COARRAY INTERPOLATION-BASED COPRIME ARRAY DOA ESTIMATION VIA COVARIANCE MATRIX RECONSTRUCTION

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ABSTRACT

Coprime arrays are capable of achieving an increased number of degrees-of-freedom by operating the coarray signals. However, their non-uniform coarrays prevent the full utilization of the available signals. To address this problem, a novel coarray interpolation-based direction-of-arrival (DOA) estimation algorithm via covariance matrix reconstruction is proposed in this paper. In particular, we formulate a gridless optimization problem to reconstruct the covariance matrix of the interpolated coarray, such that all the coarray observations are fully utilized. We also investigate the rotational invariance in the coarray domain to retrieve the DOAs. Neither spatial sampling nor spectrum searching is required in the proposed algorithm, indicating the capability of resolving off-grid DOAs. Simulation results demonstrate the effectiveness of the proposed DOA estimation algorithm.

***Index Terms*—** Coarray interpolation, coprime array, covariance matrix reconstruction, DOA estimation, rotational invariance.

1. INTRODUCTION

Direction-of-arrival (DOA) estimation is one of the key techniques in radar, sonar, acoustics, speech, seismic, and wireless communications [1–12]. Recently, coprime arrays have attracted tremendous attentions due to their systematical sparse array configurations and superior estimation performance over the uniform linear arrays (ULAs) [13–17]. Nevertheless, the coarray of the coprime array is non-uniform, making the DOA estimation methods tailored for ULAs difficult to apply. A commonly used solution is to extract the maximum contiguous segment of the coarray for the subsequent signal processing [18–21]. In so doing, however, performance loss will be inevitably encountered due to the discarded sensors.

Coarray interpolation is a promising approach to generate a ULA in the coarray domain, such that all the signals of the non-uniform coarray can be utilized. In [22], a nuclear norm minimization (NNM) algorithm is proposed to complete the interpolated coprime coarray covariance matrix, where the missing correlation elements corresponding to the interpolated sensors are recovered. Furthermore, the minimum-sized coarray for completing the interpolated coarray covariance matrix is investigated in [23]. However, both approaches are developed based on the matrix completion, indicating that the correlations obtained from the sample covariance matrix are retained in the completed covariance matrix. Hence, the estimation accuracy is affected by the finite number of snapshots. In [24], the positive semidefinite (PSD) constraint is imposed to the NNM problem, based on which a unified analysis for the extrapolation error is presented. Although the abovementioned methods optimize the covariance matrix in a gridless manner, the given spectrum searching interval limits the estimation accuracy of off-grid DOAs when incorporating the coarray MUSIC [18, 22–25].

In this paper, we propose a novel DOA estimation algorithm based on covariance matrix reconstruction by exploiting the rotational invariance of the interpolated coprime coarray covariance matrix. Different from the previous methods adopting the principle of *matrix completion*, the proposed algorithm adopts the principle of *matrix reconstruction* to recover the interpolated coarray covariance matrix, where the observed correlations are used for covariance matrix fitting in a formulated gridless optimization problem. Moreover, the shift invariance of the interpolated coprime coarray is investigated, from which a closed-form solution for DOA estimation is derived based on the rotational invariance involved in the reconstructed interpolated coprime coarray covariance matrix. As a result, the time-consuming spectrum searching process is avoided. The effectiveness of the proposed algorithm is validated through numerical simulations.

2. COPRIME COARRAY SIGNAL MODEL

A coprime array is the union of a pair of sparse ULAs as

$$\mathbb{S}=\{Nmd, 0\leq m\leq M-1\}\cup\{Mnd, 1\leq n\leq N-1\}\,, \quad (1)$$

where M and N are coprime integers, and d equals to a half wavelength, i.e., $d=\lambda/2$. Due to the coprimality, there are a total number of $M+N-1$ sensors in the coprime array.

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Assuming K far-field, narrowband, and statistically uncorrelated sources impinge the coprime array from directions $\theta_k, k = 1, 2, \cdots, K$, the coprime array received signals at time slot t can be modeled as

$$\boldsymbol{x}(t) = \sum_{k=1}^{K} \boldsymbol{a}_{\mathbb{S}}(\theta_k) s_k(t) + \boldsymbol{n}(t), \tag{2}$$

where

$$\boldsymbol{a}_{\mathbb{S}}(\theta_k) = \left[1, \mathrm{e}^{-j\frac{2\pi}{\lambda} z_2 \sin(\theta_k)}, \cdots, \mathrm{e}^{-j\frac{2\pi}{\lambda} z_{M+N-1} \sin(\theta_k)}\right]^{\mathrm{T}} \tag{3}$$

is the coprime array steering vector, $s_k(t)$ is the corresponding signal waveform, and $\boldsymbol{n}(t)$ is the Gaussian white noise vector. Here, $\mathbb{S} = \{z_1, z_2, \cdots, z_{M+N-1}\}$ is the set of sensor positions with reference $z_1 = 0$, and $(\cdot)^{\mathrm{T}}$ denotes transpose. Accordingly, the covariance matrix of the signal vector received at the coprime array is

$$\boldsymbol{R}_{\mathbb{S}} = \mathrm{E}\left[\boldsymbol{x}(t)\boldsymbol{x}^{\mathrm{H}}(t)\right] = \sum_{k=1}^{K} p_k \boldsymbol{a}_{\mathbb{S}}(\theta_k) \boldsymbol{a}_{\mathbb{S}}^{\mathrm{H}}(\theta_k) + \sigma_n^2 \boldsymbol{I}, \tag{4}$$

where $\mathrm{E}[\cdot]$ is the expectation operator, $(\cdot)^{\mathrm{H}}$ stands for conjugate transpose, $p_k = \mathrm{E}\left[|s_k(t)|^2\right]$ denotes the signal power of the k-th source, and σ_n^2 denotes the noise power.

The coprime coarray $\mathbb{V}$ can be formulated as

$$\mathbb{V} = \{z_\imath - z_\jmath | z_\imath, z_\jmath \in \mathbb{S}\}, \tag{5}$$

whose corresponding coarray signals can be obtained by vectorizing $\boldsymbol{R}_{\mathbb{S}}$. After removing the repeated rows and rearranging all remaining rows, the coarray signals can be modeled as

$$\boldsymbol{u}_{\mathbb{V}} = \sum_{k=1}^{K} p_k \boldsymbol{a}_{\mathbb{V}}(\theta_k) + \sigma_n^2 \boldsymbol{i}_{\mathbb{V}}, \tag{6}$$

where $\boldsymbol{a}_{\mathbb{V}}(\theta_k)$ is the steering vector of the coprime coarray $\mathbb{V}$, and the elements in $\boldsymbol{i}_{\mathbb{V}}$ are all zeros except ones corresponding to the zeroth position in $\mathbb{V}$. Although the *second-order* coprime coarray signals $\boldsymbol{u}_{\mathbb{V}}$ has a similar form as the *first-order* signals $\boldsymbol{x}(t)$, the coprime coarray $\mathbb{V}$ offers more degrees-of-freedom (DOFs) than the coprime array $\mathbb{S}$.

3. THE PROPOSED ALGORITHM

A novel coarray interpolation-based DOA estimation algorithm is proposed in this section, including gridless covariance matrix reconstruction and a searching-free solution.

3.1. Coarray Interpolation and Matrix Reconstruction

It is revealed in [13] that there exist several holes in $\mathbb{V}$, resulting in a discontiguous coarray configuration. In order to create a ULA-based coarray model without discarding the discontiguous sensors in $\mathbb{V}$, we introduce the idea of array interpolation into the coarray domain, where the additional virtual sensors are interpolated to the positions of holes to obtain a ULA as

$$\mathbb{I} = \{\, i | \min(\mathbb{V}) \leq i \leq \max(\mathbb{V})\}\,. \tag{7}$$

In practical applications, the theoretical covariance matrix $\boldsymbol{R}_{\mathbb{S}}$ is unavailable due to the finite number of snapshots. Hence, the observed coarray signals of $\mathbb{V}$ are obtained by vectorizing the sample covariance matrix $\hat{\boldsymbol{R}}_{\mathbb{S}}$ as

$$\hat{\boldsymbol{u}}_{\mathbb{V}} = \mathrm{vec}(\hat{\boldsymbol{R}}_{\mathbb{S}}) = \mathrm{vec}\left(\frac{1}{T}\sum_{t=1}^{T} \boldsymbol{x}(t)\boldsymbol{x}^{\mathrm{H}}(t)\right), \tag{8}$$

where T is the number of snapshots. Accordingly, the interpolated coprime coarray signals can be initialized as

$$\langle \hat{\boldsymbol{u}}_{\mathbb{I}} \rangle_\ell = \begin{cases} \langle \hat{\boldsymbol{u}}_{\mathbb{V}} \rangle_\ell, & \ell \in \mathbb{V}, \\ 0, & \ell \in \mathbb{I} \backslash \mathbb{V}, \end{cases} \tag{9}$$

where $\langle \cdot \rangle_\ell$ denotes the element corresponding to the position ℓ. Here, the coarray signals corresponding to the non-uniform coprime coarray $\mathbb{V}$ are retained in $\hat{\boldsymbol{u}}_{\mathbb{I}}$, whereas those corresponding to the interpolated sensors are simply set to zeros since they are unknown in practice.

According to the relationship between the second-order coarray signals and the Toeplitz matrix structure established in [26], the initial covariance matrix of the interpolated coprime coarray signals can be directly constructed as

$$\hat{\boldsymbol{R}}_{\mathbb{I}} = \mathcal{T}\left(\hat{\boldsymbol{u}}_{\mathbb{I}^+}\right), \tag{10}$$

where $\mathcal{T}\left(\hat{\boldsymbol{u}}_{\mathbb{I}^+}\right)$ denotes a Hermitian Toeplitz matrix with $\hat{\boldsymbol{u}}_{\mathbb{I}^+}$ as its first column. Here, $\hat{\boldsymbol{u}}_{\mathbb{I}^+}$ is a subvector of $\hat{\boldsymbol{u}}_{\mathbb{I}}$ containing the initial coarray signals of $\mathbb{I}^+$, and $(\cdot)^+$ denotes the subset consisting of the non-negative elements. The initial covariance matrix $\hat{\boldsymbol{R}}_{\mathbb{I}}$ contains all the information in $\hat{\boldsymbol{u}}_{\mathbb{I}}$, where the spatial smoothing process and the matrix square root process in [18] are skipped. Meanwhile, we define a binary matrix $\boldsymbol{\Omega} \in \mathbb{R}^{|\mathbb{I}^+| \times |\mathbb{I}^+|}$ with the same dimension as $\hat{\boldsymbol{R}}_{\mathbb{I}}$ to distinguish the elements of $\hat{\boldsymbol{R}}_{\mathbb{I}}$, whose elements corresponding to the observed correlations in $\hat{\boldsymbol{u}}_{\mathbb{I}}$ are set to ones and zeros otherwise. It is obvious that the binary matrix $\boldsymbol{\Omega}$ is determined when a systematical coprime array configuration is deployed.

The goal of covariance matrix reconstruction [19, 27, 28] is to recover the covariance matrix of the interpolated coprime coarray, including the unknown elements which are initialized to be zeros in (9). In particular, taking the initial covariance matrix $\hat{\boldsymbol{R}}_{\mathbb{I}}$ as the reference, the interpolated coprime coarray covariance matrix reconstruction can be formulated as a nuclear norm minimization problem as

$$\begin{aligned} \min_{\boldsymbol{u} \in \mathbb{C}^{|\mathbb{I}^+|}} \quad & \|\mathcal{T}(\boldsymbol{u})\|_* \\ \text{subject to} \quad & \left\|\mathcal{P}_{\boldsymbol{\Omega}}\left(\mathcal{T}(\boldsymbol{u})\right) - \hat{\boldsymbol{R}}_{\mathbb{I}}\right\|_F^2 \leq \varepsilon, \\ & \mathcal{T}(\boldsymbol{u}) \succeq \boldsymbol{0}, \end{aligned} \tag{11}$$

where $\mathcal{P}_{\boldsymbol{\Omega}}(\,\cdot\,)$ denotes the projection operator by taking the Hadamard product of the binary matrix $\boldsymbol{\Omega}$ with the same dimensional matrix, $\|\cdot\|_*$ and $\|\cdot\|_F$ respectively denote the nuclear norm and Frobenius norm, ε constrains the fitting error between the observed correlations (non-zero elements) in $\hat{\boldsymbol{R}}_{\mathbb{I}}$ and the corresponding elements in $\mathcal{T}(\boldsymbol{u})$, and $\mathcal{T}(\boldsymbol{u}) \succeq \boldsymbol{0}$ ensures a Hermitian PSD Toeplitz matrix. Alternatively, the optimization problem (11) can be reformulated as

$$\begin{aligned} \min_{\boldsymbol{u}\in\mathbb{C}^{|\mathbb{I}^+|}} \quad & \frac{1}{2}\left\|\mathcal{P}_{\boldsymbol{\Omega}}\left(\mathcal{T}(\boldsymbol{u})\right) - \hat{\boldsymbol{R}}_{\mathbb{I}}\right\|_F^2 + \mu\|\boldsymbol{u}\|_* \\ \text{subject to} \quad & \mathcal{T}(\boldsymbol{u}) \succeq \boldsymbol{0}, \end{aligned} \tag{12}$$

where μ is a regularization parameter to balance the fitting error and the nuclear norm. The optimization problem (12) is convex and can be efficiently solved using interior point methods.

3.2. DOA Estimation Exploiting Rotational Invariance

The reconstructed matrix $\mathcal{T}(\hat{\boldsymbol{u}}) \in \mathbb{C}^{|\mathbb{I}^+|\times|\mathbb{I}^+|}$ can be viewed as the covariance matrix of the interpolated coprime coarray $\mathbb{I}^+$, whose uniform structure enables the investigation of rotational invariance in the coarray domain [29, 30]. In particular, we first extract a pair of shift invariant subarrays $\mathbb{I}_1^+$ and $\mathbb{I}_2^+$ from $\mathbb{I}^+$ as illustrated in Fig. 1, whose corresponding $(|\mathbb{I}^+|-1)\times K$ dimensional steering matrices can be represented as $\boldsymbol{A}_{\mathbb{I}_1^+}$ and $\boldsymbol{A}_{\mathbb{I}_2^+}$, respectively. While the pair of subarrays share the same geometry except a translational distance of d, the steering matrices can be related as

$$\boldsymbol{A}_{\mathbb{I}_2^+} = \boldsymbol{A}_{\mathbb{I}_1^+}\boldsymbol{\Phi}, \tag{13}$$

where

$$\boldsymbol{\Phi} = \operatorname{diag}\left\{\mathrm{e}^{-j\frac{2\pi}{\lambda}d\sin(\theta_1)}, \mathrm{e}^{-j\frac{2\pi}{\lambda}d\sin(\theta_2)}, \cdots, \mathrm{e}^{-j\frac{2\pi}{\lambda}d\sin(\theta_K)}\right\} \tag{14}$$

is a unitary matrix. Meanwhile, denoting $\boldsymbol{E}_{\boldsymbol{s},\mathbb{I}^+} \in \mathbb{C}^{|\mathbb{I}^+|\times K}$ as the signal subspace of $\mathcal{T}(\hat{\boldsymbol{u}})$ including the eigenvectors corresponding to its K prominent eigenvalues, the signal subspaces of the shift invariant subarray pair $\boldsymbol{E}_{\boldsymbol{s},\mathbb{I}_1^+}$ and $\boldsymbol{E}_{\boldsymbol{s},\mathbb{I}_2^+}$ can be respectively obtained by removing the last row and the first row of $\boldsymbol{E}_{\boldsymbol{s},\mathbb{I}^+}$ as

$$\boldsymbol{E}_{\boldsymbol{s},\mathbb{I}^+} = \begin{bmatrix} \boldsymbol{E}_{\boldsymbol{s},\mathbb{I}_1^+} \\ \text{last row} \end{bmatrix} = \begin{bmatrix} \text{first row} \\ \boldsymbol{E}_{\boldsymbol{s},\mathbb{I}_2^+} \end{bmatrix}. \tag{15}$$

Since $\boldsymbol{E}_{\boldsymbol{s},\mathbb{I}^+}$ shares the same subspace with those spanned by $\boldsymbol{A}_{\mathbb{I}^+}$, a unique nonsingular matrix $\boldsymbol{T} \in \mathbb{C}^{K\times K}$ exists and relates the signal subspaces and the steering matrices of each subarray as

$$\boldsymbol{E}_{\boldsymbol{s},\mathbb{I}_1^+} = \boldsymbol{A}_{\mathbb{I}_1^+}\boldsymbol{T}, \tag{16}$$

$$\boldsymbol{E}_{\boldsymbol{s},\mathbb{I}_2^+} = \boldsymbol{A}_{\mathbb{I}_2^+}\boldsymbol{T} = \boldsymbol{A}_{\mathbb{I}_1^+}\boldsymbol{\Phi}\boldsymbol{T}. \tag{17}$$

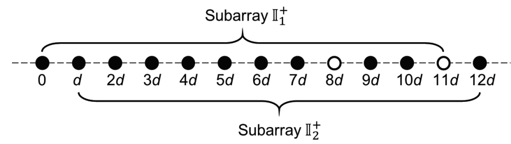

Fig. 1. Illustration of a pair of shift invariant subarrays in the coarray domain with an example of M=3 and N=5. Filled circles: coarray sensors; Hollow circles: interpolated sensors.

While both $\boldsymbol{E}_{\boldsymbol{s},\mathbb{I}_1^+}$ and $\boldsymbol{E}_{\boldsymbol{s},\mathbb{I}_2^+}$ have the full rank K, an orthogonal relationship can be established by defining the $K\times K$ dimensional matrices $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ as

$$\begin{bmatrix} \boldsymbol{E}_{\boldsymbol{s},\mathbb{I}_1^+} & \boldsymbol{E}_{\boldsymbol{s},\mathbb{I}_2^+} \end{bmatrix} \begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \end{bmatrix} = \boldsymbol{E}_{\boldsymbol{s},\mathbb{I}_1^+}\boldsymbol{F_1} + \boldsymbol{E}_{\boldsymbol{s},\mathbb{I}_2^+}\boldsymbol{F_2} = \boldsymbol{0}. \tag{18}$$

Hence, $\boldsymbol{E}_{\boldsymbol{s},\mathbb{I}_2^+} = \boldsymbol{E}_{\boldsymbol{s},\mathbb{I}_1^+}\boldsymbol{\Gamma}$ with $\boldsymbol{\Gamma} = -\boldsymbol{F}_1\boldsymbol{F}_2^{-1}$, which can be calculated as

$$\boldsymbol{\Gamma} = \boldsymbol{E}_{\boldsymbol{s},\mathbb{I}_1^+}^{\dagger}\boldsymbol{E}_{\boldsymbol{s},\mathbb{I}_2^+}. \tag{19}$$

Here, $(\,\cdot\,)^{\dagger}$ denotes the pseudo-inverse. Combining the relationships established in (16) and (17), we have

$$\boldsymbol{A}_{\mathbb{I}_1^+}\boldsymbol{\Phi}\boldsymbol{T} = \boldsymbol{A}_{\mathbb{I}_1^+}\boldsymbol{T}\boldsymbol{\Gamma}. \tag{20}$$

Because $\boldsymbol{T}$ is a nonsingular matrix and $\boldsymbol{A}_{\mathbb{I}_1^+}$ is full rank, (20) can be transformed as

$$\boldsymbol{\Phi} = \boldsymbol{T}\boldsymbol{\Gamma}\boldsymbol{T}^{-1}. \tag{21}$$

According to the definition of $\boldsymbol{\Phi}$ in (14), a closed-form solution for DOA estimation can be obtained as

$$\hat{\theta}_k = \arcsin\left(-\frac{1}{\pi}\Im(\ln\gamma_k)\right), \tag{22}$$

where $\Im(\,\cdot\,)$ denotes the imaginary part of the complex number, and γ_k is the k-th dominant eigenvalue of $\boldsymbol{\Gamma}$.

4. SIMULATION RESULTS

In our simulations, we deploy the coprime array with a pair of coprime integers $M = 3$ and $N = 5$, i.e., $\mathbb{S} =$

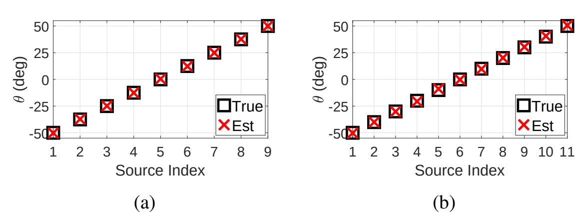

Fig. 2. The spatial spectra of the proposed algorithm with the number of sources: (a) $K = 9$; (b) $K = 11$.

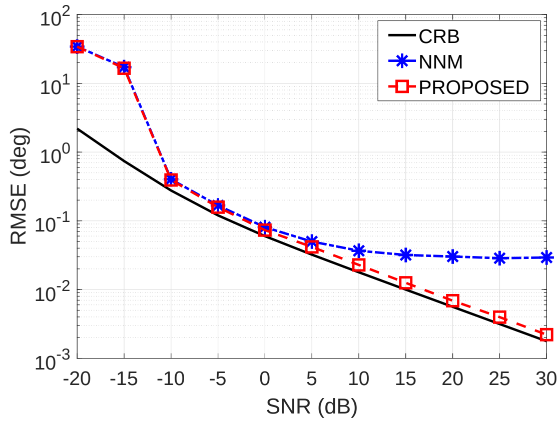

Fig. 3. Comparison of DOA estimation accuracy: RMSE versus SNR, $T = 500$.

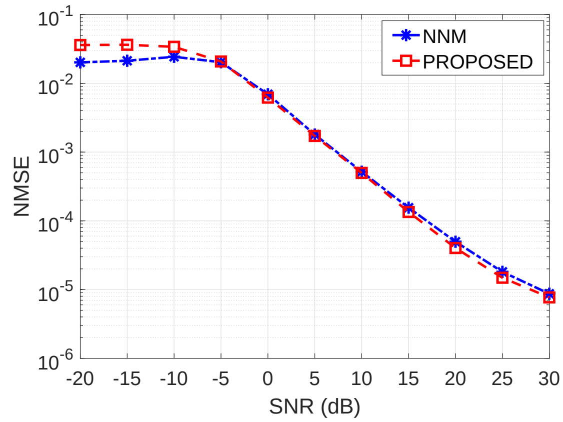

Fig. 4. Comparison of optimized covariance matrix accuracy: NMSE versus SNR, $T = 500$.

$\{0, 3d, 5d, 6d, 9d, 10d, 12d\}$. The regularization parameter μ in the proposed algorithm is chosen to be 0.25.

The available DOFs of the proposed algorithm are illustrated by depicting the spatial spectra in Fig. 2, where the sources are uniformly distributed in $[-50°, 50°]$. We first consider the case that the number of sources is $K = 9$, which exceeds the DOFs provided by the contiguous segment of $\mathbb{V}$. It is observed from Fig. 2(a) that the proposed algorithm is capable of identifying all the nine sources, indicating that the discontiguous sensors in $\mathbb{V}$ are effectively utilized. Moreover, Fig. 2(b) shows that the proposed algorithm can also resolve all $K = 11$ sources, whereas the maximum number of achievable DOFs of $\mathbb{V}$ is only ten. Hence, although the interpolated virtual sensors do not actually exist, they still provide additional DOFs via covariance matrix reconstruction.

The DOA estimation accuracy of the proposed algorithm is compared with the NNM algorithm [22], where the direction of the incident source is randomly selected from the normal distribution $\mathcal{N}(0°, (1°)^2)$. The spectrum searching interval for the NNM algorithm is given as $\Delta\theta = 0.1°$. The root mean square error (RMSE) versus the input signal-to-noise ratio (SNR) is depicted in Fig. 3, where the Cramér-Rao bound (CRB) [31] is also presented for reference. For each scenario, 1,000 Monte-Carlo trials are performed.

It is clear from Fig. 3 that the proposed algorithm outperforms the NNM algorithm when the input SNR is higher than -5 dB. This is because the estimation accuracy of the NNM algorithm is limited by the given searching interval $\Delta\theta$. In contrast, the proposed algorithm retrieves the closed-form DOAs according to the rotational invariance involved in the interpolated coprime coarray covariance matrix, where the spectrum searching process is not required. Therefore, the proposed algorithm is capable of effectively resolving off-grid DOAs. We note that the performance of the proposed algorithm in the asymptotic region is consistent with the CRB.

Furthermore, we also compare the estimation accuracy of the optimized Hermitian Toeplitz covariance matrix $\mathcal{T}(\hat{\boldsymbol{u}})$ by defining the normalized mean square error (NMSE) as

$$\text{NMSE} = \text{E}\left[\frac{\|\hat{\boldsymbol{u}}(q) - \boldsymbol{u}\|_2^2}{\|\boldsymbol{u}\|_2^2}\right], \tag{23}$$

where $\boldsymbol{u}$ is the first column of the theoretical interpolated coprime coarray covariance matrix, and $\hat{\boldsymbol{u}}(q)$ is the estimator of $\boldsymbol{u}$ in the q-th Monte-Carlo trial. The simulation parameters are the same as those in Fig. 3.

It is observed from Fig. 4 that the NMSE of the proposed algorithm outperforms that of the NNM algorithm when the input SNR is higher than 0 dB. In this case, the NNM algorithm recovers the missing elements corresponding to the interpolated sensors based on the principle of matrix completion, while the observed correlations are retained in the completed covariance matrix. In contrast, the proposed algorithm formulates an optimization problem based on the principle of matrix reconstruction, where the elements in the reconstructed covariance matrix $\mathcal{T}(\hat{\boldsymbol{u}})$ may not be the same as those in $\hat{\boldsymbol{R}}_{\mathbb{I}}$, especially in the scenario of limited snapshots.

5. CONCLUSION

In this paper, we proposed a novel coarray interpolation-based DOA estimation algorithm via covariance matrix reconstruction, such that all sensors in the discontiguous coprime coarray can be fully utilized. The idea of array interpolation is implemented in the coarray domain. A gridless optimization problem is formulated to reconstruct the interpolated coprime coarray covariance matrix, based on which the rotational invariance is then investigated to retrieve the closed-form DOAs without spectrum searching. Simulation results demonstrated the superiority of the proposed algorithm in terms of the achievable number of DOFs and the estimation accuracy.

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