K})$  there exists a  $f_a \in L^{\infty}[0, 1]$  such that

$$\int_0^1 f_a(t) \overline{\phi_k(t)} \, \mathrm{d}t = a_k$$

for all  $k \in \mathcal{K}$ .

*Proof.* " $\Rightarrow$ ": Assume that the PAPR problem is weakly solvable for  $\{\phi_n\}_{n\in\mathbb{N}}$  and  $\mathcal{K}$ . Then we have

$$\inf_{b\in\ell^2(\mathcal{K}^\complement)}\left\|\sum_{k\in\mathcal{K}}a_k\phi_k+\sum_{k\in\mathcal{K}^\complement}b_k\phi_k\right\|_{L^\infty[0,1]}<\infty.$$

It follows that there exists a  $b^* \in \ell^2(\mathcal{K}^{\complement})$  such that for

$$f_a = \sum_{k \in \mathcal{K}} a_k \phi_k + \sum_{k \in \mathcal{K}^{\mathsf{G}}} b_k^* \phi_k$$

where the convergence of the sums is in the  $L^2[0, 1]$  norm, we have  $||f_a||_{L^{\infty}[0,1]} < \infty$ . Further, since  $\{\phi_n\}_{n \in \mathbb{N}}$  is an ONS, we have

$$\int_0^1 f_a(t) \overline{\phi_k(t)} \, \mathrm{d}t = a_k$$

for all  $k \in \mathcal{K}$ .

" $\Leftarrow$ ": Let  $a \in \ell^2(\mathcal{K})$  be arbitrary but fixed. According to the assumption, there exists a  $f_a \in L^{\infty}[0, 1]$  such that

$$\int_0^1 f_a(t) \overline{\phi_k(t)} \, \mathrm{d}t = a_k$$

for all  $k \in \mathcal{K}$ . Since  $f_a \in L^{\infty}[0,1] \subset L^2[0,1]$ , the series expansion

$$\sum_{n\in\mathbb{N}}c_n\phi_n\tag{3}$$

with  $\{c_n\}_{n\in\mathbb{N}}\in\ell^2$  converges to  $f_a$  in the  $L^2[0,1]$  norm. Since the convergence of the series (3) is unconditional, the reordering

$$\sum_{k \in \mathcal{K}} a_k \phi_k(t) + \sum_{k \in \mathcal{K}^{\mathsf{G}}} b_k \phi_k(t)$$

with  $a_k = c_k$  for  $k \in \mathcal{K}$  and  $b_k = c_k$  for  $k \in \mathcal{K}^{\complement}$  also converges to  $f_a$ . Since  $f_a(t) \in L^{\infty}[0, 1]$ , this implies that

$$\left\|\sum_{k\in\mathcal{K}}a_k\phi_k+\sum_{k\in\mathcal{K}^{\complement}}b_k\phi_k\right\|_{L^{\infty}[0,1]}=\|f_a\|_{L^{\infty}[0,1]}<\infty,$$

and it follows that the PAPR problem is weakly solvable for  $\{\phi_n\}_{n\in\mathbb{N}}$  and  $\mathcal{K}$ .

In the next theorem our main result is presented.

**Theorem 1.** Let  $\{\phi_n\}_{n\in\mathbb{N}}$  be a complete ONS with  $\sup_{n\in\mathbb{N}} \|\phi_n\|_{\infty} < \infty$ , and  $\mathcal{K} \subset \mathbb{N}$ , such that the PAPR problem is weakly solvable. Then the PAPR problem is strongly solvable, i.e., there exists a constant  $C_{EX} = C_{EX}(\mathcal{K}, \{\phi_n\}_{n\in\mathbb{N}})$  such that for all  $a \in \ell^2(\mathcal{K})$  we can find  $a \in \ell^2(\mathcal{K}^{\complement})$  such that

$$\left\|\sum_{n\in\mathcal{K}}a_n\phi_n+\sum_{n\in\mathcal{K}^{\mathsf{C}}}b_n\phi_n\right\|_{L^{\infty}[0,1]}\leq C_{EX}\|a\|_{\ell^2(\mathcal{K})}.$$

For the proof of Theorem 1 we need the following lemma and the set

$$\mathcal{N}(\mathcal{K}) = \left\{ f \in L^{\infty}[0,1] : \int_{0}^{1} f(t) \overline{\phi_{n}(t)} \, \mathrm{d}t = 0 \,\,\forall n \in \mathcal{K} \right\}$$

**Lemma 2.**  $\mathcal{N}(\mathcal{K})$  is a closed subspace of  $L^{\infty}[0, 1]$ .

*Proof.* Clearly,  $\mathcal{N}(\mathcal{K})$  has a linear structure, i.e., is closed with respect to addition and multiplication with complex scalars. It remains to prove that  $\mathcal{N}(\mathcal{K})$  is closed. Let  $\{f_m\}_{m\in\mathbb{N}}$  be an arbitrary sequence in  $\mathcal{N}(\mathcal{K})$  that converges in  $L^{\infty}[0,1]$ . That is, there exists a  $f_* \in L^{\infty}[0,1]$  such that  $\lim_{m\to\infty} ||f_* - f_m||_{\infty} = 0$ . We need to show that  $f_* \in \mathcal{N}(\mathcal{K})$ . For  $m \in \mathbb{N}$ ,  $n \in \mathcal{K}$  we have

$$\left| \int_0^1 f_*(t)\overline{\phi_n}(t) \, \mathrm{d}t \right| = \left| \int_0^1 f_*(t)\overline{\phi_n(t)} \, \mathrm{d}t - \int_0^1 f_m(t)\overline{\phi_n(t)} \, \mathrm{d}t \right|$$
$$= \left| \int_0^1 (f_*(t) - f_m(t))\overline{\phi_n(t)} \, \mathrm{d}t \right|$$
$$\leq \|\phi_n\|_{L^{\infty}[0,1]} \|f_* - f_m\|_{L^{\infty}[0,1]}.$$

Letting m go to infinity, we see that

$$\left|\int_0^1 f_*(t)\overline{\phi_n(t)} \, \mathrm{d}t\right| = 0$$

for all  $n \in \mathcal{K}$ , which implies that  $f_* \in \mathcal{N}(\mathcal{K})$ .

In the proof of Theorem 1 we also employ the bounded inverse theorem, which is a consequence of the open mapping theorem [21, pp. 99]. We state the bounded inverse theorem next for convenience.

**Theorem 2** (Bounded Inverse Theorem). Let  $B_1, B_2$  be two Banach spaces. If  $T: B_1 \to B_2$  is a bounded linear operator which is also bijective then the inverse operator  $T^{-1}: B_2 \to B_1$  is bounded as well.

Now we are in the position to prove Theorem 1.

*Proof of Theorem 1.* For  $f \in L^{\infty}[0, 1]$  we define the set

$$[f] = \left\{ g \in L^{\infty}[0,1] : \int_0^1 (f(t) - g(t)) \overline{\phi_n(t)} \, \mathrm{d}t = 0 \,\,\forall n \in \mathcal{K} \right\}.$$

Let  $\mathcal{Q}_{\mathcal{K}}$  denote the quotient set  $L^{\infty}[0,1]/\mathcal{N}(\mathcal{K})$ , consisting of all the sets  $[f], f \in L^{\infty}[0,1]$ .  $\mathcal{Q}_{\mathcal{K}}$  has a linear structure: we have  $\alpha[f] = [\alpha f]$  and [f] + [g] = [f + g]. Further,

$$\|[f]\|_{\mathcal{Q}_{\mathcal{K}}} = \inf_{g \in \mathcal{N}(\mathcal{K})} \|f + g\|_{L^{\infty}[0,1]}$$

defines a norm on  $Q_{\mathcal{K}}$ . Equipped with this norm  $Q_{\mathcal{K}}$  becomes a Banach space.

Next, we consider the operator  $R_{\mathcal{K}} \colon \mathcal{Q}_{\mathcal{K}} \to \ell^2(\mathcal{K})$ , defined by

$$(R_{\mathcal{K}}[f])(k) = \int_0^1 f(t)\overline{\phi_k(t)} \, \mathrm{d}t, \quad k \in \mathcal{K}.$$

For  $r \in \mathcal{N}(\mathcal{K})$  we have

$$\begin{aligned} \|R_{\mathcal{K}}[f]\|_{\ell^{2}(\mathcal{K})} &= \left(\sum_{k\in\mathcal{K}} \left|\int_{0}^{1} f(t)\overline{\phi_{k}(t)} \,\mathrm{d}t\right|^{2}\right)^{\frac{1}{2}} \\ &= \left(\sum_{k\in\mathcal{K}} \left|\int_{0}^{1} (f(t)+r(t))\overline{\phi_{k}(t)} \,\mathrm{d}t\right|^{2}\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{k\in\mathbb{N}} \left|\int_{0}^{1} (f(t)+r(t))\overline{\phi_{k}(t)} \,\mathrm{d}t\right|^{2}\right)^{\frac{1}{2}} \\ &= \left(\int_{0}^{1} |f(t)+r(t)|^{2} \,\mathrm{d}t\right)^{\frac{1}{2}} \\ &\leq \|f+r\|_{L^{\infty}[0,1]}, \end{aligned}$$
(4)

where we used Parseval's equality in the second to last line and Hölder's inequality in the last line. Since the left hand side of (4) does not depend on r, it follows that

$$||R_{\mathcal{K}}[f]||_{\ell^{2}(\mathcal{K})} \leq \inf_{r \in \mathcal{N}(\mathcal{K})} ||f + r||_{L^{\infty}[0,1]} = ||[f]||_{\mathcal{Q}_{\mathcal{K}}}.$$

This shows that the operator  $R_{\mathcal{K}}: \mathcal{Q}_{\mathcal{K}} \to \ell^2(\mathcal{K})$  is well-defined and bounded. Further, from  $R_{\mathcal{K}}([f_1] + [f_2]) = R_{\mathcal{K}}[f_1] + R_{\mathcal{K}}[f_2]$ ,  $[f_1], [f_2] \in \mathcal{Q}_{\mathcal{K}}$ , and  $R_{\mathcal{K}}(\alpha[f]) = \alpha R_{\mathcal{K}}[f], \alpha \in \mathbb{C}, [f] \in \mathcal{Q}_{\mathcal{K}}$ , we see that  $R_{\mathcal{K}}$  is a linear operator. Let  $[f_1], [f_2] \in \mathcal{Q}_{\mathcal{K}}$  be arbitrary such that  $R_{\mathcal{K}}[f_1] = R_{\mathcal{K}}[f_2]$ . It follows that

$$(R_{\mathcal{K}}[f_1])(k) - (R_{\mathcal{K}}[f_2])(k) = \int_0^1 (f_1(t) - f_2(t))\overline{\phi_k(t)} \, \mathrm{d}t = 0$$

for all  $k \in \mathcal{K}$ . Since  $\{\phi_n\}_{n \in \mathbb{N}}$  is a complete ONS, this shows that  $f_1 = f_2$ , which in turn implies that  $[f_1] = [f_2]$ . Hence  $R_{\mathcal{K}}$  injective.

Since, according to the assumptions of the theorem, the PAPR problem is weakly solvable, we have due to Lemma 1 that for every  $a \in \ell^2(\mathcal{K})$  there exists an  $f_a \in L^{\infty}[0, 1]$  such that

$$\int_0^1 f_a(t)\overline{\phi_k(t)} \, \mathrm{d}t = a_k, \quad k \in \mathcal{K}.$$

Hence, there exists a  $[f_a] \in \mathcal{Q}_{\mathcal{K}}$  such that  $R_{\mathcal{K}}[f_a] = a$ . Since  $a \in \ell^2(\mathcal{K})$  was arbitrary, we see that  $R_{\mathcal{K}}\mathcal{Q}_{\mathcal{K}} = \ell^2(\mathcal{K})$ . That is  $R_{\mathcal{K}}$  is also surjective.

We established the fact that  $R_{\mathcal{K}}: \mathcal{Q}_{\mathcal{K}} \to \ell^2(\mathcal{K})$  is a bijective bounded linear operator. As a consequence of Theorem 2, there exists a bounded linear operator  $E_{\mathcal{K}}$  such that  $E_{\mathcal{K}} = R_{\mathcal{K}}^{-1}$ .

Let  $\epsilon > 0$  and  $a \in \ell^2(\mathcal{K})$  be arbitrary. Then we have  $E_{\mathcal{K}}(a) = [f_a]$  and

$$\|[f_a]\|_{\mathcal{Q}_{\mathcal{K}}} \le \|E_{\mathcal{K}}\|_{\ell^2(\mathcal{K}) \to \mathcal{Q}_{\mathcal{K}}}\|a\|_{\ell^2(\mathcal{K})}.$$
(5)

Further, according to the definition of  $\|\cdot\|_{\mathcal{Q}_{\mathcal{K}}}$ , there exists a  $g_{\epsilon,a} \in [f_a]$  such that

$$||g_{\epsilon,a}||_{L^{\infty}[0,1]} \le ||[f_a]||_{\mathcal{Q}_{\mathcal{K}}} + \epsilon ||a||_{\ell^{2}(\mathcal{K})}.$$
 (6)

Since  $g_{\epsilon,a} \in [f_a]$  we have

$$\int_0^1 g_{\epsilon,a}(t) \overline{\phi_k(t)} \, \mathrm{d}t = a_k, \quad k \in \mathcal{K}.$$

Combining (5) and (6), we see that

$$\|g_{\epsilon,a}\|_{L^{\infty}[0,1]} \leq (\|E_{\mathcal{K}}\|_{\ell^{2}(\mathcal{K}) \to \mathcal{Q}_{\mathcal{K}}} + \epsilon)\|a\|_{\ell^{2}(\mathcal{K})}.$$

Hence, the PAPR problem is strongly solvable with extension constant  $||E_{\mathcal{K}}||_{\ell^2(\mathcal{K})\to \mathcal{Q}_{\mathcal{K}}} + \epsilon$ .

# 6. RELATION TO PRIOR WORK

The concept of weak solvability was first discussed in Section 4.7 of the chapter "Mathematics of signal design for communication systems" in [20]. However, when [20] was published, it was unclear how the PAPR problem behaves for the concept of weak solvability. In [18] it was shown for OFDM that if the PAPR problem is not weakly solvable, then the set of  $a \in \ell^2(\mathcal{K})$  such that  $||s||_{L^{\infty}[0,1]} = \infty$  is a residual set, regardless of the choice of  $b \in \ell^2(\mathcal{K})$ . Further, it was proved in [18] that for OFDM both concepts—weak and strong solvability of the PAPR problem—are equivalent. In this paper, we generalize this result to arbitrary complete ONSs.

In [17] the general theory from this paper was applied to study the PAPR problem for CDMA transmission schemes that use the Walsh functions. For this special case, the best possible constant for the reduction of the PAPR could be determined, and optimal information sets were characterized. For future research it would be interesting to further advance the techniques from the present paper, in order to obtain similar results as in [17] also for other ONSs than the Walsh system.

# 7. REFERENCES

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