A ROBUST EVENT-TRIGGERED CONSENSUS STRATEGY FOR LINEAR MULTI-AGENT SYSTEMS WITH UNCERTAIN NETWORK TOPOLOGY

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ABSTRACT

This paper proposes a robust distributed event-triggered approach for consensus in linear multi-agent systems (MAS) with uncertain network topologies. To achieve consensus, each agent transmits its information only when a certain event-triggering condition is fulfilled. The connection weights in the network are uncertain and hence the information received by each agent is unreliable. In such an uncertain topology, the objective is to co-design robust consensus parameters (namely, the state transmission threshold and local control gains) that collectively ensure an exponential rate for consensus convergence. An objective function incorporating the transmission load and control effort is minimized to compute the design parameters. Numerical simulations quantify the effectiveness of the proposed event-triggered consensus approach in a second-order MAS.

Index Terms–Multi-agent Systems, Event-triggered Consensus, Exponential Convergence, Convex Optimization.

1. INTRODUCTION

Among cooperative behaviours in multi-agent systems (MAS), consensus has attracted considerable attention in sensor networks [1] and mobile vehicles [2, 3]. Early work in this area stipulates all agents to continuously transmit their information through the network. More recently, event-triggered consensus schemes in MAS's have been introduced to reduce the number of transmissions in bandwidth constrained environments [4–6]. In such approaches, previously received information (and not the current states of the neighbouring agents) is used by each agent to make local decisions on whether to transmit and whether to update the actuator inputs. A relatively reduced number of data exchanges in such approaches makes the system sensitive to certain environmental constraints, such as communication delay, network unreliability, and parameter uncertainties [7]. Therefore, robustness analysis of event-triggered methods is critical. Since dealing with event-based strategies imposes analytical difficulties, strong assumptions are often considered to solve the event-triggered consensus (ETC) problem. For instance, reference [8] limits its approach to undirected network configurations. A majority of other works address ETC in network topologies with ideal and time-invariant connection weights [4]; an assumption which is rarely the case in practice [9]. Necessary and sufficient conditions for ensuring robust consensus in networks with uncertain connection links are provided in [10, 11]. However, constant communication between the agents is a requirement. We also note that computing control gains and transmission thresholds in existing ETC approaches is often based on a trade-off between the communication load and convergence rate [12–14]. In order to develop a more systematic approach, reference [15] proposes an exponentially fast ETC framework. Only control inputs are event-triggered and each agent still exchanges its information at every consensus step.

Motivated by the aforementioned limitations, the paper extends the co-design ETC framework developed in our previous work [16] to meet a guaranteed exponential rate of convergence in the presence of network uncertainties. The key contributions of the paper are summarized as follows. (i) To the best of our knowledge, this is the first instance in which *robustness* to non-ideal network connectivity in design parameters for ETC is being addressed. (ii) The proposed algorithm ensures an exponential (as opposed to asymptotic) consensus convergence rate in communication constrained environment. (iii) Unlike most existing ETC implementations in which control and transmission parameters are designed as a trade-off between the convergence rate and transmission load, the optimization framework developed in this paper simultaneously minimizes a function with respect to all design parameters.

The paper is organized as follows. Section 2 introduces the preliminaries. In Section 3, we formulate the ETC problem in uncertain networks and compute ETC design parameters. We provide results from simulation examples in Section 4. Finally, Section 5 concludes the paper.

2. PRELIMINARIES

Notation: Let $A=\{a_{ij}\}_{m\times n}$ denote a $(m\times n)$ matrix. Matrix $|A|=\{|a_{ij}|\}$ is a matrix with entry-wise absolute values of A. The Frobenius norm is denoted by $\|A\|$. Matrix $A^{\dagger}$ stands for the pseudo inverse of A. Notation $A>0$ implies that A is symmetric positive definite. The minimum (maximum) eigenvalue is denoted by $\lambda_{\min(\max)}(A)$. Row vector $a_{(i,\bullet)}$ is row i of matrix A, i.e., $a_{(i,\bullet)}=[a_{i1},\ldots,a_{in}]$. Similarly, $a_{(\bullet,i)}$ is column i in A. Symbol I denotes the Identity matrix of appropriate order. Column vector of order n with unit entries is expressed by $\mathbf{1}_n$. Notations $\otimes$ and $\circ$, respectively, denote the Kronecker and Hadamard products. For two vectors $\boldsymbol{u}\in\mathbb{R}^n$ and $\boldsymbol{v}\in\mathbb{R}^n$, $\boldsymbol{u}\leq\boldsymbol{v}$ refers to entry-wise inequalities $u_i\leq v_i$, $(1\leq i\leq n)$. Notation $*$ in a symmetric matrix is the transpose of its corresponding blocks from upper triangle.

Graph Theory: A weighted digraph $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ consists of a set of vertices (nodes) $\mathcal{V}=\{v_1,...,v_N\}$, a set of edges $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ and a weighted adjacency matrix $\mathcal{A}=\{a_{ij}\}_{N\times N}$. Notation $\mathcal{N}_i$ is the neighbouring set for node v_i. Matrix L is the Laplacian matrix which is obtained from $\mathcal{A}$ [17].

State Dynamics: Consider a multi-agent network system comprising of N agents with the following state model

$$\dot{\boldsymbol{x}}_i(t) = A\boldsymbol{x}_i(t) + B_i\boldsymbol{u}_i(t), \quad 1 \le i \le N. \tag{1}$$

where $\boldsymbol{x}_i(t) \in \mathbb{R}^n$ is the state vector at time instant t, and $\boldsymbol{u}_i(t) \in \mathbb{R}^m$ is the external control signal. Matrices A and B_i, respectively, represent the system and control input matrix. The network configuration is directed and strongly connected though the connections between the nodes change over time.

Definition 1. Given any initial conditions, a proposed distributed control law $\boldsymbol{u}_i(t)$ is said to solve the consensus problem if and only if $\lim_{t\to\infty}\|\boldsymbol{x}_i(t)-\boldsymbol{x}_j(t)\| = 0, (1 \le i, j \le N)$ [18].

To reach consensus, agent i transmits its state value to its neighbouring agents at time instants $t_0^i, t_1^i, \ldots$, which are determined by a proposed local triggering condition (that will be introduced later). We define the most recently broadcasted state of agent i as $\hat{\boldsymbol{x}}_i(t) = \boldsymbol{x}_i(t_{k_i}^i)$, $t \in [\, t_{k_i}^i, t_{k_i+1}^i\,)$ where $k_i = 0, 1, 2, \ldots$ defines the transmission counter for agent i. Motivated by [12], the control law to solve the ETC in the presence of network uncertainty is proposed below

$$\boldsymbol{u}_i(t) = K_i \sum_{j\in\mathcal{N}_i} \bar{a}_{ij} \Big(e^{A(t-t_{k_i}^i)}\hat{\boldsymbol{x}}_i(t) - e^{A(t-t_{k_j}^j)}\hat{\boldsymbol{x}}_j(t) \Big), \tag{2}$$

Matrix $K_i \in \mathbb{R}^{m\times n}$ is the control gain to be designed specifically for agent i. Scalar $\bar{a}_{ij} = a_{ij} + \delta_{a_{ij}}(t)$ is the uncertain weight for edge $\mathcal{E}_{ij}$ with uncertain term $\delta_{a_{ij}}(t)$. The uncertainty in adjacency matrix $\mathcal{A}$ reflects missing information on the control gains [11], actuator bias [9], and link failures [19]. The uncertain Laplacian matrix $\bar{L} = \{\bar{l}_{ij}\}$ $(1 \le i, j \le N)$ can be written as a function of the nominal Laplacian matrix L and its multiplicative uncertainty $\Delta_L(t) = \{\delta_{L_{ij}}(t)\}_{N\times N}$, i.e.,

$$\bar{L} = (I + \Delta_L(t))L, \tag{3}$$

Similar to [11], in this work the uncertain Laplacian matrix corresponding to the network maintains the diffusion property, i.e., $\sum_{j=1}^{N} \bar{l}_{ij} = 0$, $(1 \le i \le N)$. We note that the exponential term $\exp(A(t - t_{k_i}^i))$ in (2) reduces the number of event-triggered transmissions and helps in the exclusion of Zeno behaviour [12]. The following condition is considered for $\Delta_L(t)$.

Assumption 1. Matrix $\Delta_L(t)$ satisfies $\|\,\Delta_L(t)\,\| \le \eta_L$.

In practice, an uncertain network The objective of this work is to incorporate the design of consensus parameters (namely the state transmission threshold (STT) that will be introduced later, and control gains K_i's) to obtain optimal parameters with respect to a proposed objective function. Using the computed parameters, consensus is reached with a guaranteed exponential rate of convergence in the presence of uncertainty in the network topology.

3. PROBLEM FORMULATION

Let $\boldsymbol{e}_i(t) = e^{A(t-t_{k_i}^i)}\hat{\boldsymbol{x}}_i(t) - \boldsymbol{x}_i(t)$ denote the measurement error between the most recently transmitted state and its instantaneous value for agent i. In addition we define $\boldsymbol{x}(t) = [\,\boldsymbol{x}_1^T(t), \ldots, \boldsymbol{x}_N^T(t)\,]^T$, $\hat{\boldsymbol{x}}(t) = [\,\hat{\boldsymbol{x}}_1^T(t), \ldots, \hat{\boldsymbol{x}}_N^T(t)\,]^T$, and $\boldsymbol{e}(t) = [\,\boldsymbol{e}_1^T(t), \ldots, \boldsymbol{e}_N^T(t)\,]^T$. Collectively, let $\boldsymbol{e}(t) = \Lambda\hat{\boldsymbol{x}}(t) - \boldsymbol{x}(t)$, where $\Lambda = \operatorname{diag}(\Lambda_1, \ldots, \Lambda_N)$ and $\Lambda_i = e^{A(t-t_{k_i}^i)}$. Combining (1) with the proposed control law (2) leads to the following augmented system

$$\dot{\boldsymbol{x}}(t) = (A_{[N]} + BK\bar{L}_{[n]})\boldsymbol{x}(t) + BK\bar{L}_{[n]}\boldsymbol{e}(t), \tag{4}$$

where $A_{[N]} = I_N \otimes A$, $B = \operatorname{diag}(B_1, \ldots, B_N)$, $\bar{L}_{[n]} = (I + \Delta_L^{[n]}(t))L_{[n]}$, with $L_{[n]} = L \otimes I_n$, and $\Delta_L^{[n]}(t) = \Delta_L(t) \otimes I_n$. Similar to the reasons mentioned in [16], the consensus problem for system (4) is transformed to the stability problem of an equivalent system. This approach also facilitates the use of Lyapunov method, which incorporates design specifications and constraints with stability of the system. The following transformation is used for conversion to the stability problem [20]

$$\boldsymbol{x}_{\mathrm{r}}(t) = \hat{L}_{[n]}\,\boldsymbol{x}(t), \tag{5}$$

where $\hat{L}_{[n]} = \hat{L} \otimes I_n$, and $\hat{L} = \{\hat{l}_{ij}\} \in \mathbb{R}^{(N-1)\times N}$ is obtained by removing any arbitrary row of L. According to Lemma 1 given in [16], the consensus problem for system (4) is equivalent to the stability problem of the system expressed as (5). We use (5) to convert (4) to the following reduced-order system

$$\dot{\boldsymbol{x}}_{\mathrm{r}}(t) = (\,A_{[N-1]} + \mathbb{A} + \Delta_{\mathbb{A}}\,)\,\boldsymbol{x}_{\mathrm{r}}(t) + (\,\mathbb{A} + \Delta_{\mathbb{A}}\,)\,\boldsymbol{e}_{\mathrm{r}}(t), \tag{6}$$

where new variables are $A_{[N-1]} = I_{N-1} \otimes A$, $\mathbb{A} = \hat{L}_{[n]}BK\mathbb{L}$, $\Delta_{\mathbb{A}} = \hat{L}_{[n]}BK\Delta_L^{[n]}(t)\mathbb{L}$, and $\boldsymbol{e}_{\mathrm{r}}(t) = \hat{L}_{[n]}\boldsymbol{e}(t)$, with $\mathbb{L} = L_{[n]}\hat{L}_{[n]}^{\dagger}$. It also follows from (6) that $\boldsymbol{e}_{\mathrm{r}}(t) = \hat{L}_{[n]}\Lambda\hat{\boldsymbol{x}} - \boldsymbol{x}_{\mathrm{r}}(t)$. Without losing generality and for the sake of brevity in notation, we remove row N from L, to derive $\hat{L}$. Unlike our previous work [16] where only an asymptotic convergence is guaranteed, in this work we extend the approach to ensure an exponential rate of convergence for (6).

Definition 2. Given damping coefficient $\zeta > 0$, system (6) is ζ-exponentially stable if there exists a positive scalar c such that $\|\,\boldsymbol{x}_{\mathrm{r}}(t)\,\| \le ce^{-\zeta t}\|\,\boldsymbol{x}_{\mathrm{r}}(0)\,\|$, $t \ge 0$ for any initial conditions $\boldsymbol{x}_{\mathrm{r}}(0)$ [21].

The MAS (4) can reach ETC exponentially fast with the least decay rate ζ if the condition given in Definition 2 is fulfilled for (6). We proceed by introducing the event-triggering mechanism and derive sufficient conditions to incorporate the uncertain communication constraint in the design stage. We define $\mathbb{X}_i(t) = \bar{l}_{(i,\bullet)}^{[n]}\Lambda\hat{\boldsymbol{x}}(t)$, where $\bar{l}_{(i,\bullet)}^{[n]} = \bar{l}_{(i,\bullet)} \otimes I_n$ with $\bar{l}_{(i,\bullet)}$ denoting row i in perturbed Laplacian matrix $\bar{L}$. Let $\mathbb{X}(t) = [\,\mathbb{X}_1^T(t), \ldots, \mathbb{X}_N^T(t)\,]^T$. Given $t_{k_i}^i$, the next event for agent i is triggered locally from the following condition

$$t_{k_i+1}^i = \inf\{\, t > t_{k_i}^i : \mathcal{T}_i \ge 0\}, \tag{7}$$

where $\mathcal{T}_i = \|\boldsymbol{e}_i(t)\| - \phi\,\|\mathbb{X}_i(t)\|$. Scalar $\phi > 0$ is the state transmission threshold (STT) to be designed. In a collective manner, the following inequality is derived from (7)

$$\boldsymbol{e}^{[\mathrm{Nr}]} \le \phi\,\mathbb{X}^{[\mathrm{Nr}]}, \tag{8}$$

with new terms defined as $\boldsymbol{e}^{[\mathrm{Nr}]} = [\,\|\boldsymbol{e}_1(t)\|, \ldots, \|\boldsymbol{e}_N(t)\|\,]^T$ and $\mathbb{X}^{[\mathrm{Nr}]} = [\,\|\mathbb{X}_1(t)\|, \ldots, \|\mathbb{X}_N(t)\|\,]^T$. In order to incorporate the design of ϕ with K_i's, the event-triggering condition (8) needs to be expressed in terms of $\boldsymbol{x}_{\mathrm{r}}(t)$ and $\boldsymbol{e}_{\mathrm{r}}(t)$. In [16], we use two lemmas to convert (8) to an equivalent constraint expressed by $\boldsymbol{x}_{\mathrm{r}}(t)$ and $\boldsymbol{e}_{\mathrm{r}}(t)$ (inequality (10) in [16]). However, that condition is not applicable for uncertain networks. Considering an uncertain network, the following lemma is given to convert (8) to one equivalent constraint to be used in the design stage. For the sake of readability, we remove time index t from all time-varying vectors.

Lemma 1. If a certain value ϕ satisfies the condition

$$(\boldsymbol{e}_{\mathrm{r}} + \Delta_{\hat{L}}\boldsymbol{e}_{\mathrm{r}})^T(\boldsymbol{e}_{\mathrm{r}} + \Delta_{\hat{L}}\boldsymbol{e}_{\mathrm{r}}) \le (\boldsymbol{x}_{\mathrm{r}} + \boldsymbol{e}_{\mathrm{r}})^T\bar{M}_{[n]}^T\phi^2\bar{M}_{[n]}(\boldsymbol{x}_{\mathrm{r}} + \boldsymbol{e}_{\mathrm{r}}) \tag{9}$$

for $t > 0$, the same value ϕ is applicable to be used by the event-triggering conditions (8) to determine local events. Undefined parameters in (9) are given below

$\Delta_{\hat{L}} = \{\, \delta_{\hat{L}_{ij}} \,\} = \{\, \delta_{L_{ij}} + \alpha_j \delta_{L_{iN}} \,\}$, $(1 \le i, j \le N-1)$,
$\bar{M}_{[n]} = (M + \Delta_M) \otimes I_n$,
$M = \{\, m_{ij} \,\} = \{\, l_{ij} + \alpha_j l_{iN} \,\}$, $(1 \le i, j \le N-1)$,
$\boldsymbol{\alpha} = [\, \alpha_1, \dots, \alpha_{N-1} \,] = l_{(N,\bullet)} \hat{L}^{\dagger}$, and
$\Delta_M = 2\Delta_{\hat{L}} + \hat{l}_{(\bullet,N)} l_{(N,\bullet)} \hat{L}^{\dagger} \Delta_{\hat{L}}$.

We define the upper norm bounds for above uncertain matrices as $\|\Delta_{\hat{L}}\| \le \eta_{\hat{L}}$ and $\|\Delta_M\| \le \eta_M$. The proof of Lemma 1 is omitted to save on space. Inequality (9) is the sufficient event-triggering constraint to guarantee stability in the design stage (to be presented later). The following theorem computes the optimal values for control gains K_i $(1 \le i \le N)$ and the STT ϕ with respect to the objective function used in the optimization problem.

Theorem 1. Given scalar ζ, the optimal STT ϕ and control gain K_i's are computed from

$$\phi = \sqrt{\tau_1{}^{-1} \mu^{-1}}, \text{ and } K_i = B_i^{\dagger} \mathcal{P}^{-1} \Theta_i, \quad (1 \le i \le N) \tag{10}$$

which are conditioned on the existence of matrices $\Theta_i \in \mathbb{R}^{n \times n}$ $(1 \le i \le N)$, symmetric positive definite matrix $\mathcal{P} \in \mathbb{R}^{n \times n}$, and positive scalars τ_j $(1 \le j \le 4)$, μ, ϵ, ω_{τ_1}, ω_μ, $\omega_{\mathcal{P}}$, ω_{θ_i} $(1 \le i \le N)$, satisfying the following convex minimization problem

$$\min_{\Theta_i, \mu, \epsilon, \tau_j, \mathcal{P}, \omega_{\tau_1}, \omega_\mu, \omega_{\mathcal{P}}, \omega_{\theta_i}} f = \omega_{\tau_1} + \omega_\mu + \mathrm{Tr}(\boldsymbol{\omega}_{\mathcal{P}}) + \mathrm{Tr}(\boldsymbol{\omega}_\theta) \tag{11}$$

$$\text{subject to} \quad \Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \\ * & \Pi_3 \end{bmatrix} < 0, \; \begin{bmatrix} \omega_{\mathcal{P}} I & I \\ * & \mathcal{P} \end{bmatrix} > 0,$$

$$\begin{bmatrix} -\boldsymbol{\omega}_\theta & \Theta^T \\ * & -I \end{bmatrix} < 0, \begin{bmatrix} -\omega_{\tau_1} & \tau_1 \\ * & -1 \end{bmatrix} < 0, \begin{bmatrix} -\omega_\mu & \mu \\ * & -1 \end{bmatrix} < 0,$$

where

$$\Pi_1 = \begin{bmatrix} \pi_{11} & \Xi \mathbb{L} + \epsilon \eta_M^2 I \\ * & \pi_{22} \end{bmatrix}, \Pi_2 = \begin{bmatrix} 0 & \Xi & \Xi & M_{[n]}^T & 0 \\ -\tau_2 I & 0 & 0 & M_{[n]}^T & 0 \end{bmatrix},$$

$$\Pi_3 = \mathrm{diag}\,(\, -(\tau_1 + \tau_2) I, -\tau_3 I, -\tau_4 I, -\mu I, -\epsilon I) + \mathcal{I}^T \mathcal{I},$$

$$\pi_{11} = A_{[N-1]}^T P + P A_{[N-1]} + \Xi \mathbb{L} + \mathbb{L}^T \Xi^T + 2\zeta P + \tau_3 \eta_L^2 \mathbb{L}^T \mathbb{L} + \epsilon \eta_M^2 I,$$

$$\pi_{22} = \tau_4 \eta_L^2 \mathbb{L}^T \mathbb{L} - \tau_1 I + \epsilon \eta_M^2 I + \tau_2 \eta_{\hat{L}}^2 I, \qquad M_{[n]} = M \otimes I_n,$$

$$\Xi = (\hat{L} \otimes \mathbf{1}_n \mathbf{1}_n^T\,) \circ (\, \mathbf{1}_{N-1} \otimes [\, \Theta_1, \dots, \Theta_N \,]\,),$$

$$\Theta = \mathrm{diag}\,(\, \Theta_1, \dots, \Theta_N), \quad \boldsymbol{\omega}_\theta = \mathrm{diag}(\omega_{\theta_1} I_n, \dots, \omega_{\theta_N} I_n),$$

$$\mathcal{I} = \begin{bmatrix} 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & I & 0 \end{bmatrix}, \quad \boldsymbol{\omega}_{\mathcal{P}} = \omega_{\mathcal{P}} I_n, \;\; P = I_{N-1} \otimes \mathcal{P}. \tag{12}$$

Using parameters (10), consensus is reached at the given $\zeta-$exponential convergence rate, i.e., $\|\boldsymbol{x}_{\mathrm{r}}(t)\| < c\, e^{-\zeta t} \|\boldsymbol{x}_{\mathrm{r}}(0)\|$ with $c = \sqrt{\lambda_{\max}(\mathcal{P}) \lambda_{\min}^{-1}(\mathcal{P})}$. Decision variables ω_{τ_1}, ω_μ, $\omega_{\mathcal{P}}$, and ω_{θ_i} bound design parameters ϕ and K_i defined in (10) leading to the following inequalities for the minimized objective function f

$$\phi \ge (\omega_{\tau_1} \omega_\mu)^{\frac{-1}{4}}, \quad K_i^T K_i \le \omega_{\theta_i} \omega_{\mathcal{P}}^2 B_i^{\dagger} B_i^{\dagger\, T} \; (1 \le i \le N). \tag{13}$$

Proof. To derive exponential stability conditions for (6), we consider the following inequality

$$\dot{V}(t) + 2\zeta V(t) < 0, \tag{14}$$

where $V(t) = \boldsymbol{x}_{\mathrm{r}}^T(t) P \boldsymbol{x}_{\mathrm{r}}(t)$ is the Lyapunov candidate. The condition defined in (14) is equivalent to $V(t) < V(0) e^{-2\zeta t}$. Considering $V(t)$, one can obtain the sequence of inequalities $\lambda_{\min}(\mathcal{P}) \|\boldsymbol{x}_{\mathrm{r}}(t)\|^2 \le V(t) < V(0) e^{-2\zeta t} \le \lambda_{\max}(\mathcal{P}) e^{-2\zeta t} \|\boldsymbol{x}_{\mathrm{r}}(0)\|^2$, which leads to $\|\boldsymbol{x}_{\mathrm{r}}(t)\| < c e^{-\zeta t} \|\boldsymbol{x}_{\mathrm{r}}(0)\|$, with c defined in Theorem 1. Therefore,(14) is the sufficient condition to ensure the ζ-exponential stability for a given ζ. Let $\boldsymbol{\sigma}_1 = \Delta_{\hat{L}} \boldsymbol{e}_{\mathrm{r}}$, $\boldsymbol{\sigma}_2 = \Delta_L \mathbb{L} \boldsymbol{x}_{\mathrm{r}}$, and $\boldsymbol{\sigma}_3 = \Delta_L \mathbb{L} \boldsymbol{e}_{\mathrm{r}}$. We define $\Omega = [\, \boldsymbol{x}_{\mathrm{r}}^T, \boldsymbol{e}_{\mathrm{r}}^T, \boldsymbol{\sigma}_1^T, \boldsymbol{\sigma}_2^T, \boldsymbol{\sigma}_3^T \,]^T$ and expand (14) with respect to (6)

$$\Omega^T \begin{bmatrix} \hat{\pi}_{11} & P\mathbb{A} & 0 & P\hat{L}_{[n]} BK & P\hat{L}_{[n]} BK \\ * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix} \Omega < 0, \tag{15}$$

with $\hat{\pi}_{11} = (A_{[N-1]} + \mathbb{A})^T P + P(A_{[N-1]} + \mathbb{A}) + 2\zeta P$. The following conditions hold for $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, and $\boldsymbol{\sigma}_3$ according to Assumption 1

$$\boldsymbol{\sigma}_1^T \boldsymbol{\sigma}_1 = \boldsymbol{e}_{\mathrm{r}}^T \Delta_{\hat{L}}^2 \boldsymbol{e}_{\mathrm{r}} \le \eta_{\hat{L}}^2 \boldsymbol{e}_{\mathrm{r}}^T \boldsymbol{e}_{\mathrm{r}} \tag{16}$$

$$\boldsymbol{\sigma}_2^T \boldsymbol{\sigma}_2 = \boldsymbol{x}_{\mathrm{r}}^T \mathbb{L}^T \Delta_L^2 \mathbb{L} \boldsymbol{x}_{\mathrm{r}} \le \eta_L^2 \boldsymbol{x}_{\mathrm{r}}^T \mathbb{L}^T \mathbb{L} \boldsymbol{x}_{\mathrm{r}}, \tag{17}$$

$$\text{and} \quad \boldsymbol{\sigma}_3^T \boldsymbol{\sigma}_3 = \boldsymbol{e}_{\mathrm{r}}^T \mathbb{L}^T \Delta_L^2 \mathbb{L} \boldsymbol{e}_{\mathrm{r}} \le \eta_L^2 \boldsymbol{e}_{\mathrm{r}}^T \mathbb{L}^T \mathbb{L} \boldsymbol{e}_{\mathrm{r}}. \tag{18}$$

The constraints derived in (9), (16), (17), and (18) need be included in (15). Repeatedly using Lemma S - procedure [22], the above-mentioned constraints along with new slack variables τ_1 to τ_4 appear in the following inequality

$$\bar{\Pi} = \begin{bmatrix} \bar{\pi}_{11} & \bar{\pi}_{12} & 0 & P\hat{L}_{[n]} BK & P\hat{L}_{[n]} BK \\ * & \bar{\pi}_{22} & -\tau_2 I & 0 & 0 \\ * & * & -(\tau_2 + \tau_1) I & 0 & 0 \\ * & * & * & -\tau_3 I & 0 \\ * & * & * & * & -\tau_4 I \end{bmatrix} < 0, \tag{19}$$

with $\bar{\pi}_{11} = \hat{\pi}_{11} + \tau_3 \eta_L^2 \mathbb{L}^T \mathbb{L} + \tau_1 \phi^2 \bar{M}_{[n]}^T \bar{M}_{[n]}$, $\bar{\pi}_{12} = P\mathbb{A} + \tau_1 \phi^2 \bar{M}_{[n]}^T \bar{M}_{[n]}$, and $\bar{\pi}_{22} = -\tau_1 I + \tau_4 \eta_L^2 \mathbb{L}^T \mathbb{L} + \tau_2 \eta_{\hat{L}}^2 I + \tau_1 \phi^2 \bar{M}_{[n]}^T \bar{M}_{[n]}$. Next, we apply Schur complement Lemma [22] to separate the term $\tau_1 \phi^2 \bar{M}_{[n]}^T \bar{M}_{[n]}$. Then, pre- and post-multiplying the resulting matrix with $Q = \mathrm{diag}\,(I, I, I, I, I, \tau_1^{-1} \phi^{-1} I)$ leads to the following inequality

$$\tilde{\Pi} + \boldsymbol{\Delta}^T \mathbf{I} + \mathbf{I}^T \boldsymbol{\Delta} < 0 \tag{20}$$

where

$$\tilde{\Pi} = \begin{bmatrix} \tilde{\pi}_{11} & P\mathbb{A} & 0 & P\hat{L}_{[n]} BK & P\hat{L}_{[n]} BK & M_{[n]}^T \\ * & \tilde{\pi}_{22} & -\tau_2 I & 0 & 0 & M_{[n]}^T \\ * & * & -(\tau_2 + \tau_1) I & 0 & 0 & 0 \\ * & * & * & -\tau_3 I & 0 & 0 \\ * & * & * & * & -\tau_4 I & 0 \\ * & * & * & * & * & -\tau_1^{-1} \phi^{-2} I \end{bmatrix},$$

$\mathbf{I} = [\, 0, 0, 0, 0, 0, I \,]$, and $\boldsymbol{\Delta} = [\, \Delta_M, \Delta_M, 0, 0, 0, 0 \,]$, with $\tilde{\pi}_{11} = \bar{\pi}_{11} - \tau_1 \phi^2 \bar{M}_{[n]}^T \bar{M}_{[n]}$, and $\tilde{\pi}_{22} = \bar{\pi}_{22} - \tau_1 \phi^2 \bar{M}_{[n]}^T \bar{M}_{[n]}$. Accord-

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Algorithm 1 : The ER-ETC algorithm

Input: Adjacency Matrix $\mathcal{A} = \{a_{ij}\}$, Agents' dynamics (1), decay rate ζ, Laplacian uncertainty upper bound η_L.

Output: Robust Distributed Event-triggered Consensus with Exponential Convergence Rate.

Preliminaries: Determine L from $\mathcal{A}$. Remove N^{th} row of L in order to obtain $\hat{L}$. From Lemma 1, determine matrix M, and upper bounds $\eta_{\hat{L}}$ and η_M .

Off-line Parameter Design (D1–D2)

D1. *Solving the minimization LMIs*: Using a convex optimization solver, solve (11) for given value ζ.

D2. *Feasibility Verification*: If a solution exists for (11), compute optimal ϕ, and K_i's from (10). Otherwise, decrease parameter ζ and repeat step D1.

Consensus steps: (C1-C2)

C1. *Initialization*: Initialize the process by allowing all agents to transmit their initial states $\boldsymbol{x}_i(0)$ to their neighbours.

C2. *Consensus Iterations*: With computed K_i from D1, agent i in (1) is excited by (2). The event-triggering function (7) with the computed ϕ from D1, determines whether or not to transmit $\boldsymbol{x}_i(t)$ at a consensus iteration.

Table 1: Consensus performance for varying ζ.

decay rate ζ	Number of transmissions per agent						Consensus time (sec)	Objective function f
	1	2	3	4	5	6		
0.2	262	295	333	318	369	321	10.57	401.19
0.3	133	154	175	180	164	142	4.81	406.84
0.4	68	58	95	194	50	68	3.51	411.27

ing to Lemma 1 given in [23], condition (20) is satisfied if there exists a scalar $\epsilon > 0$ such that

$$\tilde{\Pi} + \epsilon \mathbf{\Delta}^T \mathbf{\Delta} + \epsilon^{-1} \mathbf{I}^T \mathbf{I} < 0. \tag{21}$$

The non-zero entries in $\mathbf{\Delta}^T \mathbf{\Delta}$, i.e., $\Delta_M^T \Delta_M$, satisfies $\Delta_M^T \Delta_M \leq \eta_M^2 I$. Therefore, the term $\epsilon \eta_M^2 I$ is placed in corresponding blocks in (21). Inequality (21) is not linear since the decision variables are multiplied by each other. To derive a linear matrix constraint, we expand $P\mathbb{A}$ in what follows

$$P\mathbb{A} = (\hat{L} \otimes \mathbf{1}_n \mathbf{1}_n^T) \circ (\mathbf{1}_{N-1} \otimes [\,\mathcal{P}B_1K_1\,,\,\ldots\,,\,\mathcal{P}B_NK_N\,])\,\mathbb{L}.$$

Defining $\Theta_i = \mathcal{P}B_iK_i$ $(1 \leq i \leq N)$ as alternative variables, inequality (20) becomes linear with respect to Θ_i's and the Ξ given in (11) is obtained. The same procedure is used to define $\mu = \tau_1^{-1}\phi^{-2}$. The objective function in this problem would maximize the STT (to minimize the number of transmissions) and minimize the norm of control gains (to minimize control effort). The change of variables used to derive Π makes such an objective function nonlinear. Motivated by [24], parameters K_i's and ϕ are derived with respect to a modified objective function which minimizes the decision variables involved in obtaining these parameters. In this regard, the inequalities $\mathcal{P}^{-1} < \omega_{\mathcal{P}} I$, $\omega_{\mathcal{P}} > 0$, $\Theta_i^T \Theta_i < \omega_{\theta_i} I$, $\omega_{\theta_i} > 0$, ${\tau_1}^2 < \omega_{\tau_1}$, $\omega_{\tau_1} > 0$, and $\mu^2 < \omega_\mu$, $\omega_\mu > 0$ are considered for the minimized sum of $\omega_{\mathcal{P}}$, ω_{θ_i}, ω_{τ_1}, and ω_μ for all $(1 \leq i \leq N)$. The Schur complement is used to convert above inequalities into LMI structures. Once the optimization problem (11) is solved, τ_1, $\mathcal{P}$, Θ_i, and μ are obtained. Consensus variables can be derived reversely and that completes the proof. □

The proposed Exponential Robust ETC algorithm, denoted by ER-ETC, is summarized in Algorithm 1.

4. NUMERICAL SIMULATIONS

Consider a network of six second-order heterogeneous agents with the following dynamics [25]

$$\begin{aligned} \dot{r}_i(t) &= v_i(t), \\ m_i \dot{v}_i(t) &= u_i(t), \quad (1 \leq i \leq 6), \end{aligned} \tag{22}$$

where $r_i(t) \in \mathbb{R}$, and $v_i(t) \in \mathbb{R}$, respectively, denotes the position and velocity for agent i. We consider $m_i = 1$, $(1 \leq i \leq 6)$, as in [25]. The state space representation for (22) with respect to (1) is given by $\boldsymbol{x}_i(t) = [\,r_i(t), v_i(t)\,]^T$, $A = [0, 1;\, 0, 0]$, $B_i = [0,\ 1]^T$, $(1 \leq i \leq 6)$. The directed network configuration for to MAS (22) is described by the asymmetric Laplacian matrix L in (23). We assume that the connection link between {agent 1 and agent 4}, and {agent 6 and agent 3} are weak. These two links fail in every odd consensus iterations, i.e., $a_{14} = 0$, $a_{63} = 0$. The perturbed Laplacian $\bar{L}$ is given below.

$$L = \begin{bmatrix} 2.5 & 0 & 0 & -0.5 & -1 & -1 \\ 0 & 2 & 0 & 0 & -1 & -1 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 0 & 2 & 0 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ -1 & -1 & -0.5 & 0 & 0 & 2.5 \end{bmatrix} \longleftrightarrow \bar{L} = \begin{bmatrix} 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 0 & -1 & -1 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 0 & 2 & 0 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 \end{bmatrix} \tag{23}$$

From (3), we obtain $\Delta_L(t) = (\bar{L} - L)L^\dagger$. In this experiment $\|\Delta_L(t)\| = \eta_L = 0.355$. Moreover, from Lemma 1 one can obtain $\eta_{\hat{L}} = 0.295$ and $\eta_M = 0.681$. To solve the consensus problem using Theorem 1, we initialize the LMI optimization with

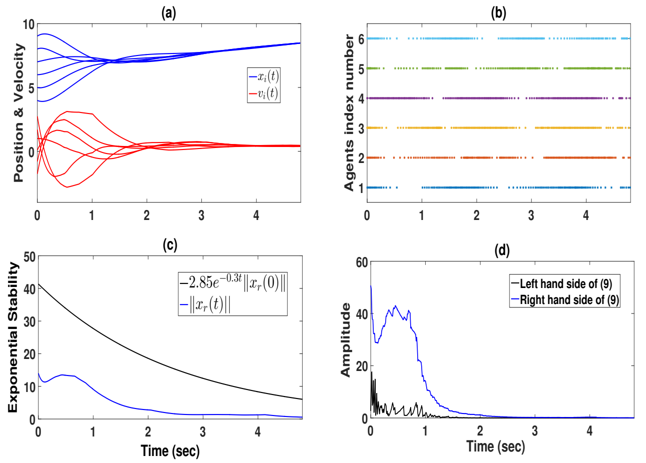

Fig. 1: Trajectories of the MAS (22)

$\zeta = 0.3$. Using the YALMIP parser and SDPT3 solver [26], we solve (11) with the aforementioned values. The control gains are derived from (10) as follows. $K_1 = -[\,1.63, 0.98\,]$, $K_2 = -[\,1.83, 1.10\,]$, $K_3 = -[\,1.99, 1.19\,]$, $K_4 = -[\,1.95, 1.17\,]$, $K_5 = -[\,2.36, 1.41\,]$, and $K_6 = -[\,1.81, 1.08\,]$. The STT is computed as $\phi = 0.0223$. In order to observe the state trajectories of the MAS (22) with the designed parameters, we pick initial values for $\boldsymbol{x}_i(0) = [\,i+3, i-3\,]^T$, $(1 \leq i \leq 6)$. Computed with discretization intervals $T_s = 0.01$sec, the state trajectories of the six agents are shown in Fig. 1(a). The triggering moments for each agent is shown in Fig. 1(b). Fig. 1(c) is included to verify that the obtained parameters, i.e., K_i's and ϕ, are capable of ensuring ζ-exponential convergence among the agents for $\zeta = 0.3$. In Fig. 1(d), we investigate the conservation imposed by Lemmas 1 which is used to convert local event-triggering conditions (8) to inequality (9). According to Fig. 1(d), the cumulative measurement errors (the left hand side of (9)) is closely upper-bounded by the right hand side of (9). It takes 481 consensus iterations to reach consensus in this experiment with a termination level of 0.01, i.e., $\|\boldsymbol{x}_r(t)\| \leq 0.01\|\boldsymbol{x}_r(0)\|$. The six agents, respectively, transmit their states on 133, 154, 175, 180, 164, and 142 occasions during the process.

Next, we study the effect of ζ on the consensus performance. To this end, we vary ζ and solve (11) while the remaining values of the system remain the same. The results are summarized in Table 1. According to Table 1, as ζ is increased, the consensus time constantly gets reduced. Faster convergence rate is achieved with a higher minimized objective function f, which implies deriving larger control gains and/or a smaller STT. Therefore, the consensus process is accomplished with more control and communication cost.

5. CONCLUSION

This paper addresses the problem of event-triggered consensus (ETC) in linear multi-agent systems (MAS) with uncertain topologies. The closed-loop system is transformed to an equivalent reduced order system. The Lyapunov stability theorem is then used to compute optimal consensus parameters (control gains and a state transmission threshold (STT)), which guarantee consensus with an exponential convergence rate in non-ideal network connectivity. The effectiveness of the proposed algorithm is studied through numerical simulations for second-order MAS's.

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