ON THE MODULUS OF CONTINUITY FOR NOISY POSITIVE SUPER-RESOLUTION

Heng Qiao and Piya Pal

Dept. of Electrical and Computer Engineering,
University of California, San Diego, USA
E-mail: h1qiao@eng.ucsd.edu, pipal@eng.ucsd.edu

ABSTRACT

This paper considers the problem of super-resolution with positive constraints. By utilizing the concept of Modulus of Continuity (MC), we propose a unified framework for analyzing the robustness of super-resolution reconstruction in presence of noise, which is algorithm-independent and emphasizes the role of signal structures. In contrast to earlier works, we show that incorporation of positive constraints improves the scaling factor of MC and provides tighter upper bound on the estimation error of any algorithm that exploits such structure. The unified framework is further applied to analyze convex algorithms for positive super-resolution, and the theoretical results are validated by numerical experiments. [1]

***Index Terms*—** Super-resolution, Modulus-of-Continuity, Positive Constraints, Noise Amplification, Universal Bounds.

1. INTRODUCTION

The problem of super-resolution is central to many imaging applications such as astronomy [1], medical imaging [2], microscopy [3] and radar [4]. The fundamental need for studying super-resolution in these fields arises due to the fact that the resolution of the captured image is always limited by the physical measurement process. For example, in microscopy [5, 6], the ability to identify closely located molecules is restricted by diffraction limit of the optical system. Mathematically, the point sources are blurred by a kernel/point-spread-function (PSF) [7] which implies that only low-frequency components of the underlying signal are retained in the measurements [8, 9].

The theory of super-resolution with noisy measurements was studied in the pioneering work by Donoho [9] and further developed in recent works [8, 10] where total-variation (TV) and l_1 norm based convex algorithms are used for super-resolution reconstruction. The key contribution in [8, 10] is an explicit construction of a dual polynomial (based on the Fejer kernel) whose properties can be exploited to analyze the performance of noisy super-resolution in line spectrum estimation [11] and low-rank Toeplitz covariance estimation [12].

Recently, the role of positive constraints on super resolution was studied in [14] using a new notion of Rayleigh regularity. The positive constraint is straightforward in many practical scenarios. For example, in microscopy, the measurement is based on the number of photons collected and is a positive quantity. Similarly, in magnetic resonance imaging (MRI), the signal is known to be sparse and positive [13]. Using the same dual polynomial from [8, 10], the authors in [14] propose an l_1 minimization framework with positivity constraints. Given a sparse non-negative signal $\mathbf{x} \in \mathbb{R}^N$, if the measurements retain only the $n < N$ smallest DFT coefficients of $\mathbf{x}$, then the result in [14] shows that the estimation error scales as SRF^2, where $SRF = \frac{N}{n}$ is known as the super-resolution-factor. In another recent work [7], the author considers the problem of robust recovery of positive streams of spikes. Instead of constructing dual polynomials, the author imposes strong structural requirements on the admissible blurring kernel. It should be noted that most existing analysis of noisy super-resolution requires the true signal to satisfy certain kinds of separation condition [8],[14].

In this paper, we consider the same problem setting as in [14] where the goal is to reconstruct a *sparse non-negative* discrete signal from low-frequency measurements. In contrast to previous works, our goal is to perform a unified analysis of positive super-resolution independent of particular algorithms. To achieve this, we revisit the concept of Modulus of Continuity (MC) [9, 15] which essentially provides an upper bound on the error of any algorithm, simply by leveraging the structure of signals. We study the explicit role of positivity on the Modulus of Continuity and our results show that the scaling factor of MC can be improved from $O(N^3)$ [9, 15] to $O(N^{2.5})$ where N is the dimension of the underlying signal. This improvement is due to positive constraints imposed on the desired signal, which is not considered in [9]. Finally, we apply our new bound on MC to analyze several convex algorithms for positive super-resolution, and demonstrate our claims via numerical experiments.

2. PROBLEM FORMULATION

In this paper, we consider following discrete measurement model

$$\mathbf{y} = \mathbf{Q}\mathbf{x}^\star + \mathbf{w} \qquad \mathbf{x}^\star \geqslant \mathbf{0} \tag{1}$$

where $\mathbf{x}^\star \in \mathbb{R}^N$ is sparse with positive non-zero entries, and $\mathbf{w}$ is the measurement noise. In the context of super-resolution, the measurements $\mathbf{y}$ only retain low-frequency

[1] Work supported in parts by NSF CPS Synergy 1544798, and the University of California, San Diego.

components of the signal $\mathbf{x}^\star$. Following the notations in [14], $\mathbf{Q}$ is defined by

$$\mathbf{Q} = \mathbf{F}_N^H \boldsymbol{\Lambda}_n \mathbf{F}_N \tag{2}$$

where $\mathbf{F}_N \in \mathbb{C}^N$ is given by $[\mathbf{F}_N]_{k,l} = \frac{1}{\sqrt{N}} e^{-j2\pi kl/N}$, $-N/2+1 \leqslant k \leqslant N/2$, $0 \leqslant l \leqslant N-1$ and $\boldsymbol{\Lambda}_n = \text{diag}([\lambda_{-N/2+1}, \cdots, \lambda_{N/2}])$ with

$$\lambda_k = \begin{cases} 1, & k = -\frac{n-1}{2}, \cdots, \frac{n-1}{2} \\ 0, & \text{otherwise} \end{cases}$$

We assume N is even and n is odd. Intuitively, $\mathbf{Q}$ only collects n low-frequency coefficients of the DFT of $\mathbf{x}^\star$. This model popularly arises in discrete super-resolution problems with positive constraints [14, 7, 16]. Let $\mathbf{x}^\#$ be any estimate of $\mathbf{x}^\star$. Our goal in this paper is to address following question:

(Q): How to obtain a universal upper bound on the estimation error $\|\mathbf{x}^\star - \mathbf{x}^\#\|_2$ in terms of $\|\mathbf{w}\|_2$, that will be obeyed by any algorithm? Can we improve this bound by constraining $\mathbf{x}^\#$ to be non-negative?

2.1. Modulus of Continuity and Universal Bounds

In order to address (Q), the authors in [9, 14] have used the following notion of Modulus of Continuity (MC):

Definition 1. *Let $\mathcal{X}^\star, \mathcal{X}^\# \subset \mathbb{R}^N$ be classes of signals, $\|.\|_p$ be the $p-$norm, and $\mathbf{Q}$ be a linear operator. Then, the modulus of continuity is defined as*

$$MC(\mathbf{Q}, \mathcal{X}^\star, \mathcal{X}^\#, p) = \sup_{\substack{\mathbf{x}_1 \in \mathcal{X}^\#, \mathbf{x}_2 \in \mathcal{X}^\star \\ \mathbf{x}_1 \neq \mathbf{x}_2}} \frac{\|\mathbf{x}_1 - \mathbf{x}_2\|_p}{\|\mathbf{Q}(\mathbf{x}_1 - \mathbf{x}_2)\|_p} \tag{3}$$

The sets $\mathcal{X}^\star, \mathcal{X}^\#$ capture desired structures of the signal of interest, such as sparsity, positivity etc. In estimation problems, $\mathcal{X}^\star$ often represents a class to which the true signal belongs, and $\mathcal{X}^\#$ represents the feasible set to which the estimator belongs. In most cases, either $\mathcal{X}^\# = \mathcal{X}^\star$ or $\mathcal{X}^\star \subset \mathcal{X}^\#$. In order to see how MC fundamentally controls the estimation error of any algorithm, we first need to define admissible estimates as follows:

Definition 2. *Consider the measurement model (1) with $\|\mathbf{w}\|_p \leqslant \epsilon$ and $\mathbf{x}^\star \in \mathcal{X}^\star$. Any estimate $\mathbf{x}^\#$ of $\mathbf{x}^\star$ is said to be admissible if*

$$\mathbf{x}^\# \in \mathcal{X}^\#, \quad \|\mathbf{y} - \mathbf{Q}\mathbf{x}^\#\|_p \leqslant \epsilon$$

The quantity $MC(\mathbf{Q}, \mathcal{X}^\star, \mathcal{X}^\#, p)$ then provides an upper bound on the error of any admissible estimate as follows:

Lemma 1. *Consider the model (1) with $\|\mathbf{w}\|_p \leqslant \epsilon$, and suppose $\mathbf{x}^\#$ is any admissible estimator of $\mathbf{x}^\star$. Then,*

$$\|\mathbf{x}^\star - \mathbf{x}^\#\|_p \leqslant 2\epsilon MC(\mathbf{Q}, \mathcal{X}^\star, \mathcal{X}^\#, p)$$

Remark 2.1. The Modulus of Continuity therefore determines a universal upper bound on the estimation error $\|\mathbf{x}^\star - \mathbf{x}^\#\|_p$. The value of $MC(\mathbf{Q}, \mathcal{X}^\star, \mathcal{X}^\#, p)$ is algorithm-independent and only depends on the choices of $\mathcal{X}^\star, \mathcal{X}^\#, \mathbf{Q}$ and the choice of the norm. However, exact computation of $MC(\mathbf{Q}, \mathcal{X}^\star, \mathcal{X}^\#, p)$ is a challenging task, which was first studied in the pioneering work by [9] in the context of super-resolution reconstruction of spike signals from low-frequency measurements and further developed in recent work on discrete positive super-resolution [14]. We will review this result by introducing the following class of signals that obey a separation condition [14].

Definition 3. *(Set of Signals Obeying Separation Condition) Given N and n, the set Δ_{sep} is given by*

$$\Delta_{sep} \triangleq \{\mathbf{x} \in \mathbb{C}^N \mid \rho(\frac{k}{N}, \frac{l}{N}) \geqslant \frac{4}{n-1} \quad \forall k \neq l \in supp(\mathbf{x})\}$$

where $\rho(\cdot,\cdot)$ is a wrap-around distance function [8] such that for $\forall \mu_1, \mu_2 \in [0,1]$

$$\rho(\mu_1, \mu_2) \triangleq \min(|\mu_1 - \mu_2|, |\mu_1 + 1 - \mu_2|, |\mu_2 + 1 - \mu_1|)$$

Additionally, the set Δ^+_{sep} is given by

$$\Delta^+_{sep} \triangleq \{\mathbf{x} \in \Delta_{sep}, \mathbf{x} \geqslant \mathbf{0}\}$$

If we assume $\mathcal{X}^\star = \mathcal{X}^\# = \Delta_{\text{sep}}$, then the following result provides an explicit upper bound on $MC(\mathbf{Q}, \mathcal{X}^\star, \mathcal{X}^\#, 2)$ in terms of n and N:

Lemma 2. *[9, 14] Let $\mathcal{X}^\star = \mathcal{X}^\# = \Delta_{sep}$, and let $\mathbf{Q}$ be given by (2). Then,*

$$MC(\mathbf{Q}, \Delta_{sep}, \Delta_{sep}, 2) \leqslant C(n)N^3 \tag{4}$$

where $C(n)$ is a function of only n (independent of N) implicitly defined in [9].

Given the measurement model (1), the goal of super-resolution is to reconstruct the N DFT coefficiens of sparse $\mathbf{x}^\star$ (or equivalently, the signal $\mathbf{x}^\star$) from observations that only preserve the lowest $n < N$ frequency components. If we assume that both the true signal and its estimate $\mathbf{x}^\#$ belong to Δ_{sep} (i.e., they satisfy the separation condition), then Lemma 2 and Lemma 1 show that given n, the estimation error grows as $O(N^3)$.

However, in practice, it is difficult to develop algorithms that can actually constrain $\mathbf{x}^\#$ to belong to Δ_{sep}. [2] In recent work [14], the authors developed an l_1 minimization framework for super resolution reconstruction, where they only constrained the estimate $\mathbf{x}^\#$ to be positive and developed algorithm-specific error bound (with respect to l_1 norm of the error). Inspired by this work, we will develop a new bound for $MC(\mathbf{Q}, \mathcal{X}^\star, \mathcal{X}^\#, 2)$ where $\mathcal{X}^\star$ imposes minimum separation as well as positivity on the true signal, whereas $\mathcal{X}^\#$ only imposes a positive constraint on $\mathbf{x}^\#$. Our analysis will show that this bound grows as $O(N^{2.5})$ and is therefore tighter than (4).

[2] Partly because Δ_{sep} is a non-convex set

3. NEW BOUND ON MODULUS OF CONTINUITY FOR POSITIVE SUPER-RESOLUTION AND APPLICATIONS

Our new upper bound for the modulus of continuity is based on a recent result from [12] for continuous Direction-of-Arrival estimation. For any vector $\mathbf{x}$, let $\text{Toep}(\mathbf{x})$ denote the Hermitian Toeplitz matrix with $\mathbf{x}$ as the first column. Consider $\mathbf{r}^\star \in \mathbb{C}^K$ such that

$$\text{Toep}(\mathbf{r}^\star) = \sum_{i=1}^{D} \mathbf{a}(\theta_i)\mathbf{a}^H(\theta_i)d_i \tag{5}$$

where $D < K$, $d_i > 0$, $\theta_i \in [0, 1]$ and

$$\mathbf{a}(\theta_i) = [1, e^{-j2\pi\theta_i}, \cdots, e^{-j2\pi(K-1)\theta_i}]^T$$

It can be easily seen that $\text{Toep}(\mathbf{r}^\star) \succeq \mathbf{0}$. We invoke the following result from [12]:

Theorem 1. *[12] Let $\mathbf{r}^\star$ be given by (5), and $\mathbf{r}^\# \in \mathbb{C}^K$ be any vector such that $\text{Toep}(\mathbf{r}^\#) \succeq \mathbf{0}$. If the frequencies $\{\theta_i\}_{i=1}^D$ satisfy the separation condition*

$$\min_{l \neq m} \rho(\theta_l, \theta_m) > \frac{2}{h}$$

and $h > 128$, then there exist positive constants $\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4$ such that for $h \leqslant k < K$

$$|r_k^\star - r_k^\#| \tag{6}$$
$$\leqslant \left(\bar{c}_1 + \frac{\bar{c}_2\pi k}{h} + \frac{\bar{c}_3\pi^2 k^2}{h^2}\right)\left(\frac{\bar{c}_4 D}{\sqrt{h}} + 1\right)\|\mathbf{r}_h^\star - \mathbf{r}_h^\#\|_2$$

where $\mathbf{r}_h^\star = [r_0^\star, \cdots, r_{h-1}^\star]^T, \mathbf{r}_h^\# = [r_0^\#, \cdots, r_{h-1}^\#]^T$.

Equipped with Theorem 1, the main result of this paper is given by

Theorem 2. *Let $\mathcal{X}^\star, \mathcal{X}^\#$ be chosen as*

$$\mathcal{X}^\star = \Delta_{sep}^+, \quad \mathcal{X}^\# = \mathbb{R}_+^N \triangleq \left\{\mathbf{x} \in \mathbb{R}^N : \mathbf{x} \geqslant \mathbf{0}\right\}$$

Furthermore, let the matrix $\mathbf{Q}$ be given by (2). If $n > 256$, the Modulus of Continuity is upper bounded as

$$MC(\mathbf{Q}, \mathcal{X}^\star, \mathcal{X}^\#, 2) \leqslant \sqrt{2 + (N - n + 1)\beta(n, N)} \tag{7}$$

where

$$\beta(n, N) \triangleq \left(\bar{c}_1 + \frac{\bar{c}_2\pi N}{n+1} + \frac{\bar{c}_3\pi^2 N^2}{(n+1)^2}\right)^2 \left(\bar{c}_4\sqrt{\frac{n+1}{8}} + 1\right)^2$$

Here $\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4$ are the same constants as in Theorem 1.

Proof. For $\forall \mathbf{x}^\star \in \mathcal{X}^\star, \forall \mathbf{x}^\# \in \mathcal{X}^\#$, $\mathbf{x}^\star \neq \mathbf{x}^\#$, $\mathbf{F}_N\mathbf{x}^\star$ and $\mathbf{F}_N\mathbf{x}^\#$ are symmetric and we can define $\mathbf{r}^\star, \mathbf{r}^\# \in \mathbb{C}^{\frac{N}{2}+1}$ as

$$r_i^\star = [\mathbf{F}_N\mathbf{x}^\star]_{i+\frac{N}{2}-1}, \quad r_i^\# = [\mathbf{F}_N\mathbf{x}^\#]_{i+\frac{N}{2}-1}, \quad 0 \leqslant i \leqslant \frac{N}{2}$$

Since $\mathbf{x}^\star, \mathbf{x}^\# \geqslant \mathbf{0}$, it follows that[12]

$$\text{Toep}(\mathbf{r}^\star) \succeq \mathbf{0} \qquad \text{Toep}(\mathbf{r}^\#) \succeq \mathbf{0}$$

Moreover $\mathbf{r}^\star$ has the form $\mathbf{r}^\star = \sum_{k=1}^{\|\mathbf{x}^\star\|_0} \mathbf{a}(\theta_k)\mathbf{a}^H(\theta_k)x_k^\star$ where $\theta_k = k/N, k \in \text{supp}(\mathbf{x}^\star)$. Since $\mathbf{x}^\star \in \Delta_{\text{sep}}$, this implies $\rho(\theta_k, \theta_l) \geqslant \frac{4}{n-1}$. Hence, Theorem 1 applies (by replacing K, h, and D with $\frac{N}{2}+1$, $\frac{n+1}{2}$, and $\|\mathbf{x}^\star\|_0$ respectively)

$$\sum_{i=\frac{n+1}{2}}^{\frac{N}{2}} (r_i^\star - r_i^\#)^2 \leqslant \frac{N-n+1}{2}\|\mathbf{r}_{\frac{n+1}{2}}^\star - \mathbf{r}_{\frac{n+1}{2}}^\#\|_2^2 \cdot$$
$$\left(\bar{c}_1 + \frac{\bar{c}_2\pi N}{n+1} + \frac{\bar{c}_3\pi^2 N^2}{(n+1)^2}\right)^2 \left(\bar{c}_4\sqrt{\frac{n+1}{8}} + 1\right)^2$$

where we use the fact that $\|\mathbf{x}^\star\|_0 \leqslant \frac{n+1}{4}$ owing to the separation condition. Also note that

$$\|\mathbf{F}_N(\mathbf{x}^\star - \mathbf{x}^\#)\|_2^2 \leqslant 2\|\mathbf{r}^\star - \mathbf{r}^\#\|_2^2$$
$$\leqslant 2\|\mathbf{r}_{\frac{n+1}{2}}^\star - \mathbf{r}_{\frac{n+1}{2}}^\#\|_2^2\left(1 + \frac{N-n+1}{2}\beta(n, N)\right)$$

Moreover,

$$\|\mathbf{Q}(\mathbf{x}^\star - \mathbf{x}^\#)\|_2^2 = \|\mathbf{\Lambda}_n\mathbf{F}_N(\mathbf{x}^\star - \mathbf{x}^\#)\|_2^2$$
$$= 2\|\mathbf{r}_{\frac{n+1}{2}}^\star - \mathbf{r}_{\frac{n+1}{2}}^\#\|_2^2 - (r_0^\star - r_0^\#)^2 \geqslant \|\mathbf{r}_{\frac{n+1}{2}}^\star - \mathbf{r}_{\frac{n+1}{2}}^\#\|_2^2$$

This implies

$$\frac{\|\mathbf{x}^\star - \mathbf{x}^\#\|_2^2}{\|\mathbf{Q}(\mathbf{x}^\star - \mathbf{x}^\#)\|_2^2} = \frac{\|\mathbf{F}_N(\mathbf{x}^\star - \mathbf{x}^\#)\|_2^2}{\|\mathbf{Q}(\mathbf{x}^\star - \mathbf{x}^\#)\|_2^2} \leqslant \frac{\|\mathbf{F}_N(\mathbf{x}^\star - \mathbf{x}^\#)\|_2^2}{\|\mathbf{r}_{\frac{n+1}{2}}^\star - \mathbf{r}_{\frac{n+1}{2}}^\#\|_2^2}$$
$$\leqslant 2\left(1 + \frac{N-n+1}{2}\beta(n, N)\right)$$

thereby proving the theorem $\square$

3.1. Comparison with Lemma 2

Our result in Theorem 2 significantly differs from the result in Lemma 2 in the following ways:

- In Lemma 2, the signal classes $\mathcal{X}^\star, \mathcal{X}^\#$ are identical, while in Theorem 2, they are different. In particular, $\mathcal{X}^\star$ (the set to which the true signal belongs) contains all non-negative vectors that satisfy separation condition, whereas $\mathcal{X}^\#$ (the set to which the estimate belongs) simply contains all non-negative vectors. Such distinction of $\mathcal{X}^\star, \mathcal{X}^\#$ enables better analysis of practical estimation algorithms since it is difficult for an algorithm to actually impose the constraint $\mathbf{x}^\# \in \Delta_{\text{sep}}$.

- The upper bound on MC in Lemma 2 is given by

$$MC(\mathbf{Q}, \Delta_{\text{sep}}, \Delta_{\text{sep}}, 2) \leqslant C(n)N^3 \tag{8}$$

On the other hand, Theorem 2 shows that

$$MC(\mathbf{Q}, \Delta^{+}_{\text{sep}}, \mathbb{R}^{N}_{+}, 2) \lesssim O(\frac{\sqrt{N-n}N^2}{n^{1.5}}) \tag{9}$$

When N is large (and n is fixed), Lemma 2 shows that the upper bounded is $O(N^3)$ while Theorem 2 suggests that this can be tightened to $O(N^{2.5})$. This 0.5 improvement in the exponent with respect to N is mainly due to the introduction of positive constraints. To the best of our knowledge, this improvement is the first result of its kind.

3.2. Unified Analysis of Specific Algorithms

In Lemma 1, we have shown that $MC(\mathbf{Q}, \mathcal{X}^{\star}, \mathcal{X}^{\#}, 2)$ provides an upper bound on the estimation error of any algorithm that produces an admissible estimate. To illustrate this further, we study the following three convex problems to estimate $\mathbf{x}^{\star}$ from the measurement model (1).

$$\begin{aligned}
&\text{find} \quad \mathbf{z} \geqslant \mathbf{0} \quad s.t \quad \|\mathbf{y} - \mathbf{Q}\mathbf{z}\|_2 \leqslant \varepsilon \qquad (\textbf{Algo-F})\\
&\min \|\mathbf{z}\|_1 \quad s.t. \quad \|\mathbf{y} - \mathbf{Q}\mathbf{z}\|_2 \leqslant \varepsilon, \mathbf{z} \geqslant \mathbf{0} \quad (\textbf{Algo-}l_1)\\
&\min \|\mathbf{y} - \mathbf{Q}\mathbf{z}\|_2 \quad s.t \quad \mathbf{z} \geqslant \mathbf{0} \qquad (\textbf{Algo-}l_2)
\end{aligned}$$

Applying Theorem 2 and Lemma 1, we have the following unified analysis of the preceding algorithms

Corollary 1. *Consider the noisy measurement model (1) with* $\|\mathbf{w}\|_2 \leqslant \epsilon$. *Suppose the true signal satisfies* $\mathbf{x}^{\star} \in \Delta^{+}_{sep}$. *Let* $\mathbf{x}^{\#}_{F}, \mathbf{x}^{\#}_{1}, \mathbf{x}^{\#}_{2}$ *be the optimal solutions of* (***Algo-F***), (***Algo-***l_1), *and* (***Algo-***l_2) *respectively. If* $n > 256$, *we have*

$$\begin{aligned}
&\max\{\|\mathbf{x}^{\star} - \mathbf{x}^{\#}_{F}\|_2, \|\mathbf{x}^{\star} - \mathbf{x}^{\#}_{1}\|_2, \|\mathbf{x}^{\star} - \mathbf{x}^{\#}_{2}\|_2\}\\
&\leqslant 2\epsilon\sqrt{2 + (N - n + 1)\beta(n, N)}\cdot
\end{aligned} \tag{10}$$

4. NUMERICAL RESULTS

In this section, we present numerical results to demonstrate our theoretical claims. We define the normalized empirical root mean-squared error (NRMSE) of any estimator, averaged over K Monte Carlo runs

$$NRMSE \triangleq \sqrt{\frac{1}{K}\sum_{k=1}^{K} \frac{\|\mathbf{x}^{\star}_{k} - \mathbf{x}^{\#}_{k}\|_2^2}{\|\mathbf{x}^{\star}_{k}\|_2^2}}$$

where $\mathbf{x}^{\star}_{k}$ and $\mathbf{x}^{\#}_{k}$ respectively denote the true signal and its estimate. The support of $\mathbf{x}^{\star}$ is generated at random while satisfying the separation condition. The non-zero entries of $\mathbf{x}^{\star}$ are uniform random variables in the range $(0.01, 10)$. The noise $\mathbf{w}$ is assumed to be zero-mean i.i.d Gaussian with standard deviation $\sigma_{\mathbf{w}}$. For each run, the parameter ε is chosen such that true solution is feasible. The Signal-to-Noise Ratio (SNR) is defined as

$$SNR \triangleq 10 \log \frac{\mathbb{E}\|\mathbf{x}^{\star}\|_2^2}{\mathbb{E}\|\mathbf{w}\|_2^2}$$

In Fig. 1, we compare the NRMSE of the three algorithms proposed in Sec. 3.2, as a function of SNR and $SRF = N/n$ respectively. It can be seen that when SNR is large enough, the NRMSE of different algorithms are almost orderwise identical. In another word, in high SNR regime, the underlying signal structures, rather than particular algorithms, uniformly determine the estimation error. Fig. 1 also shows that NRMSE is an increasing function of the super-resolution factor $\frac{N}{n}$.

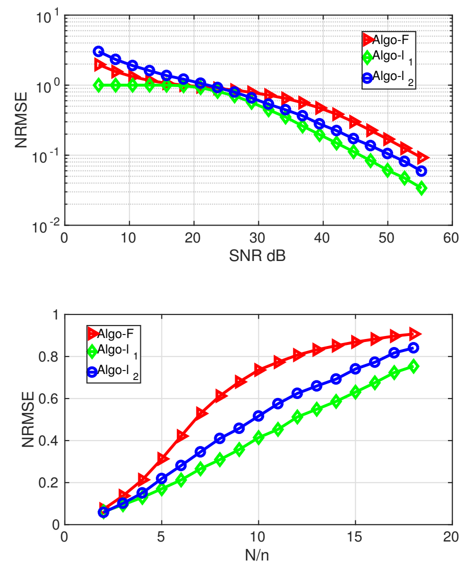

Fig. 1. $NRMSE$ as a function of SNR and N/n respectively. Results averaged over 1000 runs. (Up) $N = 256, n = 32, D = 6$ (Bottom) $n = 32, D = 6$, $\sigma_{\mathbf{w}} = 0.5$, $SNR = 35.22$dB

5. CONCLUSION

In this paper, we analyzed the problem of super-resolution where the desired sparse signal is also non-negative. We provided a universal upper bound on the estimation error of any algorithm based on the idea of Modulus of Continuity (MC). Our analysis is independent of specific algorithms and only utilizes the underlying structure of the desired signal. We show that incorporation of positive constraints strictly improves the scaling factor of MC from $O(N^3)$ to $O(N^{2.5})$. Using this unified framework, we analyzed the performance of three convex algorithms for super-resolution, and their performances are further illustrated through numerical experiments.

6. REFERENCES

[1] K. G. Puschmann and F. Kneer, "On super-resolution in astronomical imaging", *Astronomy and Astrophysics*, vol. 436, no. 1, pp. 373-378, 2005.

[2] H. Greenspan, “Super-resolution in medical imaging”, *Comput. J.*, vol. 52, no. 1, pp. 43-63, 2009.

[3] C. W. McCutchen, “Superresolution in microscopy and the Abbe resolution limit”, *J. Opt. Soc. Am.*, vol. 57, no. 10, pp. 1190-1192, 1967.

[4] R. Heckel, V. I. Morgenshtern and M. Soltanolkotabi, “Super-resolution radar”, vol. 5, no. 1, pp. 22-75, 2016.

[5] E. Betzig, G. H. Patterson, R. Sougrat, O. W. Lindwasser, S. Olenych, J. S. Bonifacino, M. W. Davidson, J. Lippincott-Schwartz and H. F. Hess, “Imaging intracellular fluorescent proteins at nanometer resolution,” *Science*, vol. 313, no. 5793, pp. 1642-1645, 2006.

[6] M. J. Rust, M. Bates and X. Zhuang, “Sub-diffraction-limit imaging by stochastic optical resconstruction microscopy (STORM),” *Nature Methods*, vol. 3, no. 10, pp. 793-795, 2006.

[7] T. Bendory, “Robust Recovery of Positive Stream of Pulses”, *IEEE Transactions on Signal Processing*, vol. 65, no. 8, pp. 2114-2122, April 2017.

[8] E. J. Candès and C. Fernandez-Granda,“Super-resolution from noisy data”, *J. Fourier Anal. Appl.*, vol. 19, no. 6, pp. 1229-1254, 2013.

[9] D. L. Donoho, “Superresolution via sparsity constraints”, *SIAM J. Math. Anal.*, vol. 23, no. 5, pp. 1309-1331, Sep. 1992.

[10] E. J. Candès and C. Fernandez-Granda, “Towards a mathematical theory of super-resolution”, *Commun. Pure Appl. Math.*, vol. 67, no. 6, pp. 906-956, June 2014.

[11] G. Tang, B. N. Bhaskar and B. Recht, “Near minimax line spectral estimation”, *IEEE Trans. Inf. Theory*, vol. 61, no. 1, pp. 499 -512, Jan. 2015.

[12] H. Qiao and P. Pal, “Gridless Line Spectrum Estimation and Low-Rank Toeplitz Matrix Compression Using Structured Samplers: A Regularization-Free Approach”, *IEEE Transactions on Signal Processing*, vol. 65, no. 9, pp. 2211-2226, May 2017.

[13] J.-D. Tournier, F. Calamante, D. G. Gadian and A. Connelly, “Direct estimation of the fiber orientation density function from diffusive-weighted MRI data using spherical deconvolution”, *NeuroImage*, vol. 23, no. 3, pp. 1176-1185, 2004.

[14] V. I. Morgenshtern and E. J. Candès, “Super-Resolution of Positive Sources: the Discrete Setup,” *SIAM J Imaging Sciences*, vol. 9, no. 1, pp. 412-444, 2016.

[15] D. L. Donoho, “Statistical Estimation and Optimal Recovery”, *Ann. Statist.*, vol. 22, no. 1, pp. 238-270, 1994.

[16] Q. Denoyelle, V. Duval and G. Peyré, “Support Recovery for Sparse Super-Resolution of Positive Measures”, *J. Fourier Anal. Appl.*, vol. 23, no. 5, pp. 1153-1194, 2017.