# Optimum Sparse Array Design for Multiple Beamformers with Common Receiver

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Abstract—The problem of optimum sparse array beamformer design to maximize output signal-to-interference-plus-noise ratio (SINR) in the case of multiple narrowband sources was recently investigated. This was based on seeking both optimum sensor placement as well as optimum a single beamformer for all sources in the array field of view. In this paper, we consider multiple beamformers with a common sparse array. That is, we deal with a more prevalent case in radar and communications where each source is assigned its own beam. This could be the case for both switched and simultaneous or staring beams. The paper considers optimum sparse array design for both narrowband and wideband sources. Analysis and simulation examples demonstrate that the optimum sparse array configuration depends on both the arrival angle and the frequency of the incoming signal and it plays a vital role in determining the performance of multiple beamformer receivers.

Keywords—Sparse Array, Multiple Beamformers, Wideband

#### I. INTRODUCTION

Adaptive array processing strives to counteract interferences while providing high sensitivity towards the desired sources. It finds applications in radar, sonar, wireless communications, radio astronomy, and satellite navigation, to list a few [1]–[11]. The beamforming performance is not only dependent on the weight coefficients but also on the array configuration [12]–[14]. For the same number of antennas, different array structures yield different output signal-to-interference-plusnoise ratios (SINRs). As such, optimum sparse array design should fully utilize both the array structure and array weights towards achieving the highest possible SINR. The problem of sparse array design is typically cast as an optimum placement of a given number of antennas on a uniform grid points or equivalently selecting a subset from a large set of uniformly spaced antennas to connect with front-end receivers. The antenna selection perspective of this problem relies on lowcomplexity Radio Frequency (RF) switches [15]-[18], with the fundamental goal of reducing hardware cost associated with expensive RF chains. The optimum sparse array in a changing environment can be adaptively reconfigured through switching on/off antennas.

The problem of optimum sparse array design for interference mitigation in the case of mutiple narrowband sources was investigated in our previous work [19], [20], where a single set of weights associated with a single beamformer is implemented, aiming at maximizing the total output SINR of all sources in the field of view (FOV). The maximum output SINR is the principal eigenvalue of the product between inverse noise covariance and source correlation matrices [21], [22]. In addition to limited practical utility, the above scheme of having a single beamformer generated by one set of weights suffers from unequal receiver sensitivities towards different sources. That is, strong, closely-spaced sources are favored compared to weak, widely separated sources.



Fig. 1. Structure of single common array receiver with multiple beamformers.

In this paper, we consider a general and practical scenario, where a single, common sparse array is used with different sets of weights, each corresponding to one source, or one task. This scenario can represent fixed or switched beam operating modes [23], [24]. The principal issue is that the common sparse array configuration influences the beams' respective gains and shapes. Therefore, optimality of antenna positions must be examined in view of the requirements placed on all beamformers and their respective SINR. The schematic of proposed single sparse array with multiple beamformers is depicted in Fig. 1. In this paper, we derive optimum sparse arrays for both narrowband and wideband signals, and demonstrate that the performance is significantly improved over the case of a single beamfomer receiver. It is noted that structured arrays, such as nested or coprime arrays are also sparse, but they are designed to meet a different objective involving coarrays [25]-[27].

The rest of this paper is organized as follows: We formulate the problem in section II. Antenna selection algorithms with a single optimum sparse array and multiple beams in both spatial and spectral domain is proposed in section III. Simulation results in section IV validate the proposed methods. Finally, conclusions are provided in section V.

# II. MATHEMATICAL MODEL

Consider a linear array of N isotropic antennas with positions specified by multiple integer of unit inter-element

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spacing  $p_n d_u, p_n \in \mathbb{N}, n = 1, \dots, N$ . Here, the unit interelement spacing is set as half the wavelength corresponding to the design frequency  $f_u$ , that is  $d_u = c/2f_u$ .

Suppose that P narrowband source signals are impinging on the array from directions  $\{\theta_1, \ldots, \theta_P\}$  and Q narrowband interfering signals with arrival angles denoted by  $\phi_1, \ldots, \phi_Q$ . The steering vectors of incoming signals can be calculated by,

$$\mathbf{u}_{p} = [1, e^{j\pi \cos \theta_{p}}, \dots, e^{j(N-1)\pi \cos \theta_{p}}]^{T}, p = 1, \dots, P \quad (1)$$
$$\mathbf{v}_{q} = [1, e^{j\pi \cos \phi_{q}}, \dots, e^{j(N-1)\pi \cos \phi_{q}}]^{T}, q = 1, \dots, Q.$$

For broadband signals with a bandwidth of B and center frequency  $f_c$ ,  $f_u = f_c + \frac{B}{2}$ . We implement the DFT based beamformer with a structure plotted in Chapter 6 of [21]. In this case, the broadband signals can be considered as a superposition of M narrowband signals separated in frequency by B/M, denoted as  $f_c + \Delta f_k = f_c + (B/M)(k - M/2), k =$  $1, \ldots, M$ . Without loss of generality, we assume M is even. It is worth noting that the narrowband signal model applies to the output of each DFT passband filter. The steering vectors of the filtered desired and interference signals with frequency  $f_c + \Delta f_k$  are expressed as,

$$\mathbf{u}_{p} = [1, e^{j\pi\cos\theta_{p}(\frac{1+b_{k}}{1+B_{f}/2})}, \dots, e^{j(N-1)\pi\cos\theta_{p}(\frac{1+b_{k}}{1+B_{f}/2})}]^{T}, \quad (2)$$
  

$$\mathbf{v}_{q} = [1, e^{j\pi\cos\phi_{q}(\frac{1+b_{k}}{1+B_{f}/2})}, \dots, e^{j(N-1)\pi\cos\phi_{q}(\frac{1+b_{k}}{1+B_{f}/2})}]^{T}, \quad (3)$$
  
where  $b_{k} = \Delta f_{k}/f_{c}, B_{f} = B/f_{c}, p = 1, \dots, P \text{ and } q = 1, \dots, Q.$ 

In the case of mixed narrowband and broadband signals, we assume the carrier frequency  $f_l$  of the *l*th narrowband signal falls into the *k*th DFT band with  $k = \left(\frac{f_l - f_c}{B/M}\right) + \frac{M}{2}$  and (•) truncating the argument to the closest integer. Suppose there are  $m_k = P + Q + L_k$  signals falling into the *k*th DFT band, comprising *P* broadband sources, *Q* broadband interferences and  $L_k$  possible narrowband sources. Clearly,  $0 \le L_k \le L$  equals to the number of narrowband sources falling into the *k*th DFT band.

In summary, all the cases considered can be decomposed into the superposition of narrowband signal models. In the sequel, we take the narrowband signal model as an example to delineate the common sparse array design associated with different beamformers. We assume all source and interference angles are known or already estimated. Denote the weight vector of the *p*th beamformer as  $\mathbf{w}_p$ ,  $p = 1, \ldots, P$ . The Capon beamformer aims at minimizing the total output variance while constraining the response towards the *p*th narrowband signal to be unity. The weight vector of Capon beamformer is well known and given by [28],

$$\mathbf{w}_p = \frac{1}{\mathbf{u}_p^H \mathbf{R}_p^{-1} \mathbf{u}_p} \mathbf{R}_p^{-1} \mathbf{u}_p, \qquad (3)$$

where  $\mathbf{R}_p$  is the covariance matrix of received signal excluding the *p*th narrowband signal of interest (SOI). It comprises of the *Q* interferences, the remaining P-1 narrowband sources and the white noise. That is,

$$\mathbf{R}_{p} = \sum_{k=1, k \neq p}^{P} \frac{\sigma_{sk}^{2}}{\sigma_{n}^{2}} \mathbf{u}_{k} \mathbf{u}_{k}^{H} + \sum_{q=1}^{Q} \frac{\sigma_{jq}^{2}}{\sigma_{n}^{2}} \mathbf{v}_{q} \mathbf{v}_{q}^{H} + \mathbf{I}, \qquad (4)$$

where  $\sigma_{jq}^2$ ,  $\sigma_{sk}^2$  and  $\sigma_n^2$  denote the *q*th interference power, the *k*th narrowband signal power and the noise power, respectively. Clearly, when the *p*th narrowband signal is the SOI, all other sources are viewed as the interfering signals. Utilizing Eq. (3), we obtain the array gain of the Capon beamformer towards the *p*th narrowband signal. That is,

$$\mathbf{G}_p = \mathbf{u}_p^H \mathbf{R}_p^{-1} \mathbf{u}_p. \tag{5}$$

Utilizing the matrix inversion lemma, the inverse covariance  $\mathbf{R}_p^{-1}$  can be written as,

$$\mathbf{R}_p^{-1} = \mathbf{I} - \mathbf{B}(\mathbf{C}_j + \mathbf{B}^H \mathbf{B})^{-1} \mathbf{B}^H,$$
(6)

where  $\mathbf{B} = [\mathbf{u}_1, \dots, \mathbf{u}_{p-1}, \mathbf{v}_1, \dots, \mathbf{v}_Q]$  and  $\mathbf{C}_j$  is a diagonal matrix with diagonal entries  $\frac{\sigma_n^2}{\sigma_{sk}^2}, k = 1, \dots, P, k \neq p$  and  $\frac{\sigma_n^2}{\sigma_{jq}^2}, q = 1, \dots, Q$ . Substituting Eq. (6) into Eq. (5) yields,

$$G_{p} = \mathbf{u}_{p}^{H} \mathbf{R}_{p}^{-1} \mathbf{u}_{p},$$
(7)  
$$= \mathbf{u}_{p}^{H} \mathbf{u}_{p} - \mathbf{u}_{p}^{H} \mathbf{B} (\mathbf{C}_{j} + \mathbf{B}^{H} \mathbf{B})^{-1} \mathbf{B}^{H} \mathbf{u}_{p},$$
$$= \frac{|\mathbf{B}_{p}^{H} \mathbf{B}_{p} + \mathbf{C}|}{|\mathbf{B}^{H} \mathbf{B} + \mathbf{C}_{j}|},$$

where  $\mathbf{B}_p = [\mathbf{B}, \mathbf{u}_p]$ , and the extended interference covariance matrix is,

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_j & \mathbf{0}_{(Q+P-1)\times 1} \\ \mathbf{0}_{1\times(Q+P-1)} & 0 \end{bmatrix}.$$
 (8)

The equivalence between the second and third lines of Eq. (7) can be proved through block matrix determinant formula.

$$\begin{vmatrix} \mathbf{B}_{p}^{H} \mathbf{B}_{p} + \mathbf{C} \end{vmatrix} = \begin{vmatrix} \mathbf{B}^{H} \mathbf{B} + \mathbf{C}_{j} & \mathbf{B}^{H} \mathbf{u}_{p} \\ \mathbf{u}_{p}^{H} \mathbf{B} & \mathbf{u}_{p}^{H} \mathbf{u}_{p} \end{vmatrix}, \qquad (9)$$
$$= |\mathbf{B}^{H} \mathbf{B} + \mathbf{C}_{j}| \mathbf{G}_{p}.$$

We can observe that the array configuration affects array gain of Capon beamformer through the SOI steering vector  $\mathbf{u}_p$  and the interference array manifold matrix **B**.

#### III. OPTIMUM SPARSE ARRAY DESIGN FOR MULTIPLE BEAMFORMERS

The optimum sparse array design can be cast as selecting K out of N candidate grid locations for antenna placement with the antenna weights determined by Capon beamforming. Denote a grid selection vector  $\mathbf{z} = [z_i, i = 1, ..., N] \in \{0, 1\}^N$  with "zero" entry for a discarded location and "one" entry for a selected one. The diagonal matrix  $\mathcal{D}(\mathbf{z})$  is the antenna selection operator with the vector  $\mathbf{z}$  populating along the diagonal. Since all N grid points are known, the full array manifold vector corresponding to the SOI and interferences can be calculated in advance. Since steering vectors are directional, the steering vectors of a sparse array, with selected K antennas in relation to the full array of N antennas, can be expressed as  $\mathcal{D}(\mathbf{z})\mathbf{u}_p$  and  $\mathcal{D}(\mathbf{z})\mathbf{v}_q$ . Correspondingly, the array gain of sparse arrays for the *p*th narrowband signal can be expressed as,

$$\mathbf{G}_{p} = \frac{\left|\mathbf{B}_{p}^{H}\mathcal{D}(\mathbf{z})\mathbf{B}_{p} + \mathbf{C}\right|}{\left|\mathbf{B}^{H}\mathcal{D}(\mathbf{z})\mathbf{B} + \mathbf{C}_{j}\right|}.$$
(10)

Multi-beamformer design assumes a common sparse array and different sets of weights for different SOI. If we set the array

gain towards each SOI to be at least  $\gamma$ , then the optimum sparse array design can be formulated as,

$$\max_{\boldsymbol{\gamma}} \quad \boldsymbol{\gamma}, \quad (11)$$

s.t. 
$$\begin{aligned} \log |\mathbf{B}_p^H \mathcal{D}(\mathbf{z}) \mathbf{B}_p + \mathbf{C}| - \log |\mathbf{B}^H \mathcal{D}(\mathbf{z}) \mathbf{B} + \mathbf{C}_j| \geq \gamma, \\ p = 1, \dots, P \\ \mathbf{z} \in \{0, 1\}^N, \ \mathbf{1}^T \mathbf{z} = K. \end{aligned}$$

Clearly, the first set of constraints in Eq. (11) belongs to the difference of two concave functions (D.C.), and the global optimizer of D.C. functions locates at the edge of the polyhedral [29], [30]. This property can be utilized to relax the boolean constraint of the selection vector  $\mathbf{z} \in \{0, 1\}^N$  to the box constraint  $0 \le \mathbf{z} \le 1$ . To further promote the boolean property of the selection vector  $\mathbf{z}$ , we modify the objective function in Eq. (11) as  $\gamma + \tau \mathbf{z}^T (\mathbf{z} - \mathbf{1})$  with the relaxed box constraints as follows.

$$\max_{\mathbf{z}} \qquad \gamma + \tau \mathbf{z}^{T}(\mathbf{z} - \mathbf{1}), \tag{12}$$
  
s.t. 
$$\log |\mathbf{B}_{p}^{H} \mathcal{D}(\mathbf{z}) \mathbf{B}_{p} + \mathbf{C}| - \log |\mathbf{B}^{H} \mathcal{D}(\mathbf{z}) \mathbf{B} + \mathbf{C}_{j}| \ge \gamma,$$
$$p = 1, \dots, P$$
$$0 < \mathbf{z} < 1, \ \mathbf{1}^{T} \mathbf{z} = K.$$

Here,  $\tau$  is a trade-off parameter that controls the relative importance between the array gain and boolean property of the selection vector **z**. Except for the D.C. constraints, the objective function in Eq. (12) also becomes non-convex. A sequential convex programming (SCP) based on iteratively linearizing the second convex function of the objective and the second concave function of the constraints is then utilized to reformulate the non-convex problem to a series of convex subproblems, each of which can be optimally solved using convex programming [31], [32]. The antenna selection in the (k+1)th iteration can be formulated based on the solution  $\mathbf{z}^{(k)}$ from the kth iteration as,

$$\max_{\mathbf{z}} \qquad \gamma + \tau [(2\mathbf{z}^{(k)} - \mathbf{1})^T \mathbf{z} - \mathbf{z}^{(k)T} \mathbf{z}^{(k)}], \qquad (13)$$
  
s.t. 
$$\log |\mathbf{B}_p^H \mathcal{D}(\mathbf{z}) \mathbf{B}_p + \mathbf{C}| - \Delta \mathbf{g}_1^T (\mathbf{z}^{(k)}) (\mathbf{z} - \mathbf{z}^{(k)}) \ge \gamma, \qquad p = 1, \dots, P$$
$$0 \le \mathbf{z} \le 1, \ \mathbf{1}^T \mathbf{z} = K.$$

where  $\Delta \mathbf{g}_1(\mathbf{z}^{(k)})$  is the gradient of the concave function  $\log |\mathbf{B}^H \mathcal{D}(\mathbf{z})\mathbf{B} + \mathbf{C}_j|$  evaluated at the point  $\mathbf{z}^{(k)}$ . That is,

$$\Delta \mathbf{g}_1 = [\mathbf{b}_{r,i}^H (\mathbf{B}^H \mathcal{D}(\mathbf{z}^{(k)}) \mathbf{B} + \mathbf{C}_j)^{-1} \mathbf{b}_{r,i}, i = 1, \dots, N]^T,$$

with  $\mathbf{b}_{r,i}$  denoting the *i*th column vector of the matrix  $\mathbf{B}^{H}$ . Note that the SCP is a local heuristic and its performance depends on the initial point  $\mathbf{z}^{(0)}$ . It is, therefore, typical to initialize the algorithm with several feasible points  $\mathbf{z}^{(0)}$  and find the one with the maximum objective value over the different runs. The Matlab embedded software package CVX [33] is utilized to solve the optimization problem in Eq. (13). The simulation results in section IV show that the obtained iterative solution agrees well with the true optimum solution from enumeration, which manifests the effectiveness of the SCP algorithm for solving D.C. problems.

### IV. SIMULATIONS

In this section, simulation results are presented to validate the proposed sparse array design for multiple beamformers.

### A. The case of multiple narrowband sources

Consider K = 10 available antennas and N = 20uniformly spaced positions with an inter-element spacing of  $d = \lambda_u/2$  and  $\lambda_u$  being the wavelength corresponding to  $f_u =$ 310MHz. Assume that three narrowband sources are impinging on the array from directions  $\theta_1 = 50^\circ, \theta_2 = 60^\circ, \theta_3 = 115^\circ$ with the SNR being [5, 2, 0] dB, respectively. There are two interfering signals arriving at  $\phi_1 = 54^\circ, \phi_2 = 122^\circ$  with the respective interference-to-noise ratios (INRs) being 10dB and 30dB. We first calculate the optimum sparse array corresponding to each source separately. The structures of three individual sparse arrays, referred to as arrays (a), (b) and (c), are shown in Fig. 2. The configuration of the common sparse array for all the three sources but separate beamformers, is the same as array (a). The reason is that the array gain of the first source is the lowest, and sparse array (a) is capable of guaranteeing a robust performance for all the three sources considered. For comparison, the optimum sparse array with a single beamformer, termed as array (d), is provided at the bottom of Fig. 2.

We calculate respective array gain of four sparse arrays towards the three sources, as displayed in Table I. Note that the maximum array gain of an arbitrary 10-antenna array is no more than 10dB. The Capon beampatterns of the sparse array (a) associated with three beamformers are plotted in Fig. 3 (i)-(iii), where each source assumes a different set of weight coefficients. We also plot the beampattern of the sparse array (d) in Fig. 3 (iv), where the common weight vector for three sources is the principal eigenvector of the product of inverse noise covariance and source correlation matrices. Clearly, the sparse arrays (a)-(c) exhibit the maximum array gain toward the first, second and third source, respectively. The sparse array (d) demonstrates extremely uneven sensitivities towards the three sources, and completely ignores the third weak and widely separated source. The beampatterns of the multi-beamformer array (a), shown in Fig. 3 (i)-(iii), exhibits three respective peaks towards the sources and four nulls against the interferences and other two sources. Note that the beampatterns exhibit unwanted high sidelobes, as the array optimization criterion is maximizing the output SINR without considering beampattern shape. The additional constraints can be imposed into the formulation in Eq. (11) for a better-shaped beampattern, while unavoidably sacrificing output SINR.

In order to verify the effectiveness of the proposed sparse array design algorithm formulated in Eq. (13), we vary the arrival angle of the third source from  $0^{\circ}$  to  $180^{\circ}$  in step of  $1^{\circ}$ . For each source arrival angle, we find the corresponding optimum sparse array per Eq. (13) using 100 different initial points. For algorithm validation, we also obtain the true optimum sparse arrays through enumeration and compare their output SINR values in Fig. 4. Clearly, the proposed algorithm can return a satisfactory sub-optimal solution with output SINR difference upper bounded by 0.3dB.

#### B. The case of wideband sources

Consider the case of three wideband sources with a center frequency  $f_c = 300$ MHz and bandwidth B = 20MHz arriving from  $\theta = [50^\circ, 60^\circ, 115^\circ]$  with SNR being 0dB. The interfering signals are also wideband and other information



Fig. 2. Optimum sparse arrays (a), (b), (c) for three sources respectively and optimum sparse array (d) for single beamformer. Sparse array (a) is also the common optimum sparse array for three beamformers.

TABLE I. THE ARRAY GAIN OF SPARSE ARRAYS (A)-(D) TOWARDS THREE SOURCES



Fig. 3. (i)-(iii) Beampatterns of the array (a) associated with three beamformer; (iv) Beampattern of array (d) associated with a single beamformer.



Fig. 4. The difference of output SINR of sparse arrays obtained through the proposed algorithm and enumeration.

remains the same as above. There are M = 16 DFT frequency bins. We find the optimum sparse array for each wideband source over its spectrum. These arrays, termed (e)-(h), are depicted in Fig. 5. The optimum sparse array in the frequency bin of f = 310MHz is actually array (a) in Fig. 2. We can see that the configuration of optimum sparse arrays differs with different signal frequencies. Note that there are three beamformers associated with the sparse array in each DFT bin. The beampatterns of three wideband beamformers over the spectrum are plotted in Fig. 6. Each beamformer comprises different array configurations and weight coefficients in different DFT bins. We also plot the beampatterns of sparse array (d) with a single beamformer for comparison. We can observe that the beampatterns towards each source are almost the same in different DFT bins with the beam pointing at the designated source and nulls against the interferences and other sources. The single beamformer fails to offer equal sensitivities towards all sources in the receiver FOV.



Fig. 5. Configuration of optimum common sparse arrays (e)-(h) in different frequency bins.



Fig. 6. (i)-(iii): The beampatterns of three beamformers for three 20MHz wideband signals in 16 DFT bins. (iv) The beampatterns of the array (d) with a single beamformer.

# V. CONCLUSIONS

We examined the problem of optimum sparse array design for multiple beamformers through antenna selection in the presence of narrowband and wideband sources. Unlike the previous work dealing with a single beamformer for all sources in FOV, we deployed in this paper a single sparse array but with different sets of weights, each beamformer is designated to one SOI. Simulation results demonstrated the significant role of array configurations in determining array output SINR performance when using multiple beams whether it is for fixed or switched beam scenario.

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