# OPTIMIZED SPARSE ARRAY DESIGN BASED ON THE SUM COARRAY

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## ABSTRACT

The number of elements of a uniform linear array (ULA) is the main bottleneck in many applications in array processing in terms of cost and power consumption. This motivates the use of sparse arrays where some of the elements are removed. However, designing a sparse array configuration with the smallest number of elements that preserves the full array beam pattern is generally NP hard. In this paper we adopt work in multiple-input-multiple-output (MIMO) radar to study a sparse array composed of two sub apertures. We derive the minimal number of elements required using this design, showing that in general there are two optimal solutions. Next, we present an extension of this approach beyond two sub apertures. By optimizing the number of sub apertures, we prove that the optimal array configuration is related to the notion of prime factorization. This allows to achieve a significant reduction in the number of elements.

*Index Terms*— Sparse arrays, beam pattern, beamforming, polynomial factorization, sum coarray

## 1. INTRODUCTION

Uniform linear arrays (ULA) play an important role in diverse fields such as radar, sonar, communications, direction-of-arrival (DOA), radio astronomy, seismology and medical ultrasound [1]. The main benefits of ULAs are simple geometry, high signal-to-noise ratio (SNR), spatial selectivity and beamforming capabilities for elimination of interference signals. However, they suffer from high mutual coupling, making the sensor responses interfere with each other [2, 3]. More importantly they may become impractical due to the increase in costs and power consumption when the number of elements grows large. This motivates the use of sparse or thinned arrays obtained by removing some of the elements [4, 5].

The number of elements, their weights, and element spacing in the array generally determine the array beam pattern, characterized by the peak side lobes height and main lobe width. One of the central objectives of sparse array design is minimizing the number of elements while maintaining a beam pattern identical to that of a fully populated array [4, 6].

Finding an optimal sparse array geometry in terms of the fewest elements is a combinatorial problem which is known to be NP hard [7]. The authors in [8] formally introduced the aperture coarray as a unifying concept applicable to both coherent and incoherent imaging, showing that some of the elements can be removed while preserving the beam pattern. Yet, no optimization for the minimal number of elements required given a number of effective aperture elements was presented. In [7] several approaches for designing sparse arrays in multiple dimensions were reviewed. However, the minimal number of elements required in each method was found numerically and no closed-form solution was derived. Lockwood et al. [4] stated that the minimal number of elements needed is proportional to the root of the number of effective aperture elements. They did not, however, consider the case where the number of elements in the effective aperture is not a perfect square. Mitra et al. [5] based their design of sparse arrays on polynomial factorization. The minimal number of elements was found through an exhaustive combinatorial search over all partitions. None of the works above considered a number of sub apertures greater than two.

In this paper we adopt previous work in multiple-input-multipleoutput (MIMO) radar [9] to study a sparse array configuration composed of two sub apertures which can be used in active sensing or combined via multiplicative beamforming in passive settings [10]. This allows to reduce the number of elements while maintaining a product beam pattern identical to that of a N element ULA. Using this approach, we derive a closed-form solution for the minimal number of elements required in passive sensing to obtain an effective aperture with N elements, proving that in general there are two optimal solutions. When N is a perfect square, the two solutions coincide, leading to an expression for the minimal number of elements given by  $2\sqrt{N} - 1$ . We broaden our work beyond two sub apertures by presenting an extension of the sparse array configuration to K > 2 sub apertures. We provide a closed-form solution for the minimal number of elements needed where K is also optimized, proving that the optimal solution with the fewest elements is given by the prime factorization of N. This allows for a significant reduction in the number of elements, showing that when N is a perfect power, the number of elements needed to form an effective beam pattern of an N element ULA is on the order of log(N).

This paper is organized as follows. Section 2 describes the model and a sparse array configuration based on the sum coarray. In Section 3 we derive the minimal number of elements required using two sub apertures with the proposed approach. We generalize the results in Section 4 by optimizing the number of sub apertures.

### 2. MODEL AND ARRAY SYNTHESIS

The far-field beam pattern at an angle  $\theta$  away from the broadside of a ULA with N isotropic elements is given by [11]

$$B(\theta) = \sum_{n=0}^{N-1} w(n) \exp\left(2\pi j \frac{d\sin(\theta)}{\lambda}n\right),\tag{1}$$

where w(n) is the weight function at element position n,  $\lambda$  is the wavelength and d is the inter-element spacing. In this work we consider the case of unity weights, i.e.  $w(n) = 1, 0 \le n \le N - 1$ . Thus, we have

$$B(\theta) = \sum_{n=0}^{N-1} \exp\left(2\pi j \frac{d\sin(\theta)}{\lambda}n\right).$$
 (2)

A ULA with missing elements is termed a sparse array. Every element missing corresponds to w(n) = 0 in (1). The problem ad-

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dressed in this paper is how to design a sparse arrays with the fewest elements which has an effective beam pattern as that of a ULA with N elements, given by (2).

Denoting  $x = e^{2\pi j d \sin(\theta)/\lambda}$ , we can represent the beam pattern as a polynomial

$$B(x) = \sum_{n=0}^{N-1} x^n.$$
 (3)

Assuming N is not prime, the polynomial given by (3) can be decomposed into a product of two lower order polynomials. This concept of polynomial factorization stands in the center of the work by Mitra et al. [5, 12, 6, 13, 14, 15] who designed sparse transmit and receive arrays with a combined effective beam pattern that is equivalent to that of a full array.

Given the decomposition  $N = N_1 N_2$  where  $N_1$  and  $N_2$  are positive integers, it follows that (3) can be rewritten as

$$B(x) = B_1(x)B_2(x),$$
 (4)

where

$$B_1(x) = 1 + x + x^2 + \dots + x^{N_1 - 1},$$
  

$$B_2(x) = 1 + x^{N_1} + x^{2N_1} + \dots + x^{(N_2 - 1)N_1}.$$

The polynomials  $B_1(x)$  and  $B_2(x)$  are used to construct two sub arrays whose combined effective aperture is the convolution of the sub array aperture functions [4]. This leads to a sparse array configuration used in MIMO radar [5, 16, 9] where the spacing at the transmitter is made  $N_1$  times that at the receiver. We can define this sparse array as the composition of two sub apertures with element locations given by the following sets

$$S_1 = \{nd, \quad n = 0, 1, ..., N_1 - 1\}$$
  

$$S_2 = \{nN_1d \quad n = 0, 1, ..., N_2 - 1\}.$$
(5)

Define the sum coarray of a pair of apertures  $A_1$  and  $A_2$  as the set [8]

$$C_{A_1A_2} = \{m : m = n_1 + n_2, n_1 \in A_1 \text{ and } n_2 \in A_2\}.$$
 (6)

Then, the sum coarray of the sets defined in (5) is given by

$$C_{S_1S_2} = \{nd, \quad n = 0, 1, ..., N - 1\},\tag{7}$$

i.e., the sum coarray is a ULA, allowing to obtain the desired beam pattern, as seen in Figure 1.



Fig. 1: Sparse Array Configuration for N = 9. (a) Full uniform linear array. (b) Sub aperture  $S_1$  with  $N_1 = 3$ . (c) Sub aperture  $S_2$  with  $N_2 = 3$ . (d) Sum coarray of  $S_1$  and  $S_2$ .

The sparse arrays (5) can be configured as two separate arrays (as in MIMO radar) or as part of a single array for passive settings using multiplicative beamforming [10]. Namely, the data is first beamformed from each sub array separately and then multiplied. The

arrays (5) are similar to nested arrays proposed by Vaidyanathan and Pal [16]. However, they are synthesized from the sum coarray perspective which arises naturally in multiplicative beamforming while nested arrays are related to difference coarray that occurs in problems involving the autocorrelation of the received signal. In addition, they require a smaller physical aperture.

Using the proposed sparse array geometry, our goal is first to find the minimal number of elements required for recovering the beam pattern by optimizing  $N_1$  and  $N_2$ . Then, we extend this approach to the case where the number of sub apertures is greater than two. Thus, we aim to achieve a further reduction in the number of elements by finding the optimal number of sub apertures and the element locations forming each one of them.

#### 3. DUAL SUB APERTURES

Given an arbitrary positive integer N > 1, we seek integers  $N_1$  and  $N_2$  which minimize the total number of elements while effectively obtaining the beam pattern of a ULA with N elements. While Mitra et al. [5] found the optimal solution by an exhaustive combinatorial search over all partitions, we provide below a closed-form expression.

Our problem can be cast as the following optimization problem:

$$\min_{\substack{N_1, N_2 \in \mathbb{N}^+ \\ \text{subject to} \quad N_1 N_2 = N.}} N_1 + N_2 - 1$$
(8)

When N is a prime number there are only two feasible solutions,  $N_1 = 1$  and  $N_2 = N$  and vice versa. Both solutions are optimal and lead to the full array configuration. A closed form solution for the case when N is not prime is given by the following theorem.

**Theorem 1.** Given a full ULA with N elements, let  $D_1$  and  $D_2$  be the sets defined as follows

$$\mathcal{D}_1 = \{n|N: n \le \sqrt{N}\}, \quad \mathcal{D}_2 = \{n|N: n \ge \sqrt{N}\},$$

where n|N denotes a divisor of N. Then, the optimum values for  $N_1$ and  $N_2$  are given by

$$N_{1} = \max(\mathcal{D}_{1}), N_{2} = \min(\mathcal{D}_{2}), N_{1} = \min(\mathcal{D}_{2}), N_{2} = \max(\mathcal{D}_{1}).$$
(9)

*Proof.* From the constraint in (8),  $N_2 = \frac{N}{N_1}$ . Assuming without loss of generality that  $N_1 \leq N_2$ , we have

$$N_1 = \underset{\mathcal{D}_1}{\operatorname{argmin}} \quad d + \frac{N}{d}, \tag{10}$$

where we neglect the constant term -1.

Define the function  $f: [1, \sqrt{N}] \to \mathcal{R}^+$  over a continuous domain

$$f(x) = x + \frac{N}{x}.$$

The function f(x) is continuous and differentiable over the open interval  $(1, \sqrt{N})$ . Its derivative is given by

$$\frac{df}{dx} = 1 - \frac{N}{x^2} < 0,$$

hence, f(x) is monotonically decreasing. Since  $\mathcal{D}_1 \subset [1, \sqrt{N}]$ , denoting  $N_1 = \max(\mathcal{D}_1)$ , we have

$$f(N_1) < f(d), \quad d \in \mathcal{D}_1$$

Therefore, the optimal solution is given by  $N_1 = \max(\mathcal{D}_1)$  and  $N_2 = \frac{N}{N_1} = \min(\mathcal{D}_2)$  accordingly. By interchanging the roles of  $N_1$  and  $N_2$  we get the solution for  $N_1 \ge N_2$ , given by  $N_1 = \min(\mathcal{D}_2)$  and  $N_2 = \max(\mathcal{D}_1)$ .

Theorem 1 states that in the general case there are two optimal solutions (see Fig. 2). Note, however, that although both choices offer the same minimal number of elements, the solution where  $N_1 \leq N_2$  might be preferable since it exhibits reduced mutual coupling compared to the second option. When  $\sqrt{N}$  is an integer max $(\mathcal{D}_1) = \min(\mathcal{D}_2) = \sqrt{N}$ , leading to the following corollary:

**Corollary 1.** Assuming  $\sqrt{N} \in \mathbb{N}^+$ , problem (8) has a unique solution. The optimum values for  $N_1$  and  $N_2$  are given by

$$N_1 = N_2 = \sqrt{N}.\tag{11}$$

Corollary 1 implies that the beam pattern of an N element ULA, where N is a perfect square, can be realized using only  $2\sqrt{N} - 1$  elements. For example, as shown in Fig. 3, a beam pattern of a 16 element ULA can be achieved using only 7 elements.



**Fig. 2: Optimal Solutions for** N = 18. (a) Beam pattern of a full ULA. (b) Beam pattern of a sub aperture with  $N_1 = 3$ . (c) Beam pattern of a sub aperture with  $N_2 = 6$ . (d) Beam pattern of a combined aperture with  $N_1 = 3$ ,  $N_2 = 6$ . (e) Beam pattern of a sub aperture with  $N_1 = 6$ . (f) Beam pattern of a sub aperture with  $N_2 = 3$ . (g) Beam pattern of a combined aperture with  $N_1 = 6$ ,  $N_2 = 3$ .

#### 4. MULTIPLE SUB APERTURES

So far, we considered using only two sub apertures for a combined effective full array. However, as shown in [5], in many cases the polynomial B(x) can be decomposed to a product of K lower order polynomials where K > 2. This allows to obtain a desired beam pattern with a further reduction in the number of elements required, as we show next.

Assume N can be written as a product

$$N = \prod_{i=1}^{K} N_i, \tag{12}$$

where  $\{N_i\}_{i=1}^K$  are positive integers greater than 1. Then, the sparse



Fig. 3: Optimal Solution for N = 16. (a) Beam pattern of a full ULA. (b) beam pattern of a sub aperture with  $N_1 = 4$ . (c) beam pattern of a sub aperture with  $N_2 = 4$  (d) beam pattern of a combined aperture with  $N_1 = 4$ ,  $N_2 = 4$ .

array configuration can be extended to K:

$$S_{1} = \{nd, \quad n = 0, ..., N_{1} - 1\}$$
  

$$S_{i} = \{nd \prod_{j=1}^{i-1} N_{j} \quad n = 0, ..., N_{i} - 1, i = 2, ..., K\}.$$
(13)

Broadening the definition of the sum coarray to multiple apertures

$$C_{A_1,\dots,A_K} = \{m : m = \sum_{i=1}^K n_i, n_i \in A_i\},$$
 (14)

we have that  $C_{S_1,...,S_K}$  is a filled ULA with N elements, allowing to attain a beam pattern of a full array. Note that here we deal with passive sensing where we use a single array, defined as the union of the sub arrays given by (13), for receiving only instead of active sensing as done in MIMO radar. After reception, multiplicative beamforming is applied to obtain an effective ULA.

To find the minimal number of elements needed using this approach, we define a generalized version of problem (8):

$$\min_{K \in \mathbb{N}^{+}} \min_{\substack{N_{1}, \dots, N_{K} \in \mathbb{N}^{+} \\ N_{1}, \dots, N_{K} \ge 2}} \sum_{i=1}^{K} N_{i} - (K-1)$$
subject to
$$\prod_{i=1}^{K} N_{i} = N.$$
(15)

The solution to (15) is given by the following theorem.

**Theorem 2.** Let N be the number of elements in a full ULA, represented by its prime factorization

$$N = \prod_{i=1}^{\omega} p_i^{q_i},$$

where  $\omega$  is the number of distinct prime factors of N. Define  $\Omega = \sum_{i=1}^{\omega} q_i$ . The optimal number of sub apertures K and the number of elements in each sub aperture are given by

$$K = \Omega, \quad \left\{ N_i \right\}_{i=1}^{\Omega} = \left\{ \underbrace{p_1, ..., p_1}_{q_1 \text{ times}}, ..., \underbrace{p_{\omega}, ..., p_{\omega}}_{q_{\omega} \text{ times}} \right\}.$$

*Proof.* Suppose to the contrary that the optimal solution satisfies  $K \neq \Omega$ . The fundamental theorem of arithmetic states that every positive integer has a unique prime factorization [17], hence,  $K \leq \Omega$ . Assume  $K < \Omega$ . Then there exists  $N_i$  which is not prime and  $N_i$  can be decomposed into the product of two smaller integers  $N_{i1}$  and  $N_{i2}$  where  $N_{i1}, N_{i2} \geq 2$ . This amounts to bisecting the *i*th aperture into 2 to yield K + 1 sub apertures. Assuming  $N_{i1} \leq N_{i2}$  without loss of generality, we have that

$$N_{i1} + N_{i2} \le 2N_{i2} \le N_{i1}N_{i2}. \tag{16}$$

Hence, breaking up the *i*th aperture into two levels decreases the value of the objective function at least by 1, in contradiction to the optimality of the solution.

Following the latter, we can go on splitting the sub apertures until all  $N_i$  are irreducible, i.e., prime numbers. This, along with the fact that the prime factorization is unique, implies that the total number of sub apertures is  $K = \Omega$  and the optimal  $\{N_i\}_{i=1}^{\Omega}$  are the prime factors of N.

Theorem 2 implies that a significant reduction in the number of elements can be obtained, leading to a minimal number of elements given by  $\sum_{i=1}^{\omega} p_i q_i - \Omega + 1$ . When  $N = 2^M$ , the optimal solution is to use M sub apertures, each with two elements where all sub apertures share the same first element, leading to a total number of M + 1 elements. This means that the minimal number of elements is on the order of  $\log(N)$  (See Fig. 4). Notice that when  $N = N_1 N_2$  and  $N_1, N_2 < N$  are two different prime numbers, the optimal solution is to use K = 2 sub apertures, reducing to the case discussed in Section 3.



Fig. 4: Optimal Solution for N = 16 with Multiple Apertures. (a) Beam pattern of a full ULA. (b) Beam pattern of a sub aperture with  $N_1 = 2$ . (c) Beam pattern of a sub aperture with  $N_2 = 2$ . (d) Beam pattern of a sub aperture with  $N_3 = 2$ . (e) Beam pattern of a sub aperture with  $N_4 = 2$ . (f) Beam pattern of the combined aperture of 4 sub apertures.

### 5. CONCLUSION

In this paper, we outlined a sparse array configuration, based on the sum coarray, which achieves the same beam pattern as a full array when used with active or passive sensing. Using this approach, we derived the minimal number of elements needed in passive mode, showing that only  $2\sqrt{N} - 1$  elements can be used to attain a beam pattern of an effective aperture with N elements. We then proposed an extension of this design beyond two sub apertures and proved that the minimal number of elements in passive settings is given by the prime factorization of N. Thus, the number of elements required is on the order of  $\log(N)$  when N is a perfect power. This demonstrates that a significant reduction in the number of elements can be achieved compared to previous approaches.

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