MMSE-BASED AUTOCORRELATION SAMPLING FOR COPRIME ARRAYS

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ABSTRACT

Sampling the physical-array autocorrelations is the initial processing step in standard direction-of-arrival (DoA) estimation with coprime arrays. These samples are then organized into an autocorrelation matrix estimate of a uniform-linear virtual *coarray*, which in turn is used for DoA estimation. Existing autocorrelation-sampling approaches provide the exact coarray autocorrelation matrix for asymptotically large sample support; however, they attain arbitrary/suboptimal mean-squared estimation error (MSE) for limited/low sample support. In this work, we present a minimum-MSE (MMSE) approach for autocorrelation sampling. The proposed method offers a superior autocorrelation-matrix estimate that can attain higher DoA estimation accuracy than the standard counterparts.

Index Terms— Coprime arrays, DoA estimation, mean-squared-error, sparse arrays.

1. INTRODUCTION

Coprime arrays offer increased degrees of freedom compared to uniform linear arrays (ULAs) equipped with the same number of elements [1–15]. This is due to their particular structure and accomplished by thereto tailored intelligent receiver processing. In practice, coprime arrays allow for the estimation of significantly increased number of directions-of-arrival (DoAs). Coprime-arrays have also been studied for beamforming [16] and space-time processing [17].

DoA estimation with coprime arrays is conducted by processing the entries of the (estimated) spatial autocorrelation matrix of the physical array. Leveraging the specific array structure, intelligent processing of the autocorrelations offers a signal subspace estimate equivalent to the subspace of a larger virtual ULA, commonly referred to as "coarray." Certainly, DoA estimation can be performed by standard multiple-signal classification (MUSIC) on the coarray signal subspace, with the ability to resolve significantly increased number of DoAs. First, the coprime array receiver samples the estimated physical-array autocorrelations and, possibly, extrapolates them by means of compressive-sensing techniques [18, 19]. Then, these samples are spatially smoothed to form a matrix within the span of which the signal and noise subspaces are separable [5, 12]. The most common approach samples the autocorrelations by *selection* [1], selecting a single sample per coarray element. Other works sample by *averaging* all autocorrelation values that correspond to each coarray element [12]. The two methods coincide when applied on the nominal physical-array autocorrelations –which the receiver could only estimate with asymptotically large number of snapshots. However, their performance differs significantly for the realistic case of limited/small number of snapshots. A recent theoretical analysis formally proved that averaging-sampling exhibits superior MSE estimation performance compared to selection-sampling [20].

In this paper, we present a novel autocorrelation sampling approach that, for any number of snapshots, estimates the coarray autocorrelation matrix with minimum-MSE (MMSE). Our numerical studies show that the proposed sampling can attain improved DoA estimation accuracy.

2. SIGNAL MODEL AND PROBLEM STATEMENT

Consider coprime array of length $L \triangleq 2M + N - 1$, for coprime naturals (M, N) such that $M < N$ [1], formed by overlapping a length-N ULA with elements at positions $\{p_{M,i} = (i-1)Md\}_{i=1}^{N}$, and a length-$(2M-1)$ ULA with elements at positions $\{p_{N,i} = iNd\}_{i=1}^{2M-1}$. Here, d is the reference unit spacing (e.g., one-half wavelength). The coprime array element positions are given by the entries of $\mathbf{p} \triangleq \mathrm{sort}([p_{M,1}, \ldots, p_{M,N}, p_{N,1}, \ldots, p_{N,2M-1}]^\top)$, where $\mathrm{sort}(\cdot)$ sorts the entries of its vector argument in ascending order. We assume that narrowband signals from $K < MN + M$ sources impinge on the array with carrier frequency f_c and propagation speed c. Under far-field conditions, the signal from source $k \in \{1, 2, \ldots, K\}$ impinges on the array from direction $\theta_k \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ with respect to the broadside. The array response vector $\mathbf{s}(\theta_k)$ for source k is $\mathbf{s}(\theta_k) \triangleq \left[v(\theta_k)^{[\mathbf{p}]_1}, \ldots, v(\theta_k)^{[\mathbf{p}]_L}\right]^\top \in \mathbb{C}^L$, with $v(\theta) \triangleq \exp\left(\frac{-j2\pi f_c}{c}\sin(\theta)\right)$ for every $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. The qth received vector takes the form

$$\mathbf{y}_q = \sum_{k=1}^{K} \mathbf{s}(\theta_k)\xi_{q,k} + \mathbf{n}_q \in \mathbb{C}^{L\times 1}, \quad (1)$$

where $\xi_{q,k} \sim \mathcal{CN}(0, d_k)$ is the qth symbol for source k (power-scaled and flat-fading-channel processed) and $\mathbf{n}_q \sim \mathcal{CN}(\mathbf{0}_L, \sigma^2\mathbf{I}_L)$ is additive white Gaussian noise (AWGN). We assume that the random variables are statistically independent across different snapshots and the symbols from different sources are independent of each other and of every entry of $\mathbf{n}_q$. The receiver's goal is to estimate the source DoAs in $\Theta \triangleq \{\theta_1, \theta_2, \ldots, \theta_K\}$. Defining the source power-vector $\mathbf{d} \triangleq [d_1, d_2, \ldots, d_K]^\top \in \mathbb{R}_+^K$ and array-response matrix $\mathbf{S} \triangleq [\mathbf{s}(\theta_1), \mathbf{s}(\theta_2), \ldots, \mathbf{s}(\theta_K)] \in \mathbb{C}^{L\times K}$, the received-signal autocorrelation matrix is given by $\mathbf{R}_y \triangleq \mathbb{E}\{\mathbf{y}_q\mathbf{y}_q^H\} = \mathbf{S}\,\mathrm{diag}(\mathbf{d})\,\mathbf{S}^H + \sigma^2\mathbf{I}_L$. Accordingly, we define $\mathbf{r} \triangleq \mathrm{vec}(\mathbf{R}_y) = \sum_{i=1}^{K}\mathbf{a}(\theta_i)d_i + \sigma^2\mathbf{i}_L \in \mathbb{C}^{L^2}$, where $\mathrm{vec}(\cdot)$ returns the column-wise vectorization of its ma-

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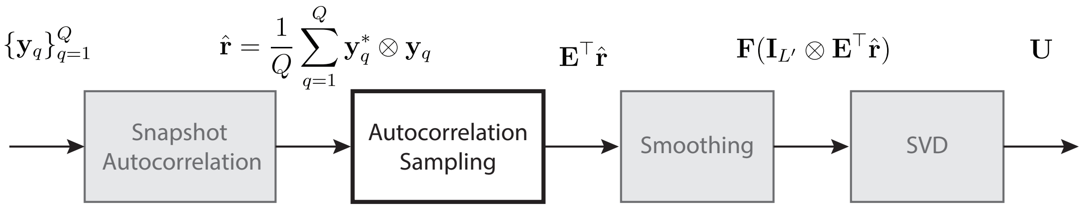

Fig. 1. Coprime processing steps: from a collection of samples $\{\mathbf{y}_q\}_{q=1}^Q$ to the estimated coarray signal-subspace basis $\mathbf{U}$.

trix argument, $\mathbf{a}(\theta_i) \triangleq \mathbf{s}(\theta_i)^* \otimes \mathbf{s}(\theta_i)$, $\mathbf{i}_L \triangleq \text{vec}(\mathbf{I}_L) \in \mathbb{R}^{L^2}$, and '$\otimes$' is the Kronecker product operator [21]. By coprime number theory [1], for every $n \in \{-L'+1, -L'+2, \ldots, L'-1\}$ with $L' \triangleq MN + M$, there exists a well-defined set of indices $\mathcal{J}_n \subset \{1, 2, \ldots, L^2\}$, such that

$$[\mathbf{a}(\theta)]_j = v(\theta)^n \;\; \forall j \in \mathcal{J}_n, \tag{2}$$

for every $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. We henceforth consider that $\mathcal{J}_n$ contains *all* j indices that satisfy (2). In view of (2), the receiver in standard coprime DoA estimation [1] selects any single index $j_n \in \mathcal{J}_n$, for $n \in \{-L'+1, \ldots, L'-1\}$, and builds the $L^2 \times (2L'-1)$ *selection* sampling matrix $\mathbf{E}_{sel} \triangleq \left[\mathbf{e}_{j_{1-L'},L^2}, \mathbf{e}_{j_{2-L'},L^2}, \ldots, \mathbf{e}_{j_{L'-1},L^2}\right]$, where, for any $p \leq P \in \mathbb{N}_+$, $\mathbf{e}_{p,P}$ is the pth column of $\mathbf{I}_P$. That is, the receiver samples the autocorrelations contained in $\mathbf{r}$, discarding by selection all duplicates (i.e., every entry with index in $\mathcal{J}_n \setminus j_n$, for every n), to form

$$\mathbf{r}_{sel} \triangleq \mathbf{E}_{sel}^\top \mathbf{r} = \sum_{i=1}^{K} \mathbf{a}_{sel}(\theta_i) d_i + \sigma^2 \mathbf{e}_{L',2L'-1}, \tag{3}$$

where, for any $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, $\mathbf{a}_{sel}(\theta) \triangleq \mathbf{E}_{sel}^\top \mathbf{a}(\theta) = [\, v(\theta)^{1-L'}, v(\theta)^{2-L'}, \ldots, v(\theta)^{L'-1} \,]^\top \in \mathbb{C}^{2L'-1}$. Thereafter, the receiver applies spatial-smoothing to organize the sampled autocorrelations in the matrix

$$\mathbf{Z}_{sel} \triangleq \mathbf{F}(\mathbf{I}_{L'} \otimes \mathbf{r}_{sel}) \in \mathbb{C}^{L' \times L'}, \tag{4}$$

where $\mathbf{F} \triangleq [\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_{L'}]$ and, for every $m \in \{1, 2, \ldots, L'\}$, $\mathbf{F}_m \triangleq [\mathbf{0}_{L' \times (L'-m)}, \mathbf{I}_{L'}, \mathbf{0}_{L' \times (m-1)}]$. Importantly, by the definitions in (3) and (4), it holds that $\mathbf{Z}_{sel} = \mathbf{S}_{co}\text{diag}(\mathbf{d})\mathbf{S}_{co}^H + \sigma^2 \mathbf{I}_{L'}$, where $[\mathbf{S}_{co}]_{m,k} \triangleq v(\theta_k)^{m-1}$, for every $m \in \{1, 2, \ldots, L'\}$ and $k \in \{1, 2, \ldots, K\}$. That is, $\mathbf{Z}_{sel}$ coincides with the autocorrelation matrix of a length-L' ULA with elements at locations $\{0, 1, \ldots, L'-1\}d$. Therefore, standard MUSIC DoA estimation can be applied on $\mathbf{Z}_{sel}$, with the ability to resolve concurrently $K < L'$ DoAs. Specifically, let the columns of $\mathbf{U}_{sel} \in \mathbb{C}^{L' \times K}$ be the dominant left-hand singular vectors of $\mathbf{Z}_{sel}$, corresponding to its K highest singular values, calculated by means of singular value decomposition (SVD). Then, $\text{span}(\mathbf{U}_{sel}) = \text{span}(\mathbf{S}_{co})$. Defining $\mathbf{v}(\theta) \triangleq \left[1, v(\theta), \ldots, v(\theta)^{L'-1}\right]^\top$ for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, we can accurately decide $\theta \in \Theta \Leftrightarrow \left(\mathbf{I}_{L'} - \mathbf{U}_{sel}\mathbf{U}_{sel}^H\right)\mathbf{v}(\theta) = \mathbf{0}_{L'}$. Equivalently, we can identify the angles in Θ by the K local minima of the MUSIC spectrum

$$P_{MU}(\theta) = \left\| \left(\mathbf{I}_{L'} - \mathbf{U}_{sel}\mathbf{U}_{sel}^H\right)\mathbf{v}(\theta) \right\|^2. \tag{5}$$

Interestingly, among other works, [12] recently proposed replacing $\mathbf{E}_{sel}$ in (3) by the $L' \times (2L'-1)$ *averaging* sampling matrix $\mathbf{E}_{avg}$, where, for every $i \in \{1, 2, \ldots, 2L'-1\}$, $[\mathbf{E}_{avg}]_{:,i} \triangleq \frac{1}{|\mathcal{J}_{i-L'}|}\sum_{j \in \mathcal{J}_{i-L'}} \mathbf{e}_{j,L^2}$. By (2) and the fact that

$$[\mathbf{i}_L]_j = \begin{cases} 1, & j \in \mathcal{J}_0 \\ 0, & j \notin \mathcal{J}_0 \end{cases}, \tag{6}$$

it holds that, for any given $n \in \{-L'+1, \ldots, L'-1\}$, $[\mathbf{r}]_j = \mathbf{e}_{j,L^2}^\top \mathbf{r}$ takes the same value for every $j \in \mathcal{J}_n$. Thus,

$$\mathbf{r}_{avg} \triangleq \mathbf{E}_{avg}{}^\top \mathbf{r} = \mathbf{r}_{sel}. \tag{7}$$

Therefore, MUSIC DoA estimation in the form of (5) can be equivalently conducted by SVD of

$$\mathbf{Z}_{avg} \triangleq \mathbf{F}(\mathbf{I}_{L'} \otimes \mathbf{r}_{avg}) = \mathbf{Z}_{sel}. \tag{8}$$

However, in the case of practical interest where $\mathbf{R}_y$ is unknown and estimated by a finite collection of snapshots, the two autocorrelation sampling methods do not coincide and, therefore, neither do their corresponding MUSIC spectra. Specifically, $\mathbf{R}_y$ is practically estimated by a collection of Q snapshots, $\{\mathbf{y}_q\}_{q=1}^Q$, as $\hat{\mathbf{R}}_y \triangleq \frac{1}{Q}\sum_{q=1}^{Q} \mathbf{y}_q \mathbf{y}_q^H$. Accordingly, $\mathbf{r}$ is estimated by $\hat{\mathbf{r}} \triangleq \text{vec}(\hat{\mathbf{R}}_y) = \frac{1}{Q}\sum_{q=1}^{Q} \mathbf{y}_q^* \otimes \mathbf{y}_q$. Then, from (3) and (7), the *selection* and *averaging* sampled autocorrelation vectors are estimated by

$$\hat{\mathbf{r}}_{sel} \triangleq \mathbf{E}_{sel}^\top \hat{\mathbf{r}} \;\text{ and }\; \hat{\mathbf{r}}_{avg} \triangleq \mathbf{E}_{avg}^\top \hat{\mathbf{r}}, \tag{9}$$

respectively. Similarly, MUSIC can be applied using the K dominant left-hand singular vectors of either

$$\hat{\mathbf{Z}}_{sel} \triangleq \mathbf{F}(\mathbf{I}_{L'} \otimes \hat{\mathbf{r}}_{sel}), \text{ or } \hat{\mathbf{Z}}_{avg} \triangleq \mathbf{F}(\mathbf{I}_{L'} \otimes \hat{\mathbf{r}}_{avg}). \tag{10}$$

Indeed, both $\hat{\mathbf{Z}}_{sel}$ and $\hat{\mathbf{Z}}_{avg}$ have been employed before in the literature [3,12]. From (7) and the fact that $\hat{\mathbf{r}}$ is a Maximum-Likelihood, unbiased, consistent estimate of $\mathbf{r}$ [22], it naturally follows that as the sample support Q increases asymptotically, both $\hat{\mathbf{Z}}_{sel}$ and $\hat{\mathbf{Z}}_{avg}$ tend to $\mathbf{Z}_{sel}$. However, for finite values of sample-support Q, the two methods differ significantly. In [20], we proved that averaging-sampling outperforms selection-sampling in mean autocorrelation-matrix estimation performance, for any given Θ.

In Fig. 1, we offer a schematic illustration of the coprime processing procedure presented above. In the sequel, we extend the results in [20] and present a novel autocorrelation sampling approach that, for any value of Q and any system configuration parameters ($M, N, K, \mathbf{d}, \sigma$), offers the minimum-MSE (MMSE) estimate of the coarray autocorrelation matrix.

3. PROPOSED MMSE AUTOCORRELATION SAMPLING

In the second step of coprime-array processing (see Fig. 1), the receiver applies linear sampling in the autocorrelation entries of $\hat{\mathbf{r}}$, by means of a sampling matrix $\mathbf{E}$. Arguably, a preferred sampling matrix will attain consistently (i.e., for any possible configuration of DoA Θ) low squared-estimation error $\|\mathbf{E}^\top\hat{\mathbf{r}} - \mathbf{r}_{sel}\|_2$ –if this error is zero, then MUSIC will provide the exact DoAs in Θ. Following this criterion, the proposed receiver uses the snapshots $\{\mathbf{y}_q\}_{q=1}^Q$ to calculate the minimum mean-squared-error sampling matrix $\mathbf{E}$, assuming that, in the most general case and in lieu of any pertinent prior information, θ_k follows a uniform distribution independently of θ_l, for every $k \neq l$; i.e., $\theta_k \sim \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})$.

Before we proceed with deriving the MMSE sampling matrix, let us start with some helpful preliminary definitions. Defining $\mathbf{A} \triangleq \left[\mathbf{S}\text{diag}([\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_K}]^\top), \sigma\mathbf{I}_L\right]$, it holds $\mathbf{R}_y = \mathbf{A}\mathbf{A}^H$. Accordingly, $\mathbf{r}$ can be written as $\mathbf{r} = (\mathbf{A}^* \otimes \mathbf{A})\,\text{vec}\,(\mathbf{I}_{K+L})$. Defining $\mathbf{V} \triangleq \mathbf{A}^* \otimes \mathbf{A} \in \mathbb{C}^{L^2 \times (K+L)^2}$ and $\mathbf{i} \triangleq \text{vec}\,(\mathbf{I}_{K+L}) \in \mathbb{R}^{(K+L)^2}$, then it holds

$$\mathbf{r} = \mathbf{V}\mathbf{i}. \tag{11}$$

Naturally, $\mathbf{r}_{sel}$ and $\mathbf{r}_{avg}$ in (3) and (7), respectively, take the equivalent form $\mathbf{r}_{sel} = \mathbf{E}_{sel}^\top\mathbf{V}\mathbf{i} = \mathbf{E}_{avg}^\top\mathbf{V}\mathbf{i} = \mathbf{r}_{avg}$. Next, we notice that for the qth snapshot $\mathbf{y}_q \in \mathcal{CN}(\mathbf{0}, \mathbf{R}_y)$ in (1), there exists some $\mathbf{x}_q \sim \mathcal{CN}(\mathbf{0}_{K+L}, \mathbf{I}_{K+L})$ such that $\mathbf{y}_q = \mathbf{A}\mathbf{x}_q$. Moreover, $\mathbf{x}_q$ is independent from $\mathbf{x}_p$ for any pair (p, q) such that $p \neq q$. Defining $\mathbf{W} \triangleq \frac{1}{Q}\sum_{q=1}^Q \mathbf{x}_q\mathbf{x}_q^H$, the estimated received-signal autocorrelation matrix $\hat{\mathbf{R}}_y = \frac{1}{Q}\sum_{q=1}^Q \mathbf{y}_q\mathbf{y}_q^H$ becomes $\hat{\mathbf{R}}_y = \mathbf{A}\mathbf{W}\mathbf{A}^H$. Thus, it holds

$$\hat{\mathbf{r}} = \mathbf{V}\mathbf{w}, \tag{12}$$

where $\mathbf{w} \triangleq \text{vec}\,(\mathbf{W})$. As discussed above, in this work we propose to employ sampling matrix $\mathbf{E}_{prop}$ that solves

$$\min_{\mathbf{E} \in \mathbb{C}^{L^2 \times 2L'-1}} \underset{\Theta, \mathbf{w}}{\mathbb{E}} \left\{ \left\| \mathbf{E}^H\mathbf{V}\mathbf{w} - \mathbf{E}_{sel}^\top\mathbf{V}\mathbf{i} \right\|_2^2 \right\}. \tag{13}$$

That is, we propose to sample $\hat{\mathbf{r}}$ in a way such that the MSE of the estimated samples from the nominal[1] samples is minimized, under the generic assumption that $\theta_k \sim \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})$. Of course, the problem in (13) could facilitate any specific distribution for θ_k different than the uniform. In the sequel, for space economy, we present the solution of (13) in 4 steps.

Step 1: Define $\mathbf{G} \triangleq \mathbf{V}\mathbf{w}\mathbf{w}^H\mathbf{V}^H$ and $\mathbf{H} \triangleq \mathbf{V}\mathbf{w}\mathbf{i}^\top\mathbf{V}^H$. Then, (13) becomes

$$\min_{\mathbf{E}} \underset{\Theta, \mathbf{w}}{\mathbb{E}} \Big\{ \text{Tr}\left(\mathbf{E}^H\mathbf{G}\mathbf{E}\right) - 2\Re\Big\{ \text{Tr}\left(\mathbf{E}^H\mathbf{H}\mathbf{E}_{sel}\right) \Big\}\Big\}, \tag{14}$$

where $\Re\{\cdot\}$ extracts the real part of its argument and $\text{Tr}\,(\cdot)$ operator returns the sum of the diagonal entries of its argument.

Step 2: Define $\mathbf{G}_{\mathbb{E}} \triangleq \mathbb{E}_{\Theta, \mathbf{w}}\{\mathbf{G}\}$, and $\mathbf{H}_{\mathbb{E}} \triangleq \mathbb{E}_{\Theta, \mathbf{w}}\{\mathbf{H}\}$; then, it can be shown (omitted from this paper due to lack of space) that (14) becomes

$$\min_{\mathbf{E}} \text{Tr}\left(\mathbf{E}^H\mathbf{G}_{\mathbb{E}}\mathbf{E}\right) - 2\Re\Big\{ \text{Tr}\left(\mathbf{E}^H\mathbf{H}_{\mathbb{E}}\mathbf{E}_{sel}\right) \Big\}. \tag{15}$$

[1] Recall that by "nominal", we refer to the samples obtained by applying selection (or, averaging) sampling on the autocorrelations of $\mathbf{r}$, when $\mathbf{R}_y$ (and thus, $\mathbf{r}$), is exactly known to the receiver.

$\gamma_j^{(i,m)}$	Condition on (i, j, m)
$[\mathbf{d}]_{[\ddot{\mathbf{u}}]_j}[\mathbf{d}]_{[\dot{\mathbf{u}}]_j}\mathcal{B}(\ddot{\omega}_{m,i} + \dot{\omega}_{i,m})$	$[\dot{\mathbf{u}}]_j = [\ddot{\mathbf{u}}]_j \leq K$,
$[\mathbf{d}]_{[\ddot{\mathbf{u}}]_j}[\mathbf{d}]_{[\dot{\mathbf{u}}]_j}\mathcal{B}(\ddot{\omega}_{m,i})\mathcal{B}(\dot{\omega}_{i,m})$	$[\dot{\mathbf{u}}]_j, [\ddot{\mathbf{u}}]_j \leq K$; $[\ddot{\mathbf{u}}]_j \neq [\dot{\mathbf{u}}]_j$
$\sigma^2[\mathbf{d}]_{[\dot{\mathbf{u}}]_j}\mathcal{B}(\ddot{\omega}_{m,i})$	$[\dot{\mathbf{u}}]_j \leq K$; $[\ddot{\mathbf{u}}]_j - K = [\ddot{\mathbf{v}}]_i = [\ddot{\mathbf{v}}]_m$
$\sigma^2[\mathbf{d}]_{[\ddot{\mathbf{u}}]_j}\mathcal{B}(\dot{\omega}_{i,m})$	$[\ddot{\mathbf{u}}]_j \leq K$; $[\dot{\mathbf{u}}]_j - K = [\dot{\mathbf{v}}]_i = [\dot{\mathbf{v}}]_m$
σ^4	$[\dot{\mathbf{u}}]_j - K = [\dot{\mathbf{v}}]_i = [\dot{\mathbf{v}}]_m$; $[\ddot{\mathbf{u}}]_j - K = [\ddot{\mathbf{v}}]_i = [\ddot{\mathbf{v}}]_m$
0	otherwise

Auxiliary variables used in the above conditions:
$\dot{\omega}_{i,m} = [\dot{\mathbf{p}}]_i - [\dot{\mathbf{p}}]_m, \ddot{\omega}_{i,m} = [\ddot{\mathbf{p}}]_i - [\ddot{\mathbf{p}}]_m, \mathbf{s}_x = [1, 2, \ldots, x]^\top$,
$\dot{\mathbf{u}} = \mathbf{1}_{K+L} \otimes \mathbf{s}_{K+L}, \ddot{\mathbf{u}} = \mathbf{s}_{K+L} \otimes \mathbf{1}_{K+L}, \dot{\mathbf{v}} = \mathbf{1}_L \otimes \mathbf{s}_L, \ddot{\mathbf{v}} = \mathbf{s}_L \otimes \mathbf{1}_L$.

Table 1. Value of $\gamma_j^{(i,m)}$ based on conditions on (j, i, m).

Step 3: In this step, we calculate $\mathbf{H}_{\mathbb{E}}$ and $\mathbf{G}_{\mathbb{E}}$ in closed form. First, we observe that for any $x \in \mathbb{R}$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, it holds

$$\underset{\theta}{\mathbb{E}}\{v(\theta)^x\} = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left(-x\frac{j2\pi f_c}{c}\sin(\theta)\right)d\theta = J_0\left(x\frac{2\pi f_c}{c}\right), \tag{16}$$

where J_0 denotes the 0-th order Bessel function of the first kind [23]. For ease of notation, we define $\mathcal{B}(x) \triangleq J_0(x\frac{2\pi f_c}{c})$. Also, we define $\dot{\mathbf{p}} \triangleq \mathbf{1}_L \otimes \mathbf{p}$, $\ddot{\mathbf{p}} \triangleq \mathbf{p} \otimes \mathbf{1}_L$, and $\omega_i \triangleq [\dot{\mathbf{p}}]_i - [\ddot{\mathbf{p}}]_i$. Then, with few algebraic manipulations, we find that, for every $(i, m) \in \left\{1, 2, \ldots, L^2\right\}^2$, the (i, m)-th entry of $\mathbf{H}_{\mathbb{E}}$ is given by[2]

$$\begin{aligned} [\mathbf{H}_{\mathbb{E}}]_{i,m} &= \|\mathbf{d}\|_2^2\,\mathcal{B}\left(\omega_i - \omega_m\right) + \sigma^4\delta(\omega_i)\delta(\omega_m) & (17)\\ &+ \sigma^2\left(\mathbf{1}_K^\top\mathbf{d}\right)\Big(\delta(\omega_i)\mathcal{B}(-\omega_m) + \mathcal{B}(\omega_i)\delta(\omega_m)\Big) & (18)\\ &+ \mathcal{B}(\omega_i)\mathcal{B}(-\omega_m)\left((\mathbf{1}_K^\top\mathbf{d})^2 - \|\mathbf{d}\|_2^2\right). & (19) \end{aligned}$$

Clearly, (19) is easily calculable at the receiver, if the transmission powers in $\mathbf{d}$ are known (an assumption made for simplicity in this paper). Next, if $\mathbf{P} \triangleq \mathbb{E}_{\Theta}\{\mathbf{V}\mathbf{V}^H\}$, we can show that

$$\mathbf{G}_{\mathbb{E}} = \mathbf{H}_{\mathbb{E}} + \frac{1}{Q}\mathbf{P}. \tag{20}$$

For $\mathbf{P}$ it holds that $[\mathbf{P}]_{i,m} = \sum_{j=1}^{(k+L)^2}\gamma_j^{(i,m)}$ for any $(i, j) \in \{1, 2, \ldots, L^2\}^2$. The value $\gamma_j^{(i,m)}$ for every triplet (i, j, m) is offered in Table 1. The proofs of (19), (20) and Table 1 are omitted from this paper due to lack of space.

Step 4: In view of (19) and (20), setting to $\mathbf{0}$ the derivate of the objective function in (15) with respect to $\mathbf{E}$ yields

$$\left(\mathbf{H}_{\mathbb{E}} + \frac{1}{Q}\mathbf{P}\right)\mathbf{E}_{prop} = \mathbf{H}_{\mathbb{E}}\mathbf{E}_{sel}. \tag{21}$$

Our numerical studies have shown that $\mathbf{P}$ defined above is of full column rank. Therefore, for any finite value of Q, $\left(\mathbf{H}_{\mathbb{E}} + \frac{1}{Q}\mathbf{P}\right)$ is invertible and the proposed MMSE sampling matrix is given by

$$\mathbf{E}_{prop} = \left(\mathbf{H}_{\mathbb{E}} + \frac{1}{Q}\mathbf{P}\right)^{-1}\mathbf{H}_{\mathbb{E}}\mathbf{E}_{sel}. \tag{22}$$

If Q and $\mathbf{P}$ are such that $\frac{1}{Q}\mathbf{P}$ is (close to) singular then $\left(\mathbf{H}_{\mathbb{E}} + \frac{1}{Q}\mathbf{P}\right)$ is not invertible any more and (22) cannot be formed. Instead, an iterative process such as Gradient descent [24] or Newton's method

[2] Notice that $\mathcal{B}(x) = \mathcal{B}(-x)$ and $\delta(x) = \delta(-x)$ for any $x \in \mathbb{R}$, implying that $[\mathbf{H}_{\mathbb{E}}]_{i,m} = [\mathbf{H}_{\mathbb{E}}]_{m,i}$ which, combined with the fact that $\mathbf{H}_{\mathbb{E}}$ is by definition a real-valued matrix, implies that $\mathbf{H}_{\mathbb{E}} = \mathbf{H}_{\mathbb{E}}^\top = \mathbf{H}_{\mathbb{E}}^H$.

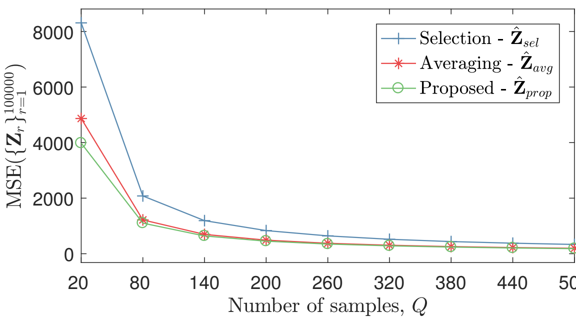

Fig. 2. MSE in estimating $\mathbf{Z}_{sel}$ versus sample support Q. At every realization, $\{\theta_k\}_{k=1}^5$ are drawn from $\mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2}]$.

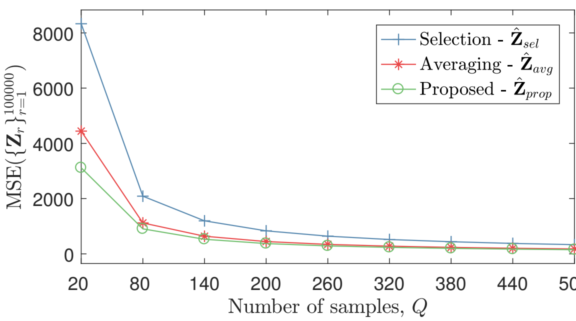

Fig. 3. MSE in estimating $\mathbf{Z}_{sel}$ versus sample support Q. Fixed $\theta_1 = -63°, \theta_2 = -45°, \theta_3 = -20°, \theta_4 = 43°, \theta_5 = 76°$.

can be employed for the acquisition of $\mathbf{E}_{prop}$ and the minimization in (15).

Given the proposed autocorrelation sampling matrix $\mathbf{E}_{prop}$, $\mathbf{r}_{sel}$ in (3) is accordingly estimated by $\hat{\mathbf{r}}_{prop} = \mathbf{E}_{prop}^\top \hat{\mathbf{r}}$. In turn, the coarray autocorrelation matrix $\mathbf{Z}_{sel}$ is estimated by

$$\hat{\mathbf{Z}}_{prop} = \mathbf{F}\left(\mathbf{I}_{L'} \otimes \hat{\mathbf{r}}_{prop}\right). \quad (23)$$

Then, similar to the state of the art, a MUSIC spectrum is formed by the K left-singular vectors of $\hat{\mathbf{Z}}_{prop}$ and, by the peaks of this spectrum, the receiver estimates the K source DoAs.

4. NUMERICAL RESULTS

Consider an $(M, N) = (2, 3)$ coprime array with $L = 6$ antenna elements. Signals from $K = 5$ sources impinge on the array with equal power $\alpha^2 = 10$ dB. Noise variance σ^2 is set to 0 dB. For each source $k \in \{1, 2, \ldots, 5\}$ it then holds that the signal-to-noise-ratio (SNR) is 10 dB.

We commence our studies by evaluating the performance of the proposed sampling matrix in terms of mean-squared-error (MSE) in estimating $\mathbf{Z}_{sel}$. That is, we let Q vary in $\{20, 80, \ldots, 500\}$ and draw 10^5 independent realizations of $\{\theta_k\}_{k=1}^5$ for $\theta_k \sim \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2}]$, and noise. At every realization r, we compute $\hat{\mathbf{Z}}_{sel,r}, \hat{\mathbf{Z}}_{avg,r}$, and $\hat{\mathbf{Z}}_{prop,r}$ as in (10) and (23) respectively, and measure the approximate MSE $\mathrm{MSE}(\{\mathbf{Z}_r\}_{r=1}^{10^5}) \triangleq \frac{1}{10^5}\sum_{r=1}^{10^5} \|\mathbf{Z}_{sel,r} - \mathbf{Z}_r\|_F^2$ for every $\mathbf{Z}_r \in \{\hat{\mathbf{Z}}_{sel,r}, \hat{\mathbf{Z}}_{avg,r}, \hat{\mathbf{Z}}_{prop,r}\}$. In Fig. 2, we plot $\mathrm{MSE}(\{\mathbf{Z}_r\}_{r=1}^{10^5})$ versus sample-support Q. Expectedly, the proposed sampling approach attains the lowest MSE across all values of Q, as it was designed to minimize the MSE in estimating $\mathbf{Z}_{sel}$. As Q increases, the gap between any two curves is diminishing.

As a second study, we evaluate $\mathrm{MSE}(\{\mathbf{Z}_r\}_{r=1}^{10^5})$ of the proposed

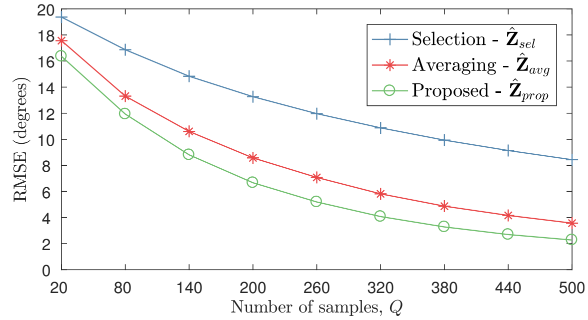

Fig. 4. Root-mean-squared-error versus sample support Q. Fixed $\theta_1 = -63°, \theta_2 = -45°, \theta_3 = -20°, \theta_4 = 43°, \theta_5 = 76°$.

coprime array autocorrelation method for a given/fixed set of DoAs. That is, we fix $\theta_1 = -63°$, $\theta_2 = -45°$, $\theta_3 = -20°$, $\theta_4 = 43°$, $\theta_5 = 76°$ across all 10^5 realizations. The rest of the parameters remain the same as in the study of Fig. 2 above. $\mathrm{MSE}(\{\mathbf{Z}_r\}_{r=1}^{10^5})$ is shown in Fig. 3. Interestingly, the MSE curves look very similar to those of Fig. 2. Also, at each realization we apply MUSIC on $\mathbf{Z}_r \in \{\hat{\mathbf{Z}}_{sel,r}, \hat{\mathbf{Z}}_{avg,r}, \hat{\mathbf{Z}}_{prop,r}\}$ and measure the root-mean-squared error (RMSE) defined as $\mathrm{RMSE} \triangleq \sqrt{\frac{1}{5}\sum_{k=1}^5 \frac{1}{10^5}\sum_{r=1}^{10^5}\left(\theta_k - \hat{\theta}_{k,r}\right)^2}$. In Fig. 4 we plot the calculated RMSE versus sample support Q. We observe that the proposed method attains, across the board, superior DoA estimation performance compared to both its counterparts. Specifically, when Q varies from 20 to 260, the proposed method exhibits 10%-20% lower RMSE than averaging sampling; for $Q > 260$ this performance gap further increases all the way to 45%. Moreover, the proposed method outperforms remarkably the widely employed selection sampling method, exhibiting up to 75% lower RMSE for $Q = 500$.

5. CONCLUSION AND FUTURE WORK

In this paper, we presented for the first time the autocorrelation sampling matrix that attains minimum MSE in the estimation of the virtual-coarray autocorrelation matrix $\mathbf{Z}_{sel}$, for uniformly distributed source DoAs. The proposed sampling matrix was shown to attain superior MSE (estimation of $\mathbf{Z}_{sel}$) and RMSE (estimation of DoAs in Θ) performance, compared to the selection-sampling and averaging-sampling counterparts. Clearly, the proposed MMSE design scheme can be applied for different prior DoA distributions (other than $\mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})$). In view of the presented findings, in the immediate future, we plan to address the following:

(i) Derive in closed form the MSE attained by $\mathbf{E}_{prop}$ for uniformly distributed DoAs.

(ii) Derive in closed form the MSE attained by $\mathbf{E}_{prop}$ for any given selection of DoAs in Θ.

(iii) Assuming the powers in $\mathbf{d}$ to be unknown to the receiver, reformulate the MSE minimization over jointly random Θ and $\mathbf{d}$.

The presented work initiates a new line of research for optimized autocorrelation-sampling in coprime arrays and is expected to unlock significant performance enhancement in DoA estimation with coprime arrays.

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