

OPTIMUM CONFIGURATIONS OF SPARSE SUBARRAY BEAMFORMERS

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ABSTRACT

The problem of optimum distribution of the available spatial degrees of freedom among two sparse antenna subarray beamformers in shared aperture receiver is investigated. The two subarrays, forming a full array, co-exist on the same platform and could perform separate RF sensing and communications tasks. The sparsity and cardinality of the subarray configurations are joint optimization variables which considerably affect the output signal-to-interference plus noise ratios (SINR) of the two beamformer outputs. A minimum output SINR figure value is imposed to guarantee minimum performance. We solve this problem by utilizing Taylor series approximation to reformulate the initial non-convex problem to a convex one. Simulation results validate the effectiveness of the proposed method.

Index Terms— Adaptive beamforming, sparse array design, SINR optimization, multiple sources.

1. INTRODUCTION

Adaptive sensor array design techniques provide an effective tool to enhance the response of the system towards the desired sources while mitigating interference and noise at the output of the receiver [1–7]. Optimum beamformer design should exploit all available degrees of freedom, which comprise the sparse array configuration and the beam pattern array coefficients, in order to maximize system performance. Optimal sparse array design for optimizing SINR has been investigated in [8, 9], showing that it provides superior performance over structured arrays, such as nested, coprime and uniform arrays. The authors in [10, 11] studied the effect of sparse array structures on the performance of adaptive beamformers.

The majority of existing sparse array design techniques utilize all available antennas for maximizing SINR towards one source. Recently, optimum sparse array configuration

was provided using a single beamformer with peaks towards the sources of interest [12, 13]. With only one set of array coefficients, unequal response towards different sources is deemed to emerge, putting some sources at disadvantage over others. The solution is to form different beamformers and designate separate sets of coefficients to each source [14]. Multi-beam scenario is common in many sensor array processing, and underpins different modes in radar including scanned, switched, and simultaneous or staring beams [15, 16]. In some applications, however, each beamformer is provided using a separate array or subarray which could be uniform or sparse. This situation of unshared antennas arises when antenna specifications and signal bandwidth differ for different beams, specifically when performing different missions and functions - a design known as shared aperture [17–19].

In this paper, we design optimal sparse subarrays with different sets of weights, or beamformers, each is designated to one look, or source. We consider a platform of N candidate uniform grid locations for N antennas with positions given by $nd, n = 1, \dots, N$, where d represents the inter-element spacing. We consider two look directions or presume that there are two sources in the far field with angles ϕ_A and ϕ_B . The main goal of this work is to jointly design two sparse subarrays with the objective of maximizing the SINR for one source, while attaining a predefined SINR for the other source. The two subarrays have their unique antenna elements and collectively span the entire full array. As such, the problem amounts to optimum distribution of the spatial degrees of freedom such that the SINR at each subarray beamformer output is maximized.

The rest of the paper is organized as: The system mathematical model is formulated in section 2. The SINR constrained sparse subarray design is examined in section 3. Simulation results and comments upon the results are presented in section 4. The final conclusions are given in section 5.

2. SYSTEM MODEL

Let K_A be the number of antennas in subarray A with locations defined by $y_{An}d, y_{An} \in \mathbb{N}, n = 1, \dots, K_A$, and

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K_B the number of antennas that form the second subarray B , placed at $y_{Bn}d$, $y_{Bn} \in \mathbb{N}$, $n = 1, \dots, K_B$. Subarray A aims to detect source A whereas subarray B does the detection of source B . Suppose that there are m interfering signals arriving at the composite array from angles $\{\phi_{i1}, \dots, \phi_{im}\}$. We consider that source B , if present, acts as an interference for subarray A and vice versa. The receive steering vectors for each subarray regarding direction ϕ can be written as:

$$\mathbf{a}(\phi) = [e^{jk_0 y_{A1} d \cos \phi}, \dots, e^{jk_0 y_{AK_A} d \cos \phi}]^T \quad (1)$$

$$\mathbf{b}(\phi) = [e^{jk_0 y_{B1} d \cos \phi}, \dots, e^{jk_0 y_{BK_B} d \cos \phi}]^T,$$

respectively, where k_0 is the wavenumber and is defined as $k_0 = 2\pi/\lambda$ with λ denoting the wavelength. The received signals for each subarray at time instant t are given by:

$$\mathbf{x}_A(t) = s_A(t)\mathbf{a}(\phi_A) + \mathbf{C}_A \mathbf{c}_A(t) + \mathbf{n}_A(t) \quad (2)$$

$$\mathbf{x}_B(t) = s_B(t)\mathbf{b}(\phi_B) + \mathbf{C}_B \mathbf{c}_B(t) + \mathbf{n}_B(t), \quad (3)$$

where $\mathbf{C}_A = [\mathbf{a}(\phi_{i1}), \mathbf{a}(\phi_{i2}), \dots, \mathbf{a}(\phi_{im})]$ and $\mathbf{C}_B = [\mathbf{b}(\phi_{i1}), \mathbf{b}(\phi_{i2}), \dots, \mathbf{b}(\phi_{im})]$ are the interference array manifold matrices with full column rank for subarrays A and B , respectively. Sources A and B signals are denoted as $s_A(t) \in \mathbb{C}$ and $s_B(t) \in \mathbb{C}$, respectively, with corresponding powers σ_{As}^2 and σ_{Bs}^2 . The vectors $\mathbf{c}_A(t) = [s_B(t), c_1(t), \dots, c_m(t)] \in \mathbb{C}^{m+1}$ and $\mathbf{c}_B(t) = [s_A(t), c_1(t), \dots, c_m(t)] \in \mathbb{C}^{m+1}$ stand for the interfering signals for subarrays A and B , respectively, with covariance matrices \mathbf{R}_{bA} and \mathbf{R}_{bB} and $\mathbf{n}_A(t) \in \mathbb{C}^K$, $\mathbf{n}_B(t) \in \mathbb{C}^{N-K}$ denote the received Gaussian noise vectors at subarrays A and B with common power σ_n^2 . The interference plus noise covariance matrices for the two subarrays are defined as $\mathbf{R}_{nA} = \mathbf{C}_A \mathbf{R}_{bA} \mathbf{C}_A^H + \sigma_n^2 \mathbf{I}_K$ and $\mathbf{R}_{nB} = \mathbf{C}_B \mathbf{R}_{bB} \mathbf{C}_B^H + \sigma_n^2 \mathbf{I}_{N-K}$, respectively.

The output of the receiver at subarray A is $\mathbf{w}_A^H \mathbf{x}_A(t)$, when filtered by the K_A -length complex weight vector \mathbf{w}_A . The output SINR responsible for the detection of source A is given by

$$\text{SINR}_{oA} = \frac{\sigma_{As}^2 |\mathbf{w}_A^H \mathbf{a}(\phi_A)|^2}{\mathbf{w}_A^H \mathbf{R}_{nA} \mathbf{w}_A}. \quad (4)$$

The minimum variance distortionless response (MVDR) beamformer maximizes the output SINR by minimizing the variance of the total output at the beamformer, subject to a distortionless response towards the direction of the desired source and is given by [20]:

$$\mathbf{w}_A = \frac{\mathbf{R}_{nA}^{-1} \mathbf{a}(\phi_A)}{\mathbf{a}(\phi_A)^H \mathbf{R}_{nA}^{-1} \mathbf{a}(\phi_A)} \quad (5)$$

By substituting (5) into (4), the output SINR of the matched MVDR beamformer regarding source A can be written as:

$$\text{SINR}_{oA} = \sigma_{As}^2 \mathbf{a}(\phi_A)^H \mathbf{R}_{nA}^{-1} \mathbf{a}(\phi_A). \quad (6)$$

In order to illustrate the effect of the subarray structure on the output SINR, we use the matrix inversion lemma and reformulate the interference plus noise covariance matrix \mathbf{R}_{nA}^{-1} as:

$$\mathbf{R}_{nA}^{-1} = \sigma_n^{-2} [\mathbf{I}_K - \mathbf{C}_A (\mathbf{R}_{mA} + \mathbf{C}_A^H \mathbf{C}_A)^{-1} \mathbf{C}_A^H] \quad (7)$$

where $\mathbf{R}_{mA} = \sigma_n^2 \mathbf{R}_{bA}^{-1}$. By defining $\text{SNR}_{iA} = \sigma_{As}^2 / \sigma_n^2$ as the input signal-to-noise ratio (SNR) at subarray A and substituting (7) into (6), the output SINR at subarray A can be written as in (8).

By following the same steps for subarray B , the output SINR is given by (9), where $\text{SNR}_{iB} = \sigma_{Bs}^2 / \sigma_n^2$ defines the SNR at subarray B and $\mathbf{R}_{mB} = \sigma_n^2 \mathbf{R}_{bB}^{-1}$. It is clear from (8) and (9) that the subarray structures and their antenna element positions affect the output SINR at both subarrays through the source steering vectors $\mathbf{a}(\phi_A)$ and $\mathbf{b}(\phi_B)$ and the interference array manifold matrices \mathbf{C}_A and \mathbf{C}_B .

3. SPARSE SUBARRAY SELECTION THROUGH SINR CONSTRAINED OPTIMIZATION

It is typical for source detection to be constrained by a certain SINR threshold which corresponds to a certain detection performance [21]. Our primary objective in this section is to design an algorithm that maximizes the detection performance of one source, while attaining a predefined SINR threshold for the other source. The proposed method provides both the optimal separable sparse subarray selections and the optimal number of antennas (K_A , K_B) for each subarray. We set an SINR threshold γ_A^* towards source A . With a total of N antennas placed on uniform grid, the optimum sparse subarray design amounts to selecting the optimal $K_A = K$ and $K_B = N - K$ non-overlapping candidate antennas towards sources A and B and jointly maximize the detection performance towards source B , while attaining the SINR threshold towards source A . Hence, we define an antenna selection vector $\mathbf{z} \in \{0, 1\}^N$, where entry "1" denotes an antenna selected for subarray A and a zero "0" entry denotes an antenna selected for subarray B . Since we have information regarding all the antenna locations, the full ULA receive steering vector towards direction ϕ is given by:

$$\hat{\mathbf{a}}(\phi) = [e^{jk_0 y_1 d \cos \phi}, \dots, e^{jk_0 y_N d \cos \phi}]^T. \quad (13)$$

Thus, the respective steering vectors for subarrays A and B towards direction ϕ can be defined as $\mathbf{a}(\phi) = \mathbf{z} \odot \hat{\mathbf{a}}(\phi)$ and $\mathbf{b}(\phi) = (\mathbf{1}_N - \mathbf{z}) \odot \hat{\mathbf{a}}(\phi)$ and discard the zero entries, where $\mathbf{1}_N$ is an all one vector of size N and \odot stands for the Hadamard product (element-wise product). In order to design the optimal separated, sparse subarrays A and B , we consider the following SINR_{oA} constrained-SINR $_{oB}$ maximization problem:

$$\max_{\mathbf{z}, K} \quad \text{SINR}_{oB} \quad (14)$$

$$\text{s.t.} \quad \text{SINR}_{oA} \geq \gamma_A^*, \quad \mathbf{1}_N^T \mathbf{z} = K, \quad \mathbf{z} \in \{0, 1\}^N$$

$$\text{SINR}_{oA} = \text{SNR}_{iA} [K_A - \mathbf{a}(\phi_A)^H \mathbf{C}_A (\mathbf{R}_{mA} + \mathbf{C}_A^H \mathbf{C}_A)^{-1} \mathbf{C}_A^H \mathbf{a}(\phi_A)] \quad (8)$$

$$\text{SINR}_{oB} = \text{SNR}_{iB} [K_B - \mathbf{b}(\phi_B)^H \mathbf{C}_B (\mathbf{R}_{mB} + \mathbf{C}_B^H \mathbf{C}_B)^{-1} \mathbf{C}_B^H \mathbf{b}(\phi_B)] \quad (9)$$

$$\max_{\mathbf{z}, K} \log |\hat{\mathbf{C}}_{aB}^H \text{diag}(\mathbf{1}_N - \mathbf{z}) \hat{\mathbf{C}}_{aB} + \mathbf{R}_B| - \log |\hat{\mathbf{C}}_B^H \text{diag}(\mathbf{1}_N - \mathbf{z}) \hat{\mathbf{C}}_B + \mathbf{R}_{mB}| \quad (10)$$

$$\text{s.t.} \quad \log |\hat{\mathbf{C}}_{aA}^H \text{diag}(\mathbf{z}) \hat{\mathbf{C}}_{aA} + \mathbf{R}_A| - \log |\hat{\mathbf{C}}_A^H \text{diag}(\mathbf{z}) \hat{\mathbf{C}}_A + \mathbf{R}_{mA}| \geq \gamma_A^*, \quad \mathbf{1}_N^T \mathbf{z} = K, \quad \mathbf{z} \in \{0, 1\}^N$$

$$\log |\hat{\mathbf{C}}_A^H \text{diag}(\mathbf{z}) \hat{\mathbf{C}}_A + \mathbf{R}_{mA}| \approx \log |\hat{\mathbf{C}}_A^H \text{diag}(\mathbf{z}^{(k)}) \hat{\mathbf{C}}_A + \mathbf{R}_{mA}| + \nabla \mathbf{g}_A^T(\mathbf{z}^{(k)}) (\mathbf{z} - \mathbf{z}^{(k)}) \triangleq T_A \quad (11)$$

$$\log |\hat{\mathbf{C}}_B^H \text{diag}(\mathbf{1}_N - \mathbf{z}) \hat{\mathbf{C}}_B + \mathbf{R}_{mB}| \approx \log |\hat{\mathbf{C}}_B^H \text{diag}(\mathbf{1}_N - \mathbf{z}^{(k)}) \hat{\mathbf{C}}_B + \mathbf{R}_{mB}| + \nabla \mathbf{g}_B^T(\mathbf{z}^{(k)}) ((\mathbf{1}_N - \mathbf{z}) - (\mathbf{1}_N - \mathbf{z}^{(k)})) \triangleq T_B \quad (12)$$

Based on [22], we can replace the output SINRs in (14) with the logarithm of the output SINRs for both sources as in (10), where $\hat{\mathbf{C}}_{aA} = [\hat{\mathbf{C}}_A, \hat{\mathbf{a}}(\phi_A)]$ and $\hat{\mathbf{C}}_A = [\hat{\mathbf{a}}(\phi_B), \hat{\mathbf{a}}(\phi_{i1}), \dots, \hat{\mathbf{a}}(\phi_{im})]$. Likewise, $\hat{\mathbf{C}}_{aB} = [\hat{\mathbf{C}}_B, \hat{\mathbf{a}}(\phi_B)]$ and $\hat{\mathbf{C}}_B = [\hat{\mathbf{a}}(\phi_A), \hat{\mathbf{a}}(\phi_{i1}), \dots, \hat{\mathbf{a}}(\phi_{im})]$ and

$$\mathbf{R}_A = \begin{bmatrix} \mathbf{R}_{mA} & \mathbf{0}_{1 \times m} \\ \mathbf{0}_{m \times 1} & 0 \end{bmatrix}, \quad \mathbf{R}_B = \begin{bmatrix} \mathbf{R}_{mB} & \mathbf{0}_{1 \times m} \\ \mathbf{0}_{m \times 1} & 0 \end{bmatrix}.$$

We relax the non-convex binary selection constraints $\mathbf{z} \in \{0, 1\}^N$ of (10) to a box constraint $0 \leq \mathbf{z} \leq 1$, as the global optimizer of the difference of two concave functions locates at the extreme points of the polyhedron [23]. However, (10) remains non-convex since the objective function and the first constraint are differences of two concave functions. To overcome this problem, we utilize first order Taylor series that can iteratively approximate the negative logarithm terms of the optimization problem, which cause the non-convexity of (10). The $(k+1)_{th}$ Taylor approximations of those terms based on the previous solution $\mathbf{z}^{(k)}$ are shown in (11) and (12), where $\nabla \mathbf{g}_A(\mathbf{z}^{(k)})$ and $\nabla \mathbf{g}_B(\mathbf{z}^{(k)})$ represent the gradients of the logarithmic functions $\log |\hat{\mathbf{C}}_A^H \text{diag}(\mathbf{z}) \hat{\mathbf{C}}_A + \mathbf{R}_{mA}|$ and $\log |\hat{\mathbf{C}}_B^H \text{diag}(\mathbf{1}_N - \mathbf{z}) \hat{\mathbf{C}}_B + \mathbf{R}_{mB}|$ evaluated at $\mathbf{z}^{(k)}$, respectively. By exploiting sequential convex programming (SCP) techniques, the initially non-convex problem is reformulated to a series of convex subproblems, that can be optimally solved via convex optimization methods [24]. By substituting (11) and (12) in (10), we have the following approximated convex optimization problem:

$$\max_{\mathbf{z}, K} \log |\hat{\mathbf{C}}_{aB}^H \text{diag}(\mathbf{1}_N - \mathbf{z}) \hat{\mathbf{C}}_{aB} + \mathbf{R}_B| - T_B \quad (15)$$

$$\text{s.t.} \quad \log |\hat{\mathbf{C}}_{aA}^H \text{diag}(\mathbf{z}) \hat{\mathbf{C}}_{aA} + \mathbf{R}_A| - T_A \geq \gamma_A^* \\ \mathbf{1}_N^T \mathbf{z} = K, \quad 0 \leq \mathbf{z} \leq 1$$

Since SCP is a local heuristic, the solution of (15) is dependent on the initial selection vector $\mathbf{z}^{(0)}$. Hence, we consider several feasible initialization vectors $\mathbf{z}^{(0)}$ for (15) and keep the solution that provides the maximum objective function value.

4. SIMULATION RESULTS

In this section, simulation results are presented to validate the effectiveness of the proposed algorithms. We consider a uniform linear array (ULA) of $N = 20$ antennas with an inter-element spacing of $d = \lambda/2$. There are two source signals arriving at the ULA from $\phi_A = 135^\circ$ and $\phi_B = 50^\circ$ and with an SNR set at 0dB for both sources. An interference source is presumed, impinging on the ULA at $\phi_{i1} = 108^\circ$ with INR=20dB. We aim to design two separate subarrays, spanning the entire ULA, where the output SINR of subarray B is maximized while attaining a predefined SINR or detection threshold for source A . The algorithm not only decides on the optimal locations of the antennas but also selects the optimal number of antennas for each subarray (K_A, K_B). Fig. 1 shows the beampatterns towards source A for different preset SINR values, γ_A^* . As expected, higher γ_A^* yields a better shaped beampattern towards source A with lower sidelobes and better cancellation of interference. The optimal number of antennas for subarrays A and B and the maximum SINR_{oB} achieved from (15) for different γ_A^* are depicted in Table 1. It is clear that the higher the γ_A^* the more antennas are allocated to subarray A which leads to a drop of SINR_{oB} .

In order to demonstrate the offerings of the proposed adaptive method, we compare the performance of (15) to the case when nested and coprime arrays are utilized to design subarray A [25, 26]. We employ 8 antennas to differently

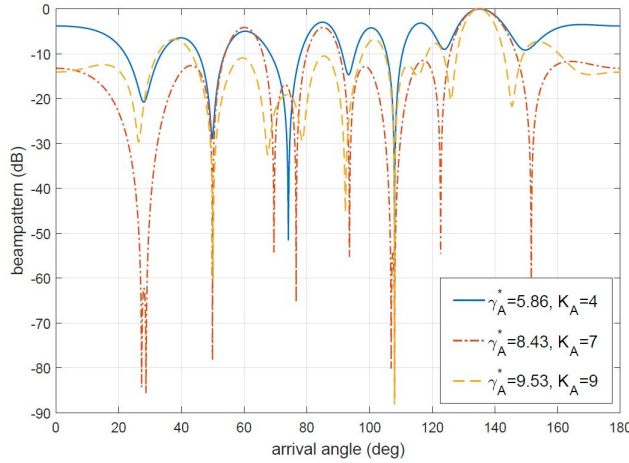


Fig. 1: Beampatterns towards source A for different γ_A^* .

Table 1: Optimal number of antennas for subarrays A and B and maximum $SINR_{oB}$ (dB) for different target $SINR$ γ_A^* .

$SINR_{oA}$ target	K_A	K_B	$SINR_{oB}$
$\gamma_A^* = 5.86$	4	16	12.02
$\gamma_A^* = 8.43$	7	13	11.12
$\gamma_A^* = 9.53$	9	11	10.39

build the nested and coprime subarray A . Since it is not feasible to simultaneously design nested or coprime subarrays A and B , subarrays B are constituted from the remaining sensors in each case. The two coprime numbers are $M_c = 5$ and $N_c = 4$, both starting from sensor 1 and hence providing 8 different sensor locations. The same settings as in the previous example are used. Subarray A structures for the proposed optimization (15) with $\gamma_A^* = 8.39$, the nested and the coprime techniques are given in Fig.2, and the corresponding beampatterns are shown in Fig.3. Table 2 illustrates the superiority of the proposed adaptive subarray selection over the nested and coprime prefixed schemes. Nested and coprime subarrays A provide less SINR than the proposed technique, which is implementing only 7 antennas. The unengaged antenna is used for maximizing the detection performance of source B . Therefore the SINR of source B is also higher for the proposed method as opposed to the prefixed techniques.

5. CONCLUSION

We examined the design of dual sparse subarrays in dual mission shared aperture passive sensing platforms. The proposed algorithm uses SINR criterion to simultaneously decide on

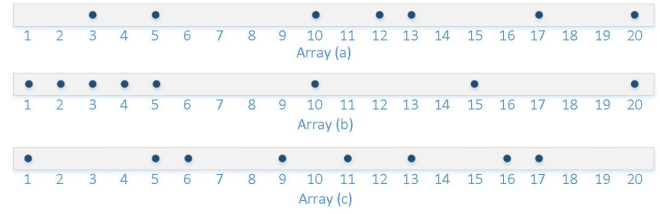


Fig. 2: Subarray A : (a) Proposed method (15) with $\gamma_A^* = 8.39$, (b) Nested array, (c) Coprime array.

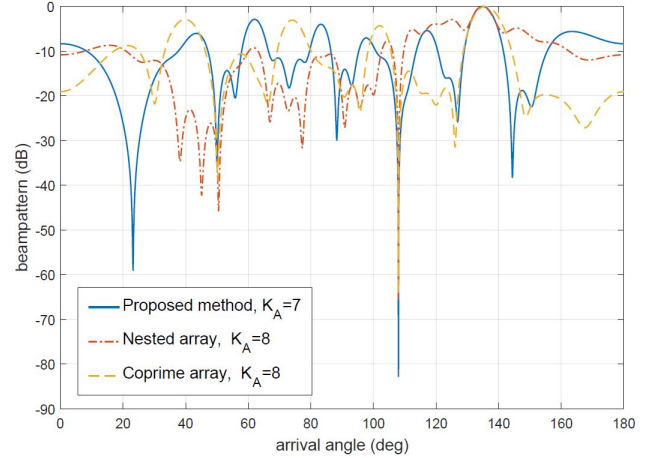


Fig. 3: Beampatterns for subarrays A in Fig.2.

the location and the cardinality of the antennas in each subarray. This could amount in radar to maximizing the detection performance of one source, while securing a desired detection threshold for the other source(s). Sequential convex programming and Taylor series approximation were utilized to render the sparse subarray design a convex problem. Simulation results demonstrated the effectiveness of the proposed algorithm regarding joint maximizing SINR performance under SINR constraint of two subarrays. The optimum design demonstrated superiority over other sparse array configurations.

Table 2: Comparison of the system performance for the proposed method, the nested and the coprime arrays schemes.

	K_A	K_b	$SINR_{oA}$	$SINR_{oB}$
Optimization (15)	7	13	8.39	11.13
Nested	8	12	8.30	10.78
Coprime	8	12	8.27	10.54

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