OPTIMUM SPARSE ARRAY DESIGN FOR MAXIMIZING SIGNAL-TO-NOISE RATIO IN PRESENCE OF LOCAL SCATTERINGS

Syed A. Hamza**, Moeness G. Amin** *and Giuseppe Fabrizio*†

* Center for Advanced Communications, Villanova University, Villanova, PA 19085, USA
† Defence Science and Technology Group, Edinburgh, SA, 5111, Australia
Emails: {shamza, moeness.amin}@villanova.edu; joe.fabrizio@dsto.defence.gov.au

ABSTRACT

Optimum sparse array design for maximum output signal-to-noise ratio (MaxSNR) and signal-to-interference ratio (MaxSINR) have been shown to yield significant performance improvement compared to random or environmental-independent structured arrays, with the same number of antennas. We examine the MaxSNR problem in presence of local scatterings which may follow specific deterministic or statistical scattering models. It is shown that if the scatterers assume a Gaussian distribution centered around the source angular position, the optimum array configuration for maximizing the SNR is the commonly used uniform linear array (ULA). If the scatterers circulate the source, the optimum design yields sparse array topologies with superior performance over ULAs. Simulation results are presented to show the effectiveness of array configurability in the case of both Gaussian and circularly spread sources.

***Index Terms*—** MaxSNR, MaxSINR, ULA, sparse array, Gaussian distribution, circularly spread sources

1. INTRODUCTION

Sparse arrays have recently attracted much attention that is motivated by switched antenna technologies and advances in constrained minimization and convex optimization techniques. There are several metrics to design sparse arrays and decide on optimum array configurations. Among those metrics, maximum signal-to-noise ratio (MaxSNR), maximum signal-to-interference and noise ratio (MaxSINR), and reduced Cramer-Rao bound (CRB) for direction-of-arrival (DOA) estimation, yield improved beamforming and direction finding performance [1–6].

Designing optimum sparse arrays for MaxSNR strives to maximize the SNR at the array output for a given source direction-of-arrival (DOA). Depending on the number of antennas and permissible antenna locations, it has been shown that a significant performance improvement can be achieved over other commonly used sparse arrays, including nested and coprime structured array [7–9]. In past contributions, point sources have typically been assumed where each source is characterized by a steering vector and provides a rank-one spatial covariance matrix at the array receiver. However, depending on the multipath environments and source signal bandwidth, these steering vectors, along with the corresponding covariance matrix rank, can significantly change [10–13].

In this paper, the effect of the spatial channel on optimum sparse array beamforming for MaxSNR is examined for the first time. Two models for local scatterings, namely, the Gaussian model and circular model, are considered. These scattering models are most suitable for dense urban environment, in which the signal encounter rich scattering prior to its arrival at the array receiver [14–17]. It is shown that the optimum sparse array always elects a configuration that seeks the highest spatial correlation values across the antennas. On the other hand for a spatial correlation that is monotonically decreasing, the sparse array would assume the minimum possible antenna separation, so as to collect the highest correlation values. This is accomplished by configuring a ULA. For those scattering models in which the correlation rises and falls with increased antenna spacing, the optimum MaxSNR sparse array configuration deviates from a ULA, and positions the antennas such that their separations are consistent with the highest sensor correlation values.

We pose the problem as optimally selecting K antennas out of N possible equally spaced locations on a uniform grid. The antenna selection problem for maximizing SNR amounts to maximizing the principal eigenvalue of the source correlation matrix [18]. In order to realize convex relaxation for this NP hard optimization problem and avoid computational burden of singular value decomposition for each possible configuration, we maximize the lower bound of output SNR instead. The lower bound optimization objective is approximated by Taylor series and formulated as an iterative linear program.

The rest of the paper is organized as follows: In section 2, we formulate the problem for maximizing the output SNR. The role of array configurability for MaxSNR for Gaussian and circular scattering models is explained in section 3. Section 4 deals with the iterative solution of finding optimum array design. Simulations and conclusion follows at the end.

The work by S. Hamza and M. Amin is supported by NSF award # AST-1547420.

2. PROBLEM FORMULATION

Consider a spatially spread source with P independently scattered components impinging on a linear array with N uniformly placed antennas. Then, the signal received at the array at time instant t is given by:

$$\mathbf{x}(t) = \sum_{k=1}^{P} (\alpha_k(t))\mathbf{a}(\theta_k) + \mathbf{n}(t), \quad (1)$$

where
$\mathbf{a}(\theta_k) = [1 \;\; e^{j(2\pi/\lambda)dcos(\theta_k)} \ldots e^{j(2\pi/\lambda)d(N-1)cos(\theta_k)}]^T$,
is the steering vector corresponding to the kth scatterer with the direction-of-arrival θ_k, d is the inter-element spacing in terms of wavelengths λ, $\alpha_k(t) \in \mathbb{C}$ represents the complex amplitute of kth scatterer and $\mathbf{n}(t) \in \mathbb{C}^N$ represents the additive Gaussian noise with variance σ_n^2 at the receiver output. The received signal $\mathbf{x}(t)$ is linearly combined by the N-antenna beamformer that strives to maximize the output SNR. The output signal $y(t)$ of the optimum beamformer for maximum SNR is given by [18],

$$y(t) = \mathbf{w_0}^H \mathbf{x}(t), \quad (2)$$

with

$$\mathbf{w_0} = \mathscr{P}\{\mathbf{R_n}^{-1}\mathbf{R}\}. \quad (3)$$

The operator $\mathscr{P}\{.\}$ computes the principal eigenvector, $\mathbf{R} = \mathbf{U R_s U^H}$ is the received source correlation matrix with $\mathbf{U} = [\mathbf{a}(\theta_1) \ldots \mathbf{a}(\theta_P)]$ and k, lth entry of $\mathbf{R_s}(k,l) = E\{\alpha_k(t)\alpha_k^H(t)\}$ for $k = l$ and zero otherwise. For spatially uncorrelated noise, $\mathbf{R_n} = \sigma_n^2\mathbf{I}$, we obtain the corresponding optimum output SNR_o as follows:

$$\mathbf{w_0} = \mathscr{P}\{\mathbf{R}\}, \quad (4)$$

$$SNR_o = \frac{\mathbf{w_0^H R w_0^H}}{\mathbf{w_0 R_n w_0}} = \frac{||\mathbf{R}||_2}{\sigma_n^2}. \quad (5)$$

Here, $||.||_2$ denotes the spectral norm or the maximum eigenvalue of the matrix. Equations (4) and (5) show that the optimum beamformer for MaxSNR is directly tied to the eigenstructure of the correlation matrix. As such, there is a need to analyze the correlation matrix under the scattering models.

3. GAUSSIAN AND CIRCULAR SCATTERING MODELS

Gaussian model assumes that the directions of arrival of the scatterers are Gaussianly distributed, $\mathscr{N}(\theta_0, \sigma_\theta)$, having mean direction of arrival θ_0 and variance σ_θ. Consequently, each element of the steering vector would be jointly Gaussian with zero mean and covariance matrix given by [14],

$$\mathbf{R}_{\mathbf{g}_{(\theta_0,\sigma_0)}} \approx (\mathbf{a}(\theta_0)\mathbf{a}^H(\theta_0)) \circ \mathbf{B}_{(\theta_0,\sigma_0)}, \quad (6)$$

where '$\circ$' denotes Hadamard product and

$$\mathbf{B}_{(\theta_0,\sigma_0)}(k,l) = e^{-2(\pi\delta(k-l))^2\sigma_0^2 cos^2(\theta_0)}. \quad (7)$$

Denote $\mathbf{z} \in \{0,1\}^N$ a selection vector whose entries 0's and 1's represents the presence and absence of corresponding antennas respectively. The steering vector corresponding to antenna selection vector $\mathbf{z}$ is given by $\mathbf{a}_{\theta_0}(\mathbf{z}) = \mathbf{a}(\theta_0) \odot \mathbf{z}$. Here '$\odot$' is the element-wise product operator which allows the selection of antenna elements according to $\mathbf{z}$. Similarly, $\mathbf{B}_{(\theta_0,\sigma_0)}(\mathbf{z}) = \mathbf{B}_{(\theta_0,\sigma_0)} \odot \mathbf{Z}$ with $\mathbf{Z} = \mathbf{z}\mathbf{z}^T$ being the corresponding antenna selection matrix. Equation (6) with selected antennas can be re-written as follows:

$$\mathbf{R}_{\mathbf{g}_{(\theta_0,\sigma_0)}}(\mathbf{z}) \approx (\mathbf{a}_{\theta_0}(\mathbf{z})\mathbf{a}_{\theta_0}^H(\mathbf{z})) \circ \mathbf{B}_{(\theta_0,\sigma_0)}(\mathbf{z}). \quad (8)$$

We note that the trace of $\mathbf{R}_{\mathbf{g}_{(\theta_0,\sigma_0)}}(\mathbf{z})$ is constant since the input source power remains the same irrespective of the array configuration. Accordingly, the sum of eigenvalues is constant for all possible correlation matrices associated with the K antenna selection problem. To be more explicit, the problem formulated in Eq. (5) can be expressed as:

$$\begin{aligned} &\max_{\mathbf{z}} \quad ||\mathbf{R}_{\mathbf{g}_{(\theta_0,\sigma_0)}}(\mathbf{z})||_2, \\ &\text{given; } \sum_{k=1}^{K} v_{\mathbf{z}}(k) = \mathbf{Tr}\left(\mathbf{R}_{\mathbf{g}_{(\theta_0,\sigma_0)}}(\mathbf{z})\right) = K\mathbf{Tr}(\mathbf{R_s}), \end{aligned} \quad (9)$$

where $\mathbf{Tr}\,(.)$ denotes the trace of the matrix, $v_{\mathbf{z}}(k)$ is the kth eigenvalue of correlation matrix $\mathbf{R}_{\mathbf{g}_{(\theta_0,\sigma_0)}}(\mathbf{z})$. Equations (6) and (7) show that the correlation drops monotonically with increased correlation lag. As shown below, this property compels the optimum sparse array to assume a ULA with minimum inter-element spacing; hence, the solution does not require any iteration-based method or enumeration.

Let $\mathbf{R}_{\mathbf{g}_{(\theta_0,\sigma_0)}}(\mathbf{u})$ be the correlation matrix for ULA "$\mathbf{u}$" and $\mathbf{R}_{\mathbf{g}_{(\theta_0,\sigma_0)}}(\mathbf{s})$ be the correlation matrix associated with the sparse "$\mathbf{s}$" configuration with the same number of antennas, K. The kth eigenvalue $v_{\mathbf{z}M}(k)$ of $\mathbf{R}^M_{\mathbf{g}(\theta_0,\sigma_0)}(\mathbf{z})$ is related to its corresponding eigenvlaue $v_{\mathbf{z}}(k)$ of $\mathbf{R}_{\mathbf{g}_{(\theta_0,\sigma_0)}}(\mathbf{z})$ by [19],

$$v_{\mathbf{z}M}(k) = v^M_{\mathbf{z}}(k), \quad \forall M \geq 0. \quad (10)$$

For the Gaussian spatial channel, $\mathbf{Tr}\;(\mathbf{R}^M_{\mathbf{g}(\theta_0,\sigma_0)}(\mathbf{u})) > \mathbf{Tr}\,(\mathbf{R}^M_{\mathbf{g}(\theta_0,\sigma_0)}(\mathbf{s}))$ (proof in Appendix) along with Eq. (10), yields

$$\sum_{k=1}^{K} v^M_{\mathbf{u}}(k) > \sum_{k=1}^{K} v^M_{\mathbf{s}}(k), \quad \forall M \geq 2. \quad (11)$$

From Eq. (11), it can be readily shown that $max\,(v_{\mathbf{u}}(k)) > max\,(v_{\mathbf{s}}(k))$. Here, we make use of the fact that all the eigenvalues of the correlation matrices are greater or equal to zero. Therefore, for the Gaussian scattering model where the correlation function is monotonically decreasing with sensors spacing, the optimum sparse array would always cluster with minimum spatial separation, configuring a ULA.

The correlation between consecutive sensors for the circular model is given by [20],

$$r_c(\theta_0, \sigma_0) = \frac{1}{P}\sum_{i=0}^{P-1} e^{-j2\pi(\delta(k-l))cos(\theta_0+\theta_i)}, \quad (12)$$

where $r_c(\theta_0, \sigma_0)$ is the k,lth element of the correlation matrix $\mathbf{R}_{\mathbf{c}(\theta_0,\sigma_0)}$ and θ_i's are circularly distributed around the source. Contrary to Gaussian model, the sensor data correlation in circular case shows oscillatory behaviour as a function of lags (Eq. (12)). We can, however, bound the upper and lower limits of the optimum SNR as follows:

$$\frac{\mathbf{Tr}\left(\mathbf{R}_{\mathbf{c}(\theta_0,\sigma_0)}(\mathbf{w})\right)}{K} \leq SNR_o \leq \mathbf{Tr}\left(\mathbf{R}_{\mathbf{c}(\theta_0,\sigma_0)}(\mathbf{o})\right). \quad (13)$$

$\mathbf{R}_{\mathbf{c}(\theta_0,\sigma_0)}(\mathbf{o})$ and $\mathbf{R}_{\mathbf{c}(\theta_0,\sigma_0)}(\mathbf{w})$ are the optimum "$\mathbf{o}$" and worst "$\mathbf{w}$" array correlation matrices, respectively. In the optimal case, the eigenvalues are maximally spread, whereas the worst possibility for MaxSNR arises when all the eigenvalues are equal. We also note that Eq. (11) in fact determines the eigen-spread of the correlation matrix asymptotically. It can be shown that this equation remains valid for circular correlation matrix for some ζ sufficiently large, such that $\sum_{k=1}^{K} v_{\mathbf{o}}^{M}(k) > \sum_{k=1}^{K} v_{\mathbf{s}}^{M}(k), \quad \forall M \geq \zeta$. Therefore, finding the optimum configuration amounts to maximizing $\sum_{k=1}^{K} v_{\mathbf{z}}^{M}(k)$ or equivalently, Tr $(\mathbf{R}^{M}_{\mathbf{c}(\theta_0,\sigma_0)}(\mathbf{z}))$ for any $M \geq \zeta$, over all posssible configurations. Though maximizing Tr $(\mathbf{R}^{M}_{\mathbf{c}(\theta_0,\sigma_0)}(\mathbf{z}))$ is computationally less expensive than maximizing the principal eigenvalue, yet it is highly computationally involved. Therefore, in the next section we resort to lower bound relaxation to design the optimum configuration.

4. OPTIMUM SPARSE ARRAY DESIGN

Following the approach in [21], we assume that we have an estimate of $\mathbf{R}_{(\theta_\mathbf{0},\sigma_\mathbf{0})}$, which is the full antenna array source correlation matrix. Then, the problem in Eq. (9) can be rewritten as follows:

$$\begin{aligned} \underset{\mathbf{z}}{\text{max}} \quad & ||\mathbf{R}_{(\theta_\mathbf{0},\sigma_\mathbf{0})}(\mathbf{z})||_2, \\ \text{s.t.} \quad & ||\mathbf{z}||_0 = K. \end{aligned} \quad (14)$$

Here, $||.||_0$ determines the cardinality of selection vector $\mathbf{z}$. Given $\mathbf{e_0}$, the principal eigenvector corresponding to the full antenna array, we approximate the thinned vector $\mathbf{e_0}(\mathbf{z})/||\mathbf{e_0}(\mathbf{z})||_2$ as the principal eigenvector of the selected K-element subarray $\mathbf{z}$ [22]. The problem in Eq. (14) can then be approximated by:

$$\begin{aligned} \underset{\mathbf{z}}{\text{max}} \quad & \frac{\mathbf{e_0}^H(\mathbf{z})\mathbf{R}_{(\theta_\mathbf{0},\sigma_\mathbf{0})}(\mathbf{z})\mathbf{e_0}(\mathbf{z})}{||\mathbf{e_0}(\mathbf{z})||_2^2}, \\ \text{s.t.} \quad & ||\mathbf{z}||_0 = K. \end{aligned} \quad (15)$$

This approximation represents a lower bound of optimum SNR. Define, $\tilde{\mathbf{R}}_{\theta,\sigma_0} = \mathbf{R}_{(\theta_0,\sigma_0)} \circ (\mathbf{e_0}\mathbf{e_0}^H)$ and $\tilde{\mathbf{e_o}} = \mathbf{e_o}^* \circ \mathbf{e_o}$. Equation (15) can be rephrased as follows:

$$\begin{aligned} \underset{\mathbf{z}}{\text{max}} \quad & \frac{\mathbf{z}^T\tilde{\mathbf{R}}_{\theta,\sigma_\mathbf{0}}\mathbf{z}}{\mathbf{z}^T\tilde{\mathbf{e_0}}}, \\ \text{s.t.} \quad & ||\mathbf{z}||_1 = K, \\ & 0 \leq \mathbf{z} \leq 1. \end{aligned} \quad (16)$$

The constraint in Eq. (15) is relaxed to affine equality constraint ($||.||_1$ denotes l^1-norm) and a box constraint, but the objective function still remains non-convex. Therefore, we resort to iterative first order Taylor approximation as follows:

$$\begin{aligned} \underset{\mathbf{z}}{\text{max}} \quad & \frac{-\mathbf{z}^{iT}\tilde{\mathbf{R}}_{\theta,\sigma_\mathbf{0}}\mathbf{z}^{i} + 2\mathbf{z}^{iT}\tilde{\mathbf{R}}_{\theta,\sigma_\mathbf{0}}\mathbf{z}}{\mathbf{z}^T\tilde{\mathbf{e_0}}}, \\ \text{s.t.} \quad & ||\mathbf{z}||_1 = K, \\ & 0 \leq \mathbf{z} \leq 1. \end{aligned} \quad (17)$$

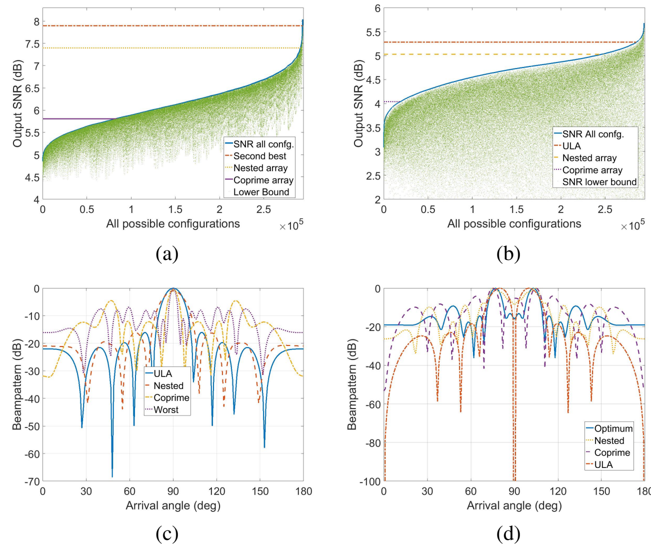

Fig. 1: Output SNR comparison for various array configurations; (a) Gaussian model; (b) circular model. Beampattern of optimum array; (c) Gaussian model; (d) circular model.

Here, i is the iteration number. This linear fractional program (LFP) can be turned into LP by simple change of variables, $\alpha = 1/\mathbf{z}^T\tilde{\mathbf{e_0}}$ and $\mathbf{v} = \mathbf{z}/\mathbf{z}^\mathbf{T}\tilde{\mathbf{e_0}}$ as follows [23],

$$\begin{aligned} \underset{\mathbf{v},\alpha}{\text{max}} \quad & -\mathbf{z}^{iT}\tilde{\mathbf{R}}_{\theta,\sigma_\mathbf{0}}\mathbf{z}^{i}\alpha + 2\mathbf{z}^{iT}\tilde{\mathbf{R}}_{\theta,\sigma_\mathbf{0}}\mathbf{v}, \\ \text{s.t.} \quad & \mathbf{1}^T\mathbf{v} - K\alpha = 0, \;\; \tilde{\mathbf{e_0}}^T\mathbf{v} = 1, \\ & \mathbf{v} \geq 0, \;\; \mathbf{v} - \alpha \leq 0, \;\; \alpha \geq 0. \end{aligned} \quad (18)$$

yielding, the estimate of the selection vector $\mathbf{z} = \mathbf{v}/\alpha$.

5. SIMULATIONS

For both scattering models, we select $K = 9$ sensors from $N = 21$ possible equally spaced locations with inter-element spacing of $\lambda/2$. The source SNR is 0 dB.

5.1. Gaussian model

Figure 1(a) plots the optimum SNR for all possible array configurations in ascending order of output SNR for Gaussian spread source with center angle of 90^0 and variance of 5^0. As expected, the optimum array emerges as a ULA with output SNR of 8 dB. The lower bound relaxation is also depicted in Fig. 1(a) and is shown to offer a good metric in the underlying case. The ULA has more than 3 dB advantage over the worst array configuration having less than 5 dB output SNR. The worst array configuration is shown in the Fig. 2(a) where "." and "$\times$" represent the presence and absence of sensor respectively. We also observe that if sensor malfunction prevents a contiguous uniformly spaced antenna array configuration, the optimum sparse array opts to minimally stretch, incorporating the next closest antenna position, as shown in

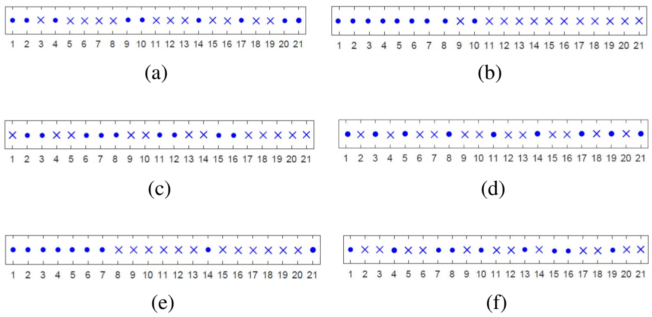

Fig. 2: (a) Worst sparse array (Gaussian); (b) Second best (Gaussian); (c) Optimum array (circular); (d) Worst sparse array (circular); (e) Nested array; (f) Coprime array

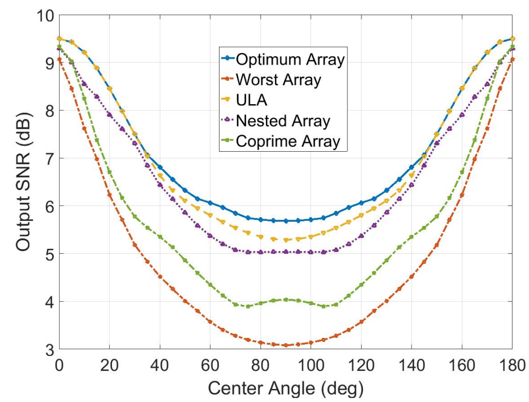

Fig. 3: Output SNR for different arrays vs center angle.

the Fig. 2(b). In order to simulate this case, we set antenna 9 as a faulty antenna. The output SNR corresponding to this new configuration is 7.9 dB which is 0.1 dB less than the optimum array configuration. This lower SNR is the price paid by including a smaller correlation value corresponding to antenna position 10 compared to that of antenna position 9. The output SNR corresponding to coprime and nested sparse arrays, as depicted in Fig. 1(a), are significantly less as compared to the ULA. Figure 1(c) shows the optimum beampattern corresponding to the different configurations with ULA giving the widest main lobe with the highest gain possible at the scatterer's center. Moreover, the sidelobes for ULA are significantly lower as compared to other sparse arrays which is highly desirable.

5.2. Circular model

Figure 1(b) shows the scenario for circular scattering model with a single source at the broadside with the spatial spread of 30^0. It shows that the optimum sparse array (shown in Fig. 2(c)) has SNR of 5.68 dB, whereas the worst sparse array (shown in Fig. 2(d)) has output SNR of 3 dB, which is more than 2.5 dB down as compared to the optimum array topology. It is informative to compare the performance of optimum array with the ULA, which is a de facto array configuration in many applications. The ULA gives output SNR of 5.28 dB which is 0.4 dB lower than the optimum performance. The significant difference however lies in the beampattern of the two arrays, as shown in Fig. 1(d). Contrary to the optimum configuration, the ULA attempts to maximize the output SNR by placing a null exactly in the center of the scatterer beam which is highly undesirable. Figure 1(b) also shows that the optimum sparse array has a clear advantage over coprime and nested array topologies in terms of output SNR. This is beacuse the optimum array manages wider mainlobe with higher gain where the scatterers are most dense (Fig. 1(d)). Figure 3 shows that better performance of the optimum sparse array is more pronounced at the broadside, whereas the ULA is the optimum array configuration for DOAs near the array end-fire location. It can be seen that the performance of the optimum array and sub optimum sparse arrays differ significantly over a wide field of view.

6. CONCLUSION

This paper considered optimum sparse array configuration for maximizing the beamformer output SNR for a single source that is seen to the receiver through its local scatterers. It is shown that for the Gaussian local scattering model, the correlation is weakened monotonically across the receiver antennas. As such, the optimum configuration, in seeking to capture the highest spatial correlation values, becomes the ULA. We showed that for a circular local scattering model, the optimum sparse array loses uniformity in quest of including the high correlation values corresponding to large antenna separations. In both cases, we solved the optimization problem both iteratively and by enumerations and showed strong agreement between the two methods.

7. APPENDIX

Proof of the result used in Eq. (11):

$$\begin{aligned}
&\mathbf{Tr}\left(\mathbf{R}^{M}_{\mathbf{g}(\theta_0,\sigma_0)}(\mathbf{z})\right) \\
&\quad\approx \mathbf{Tr}\left((\mathbf{a}_{\theta_0}(\mathbf{z})\mathbf{a}^{H}_{\theta_0}(\mathbf{z}) \circ \mathbf{B}_{(\theta_0,\sigma_0)}(\mathbf{z}))^{M}\right) \\
&\quad= \mathbf{Tr}\left(\mathbf{a}_{\theta_0}(\mathbf{z})\mathbf{a}^{H}_{\theta_0}(\mathbf{z}) \circ \mathbf{B}^{M}_{(\theta_0,\sigma_0)}(\mathbf{z})\right) \\
&\quad= \mathbf{Tr}\left(\mathbf{B}^{M}_{(\theta_0,\sigma_0)}(\mathbf{z})\right).
\end{aligned} \tag{19}$$

For Gaussian model, $\mathbf{B}_{(\theta_0,\sigma_0)}(\mathbf{u}) \geq \mathbf{B}_{(\theta_0,\sigma_0)}(\mathbf{s}) > 0$. Here, '$\geq$' means element wise comparison and strict equality holds only for diagonal entries. This implies,

$$\begin{aligned}
\mathbf{B}^{M}_{(\theta_0,\sigma_0)}(\mathbf{u}) &> \mathbf{B}^{M}_{(\theta_0,\sigma_0)}(\mathbf{s}), \qquad \forall M \geq 2 \\
\mathbf{Tr}\left(\mathbf{B}^{M}_{(\theta_0,\sigma_0)}(\mathbf{u})\right) &> \mathbf{Tr}\left(\mathbf{B}^{M}_{(\theta_0,\sigma_0)}(\mathbf{s})\right).
\end{aligned} \tag{20}$$

Combining (19) and (20), we have,

$$\mathbf{Tr}\left(\mathbf{R}^{M}_{\mathbf{g}(\theta_0,\sigma_0)}(\mathbf{u})\right) > \mathbf{Tr}\left(\mathbf{R}^{M}_{\mathbf{g}(\theta_0,\sigma_0)}(\mathbf{s})\right) \quad \forall M \geq 2$$

□

8. REFERENCES

[1] O. Mehanna, N. D. Sidiropoulos, and G. B. Giannakis, "Joint multicast beamforming and antenna selection," *IEEE Transactions on Signal Processing*, vol. 61, no. 10, pp. 2660–2674, May 2013.

[2] X. Wang, E. Aboutanios, M. Trinkle, and M. G. Amin, "Reconfigurable adaptive array beamforming by antenna selection," *IEEE Transactions on Signal Processing*, vol. 62, no. 9, pp. 2385–2396, May 2014.

[3] X. Wang, M. G. Amin, X. Wang, and X. Cao, "Sparse array quiescent beamformer design combining adaptive and deterministic constraints," *IEEE Transactions on Antennas and Propagation*, vol. PP, no. 99, pp. 1–1, 2017.

[4] N. D. Sidiropoulos, T. N. Davidson, and Z.-Q. Luo, "Transmit beamforming for physical-layer multicasting," *IEEE Transactions on Signal Processing*, vol. 54, no. 6, pp. 2239–2251, June 2006.

[5] V. Roy, S. P. Chepuri, and G. Leus, "Sparsity-enforcing sensor selection for DOA estimation," in *2013 5th IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP)*, Dec. 2013, pp. 340–343.

[6] W. Roberts, L. Xu, J. Li, and P. Stoica, "Sparse antenna array design for MIMO active sensing applications," *IEEE Transactions on Antennas and Propagation*, vol. 59, no. 3, pp. 846–858, March 2011.

[7] P. Pal and P. P. Vaidyanathan, "Nested arrays: A novel approach to array processing with enhanced degrees of freedom," *IEEE Transactions on Signal Processing*, vol. 58, no. 8, pp. 4167–4181, Aug. 2010.

[8] ——, "Coprime sampling and the MUSIC algorithm," in *2011 Digital Signal Processing and Signal Processing Education Meeting (DSP/SPE)*, Jan. 2011, pp. 289–294.

[9] S. Qin, Y. D. Zhang, and M. G. Amin, "Generalized coprime array configurations for direction-of-arrival estimation," *IEEE Transactions on Signal Processing*, vol. 63, no. 6, pp. 1377–1390, March 2015.

[10] G. G. Raleigh and V. K. Jones, "Multivariate modulation and coding for wireless communication," *IEEE Journal on Selected Areas in Communications*, vol. 17, no. 5, pp. 851–866, May 1999.

[11] H. Bolcskei, D. Gesbert, and A. J. Paulraj, "On the capacity of OFDM-based spatial multiplexing systems," *IEEE Transactions on Communications*, vol. 50, no. 2, pp. 225–234, Feb. 2002.

[12] P. Chavali and A. Nehorai, "Cognitive radar for target tracking in multipath scenarios," in *2010 International Waveform Diversity and Design Conference*, Aug. 2010, pp. 000 110–000 114.

[13] J. L. Krolik, J. Farrell, and A. Steinhardt, "Exploiting multipath propagation for GMTI in urban environments," in *2006 IEEE Conference on Radar*, April 2006, pp. 4 pp.–.

[14] R. B. Ertel, P. Cardieri, K. W. Sowerby, T. S. Rappaport, and J. H. Reed, "Overview of spatial channel models for antenna array communication systems," *IEEE Personal Communications*, vol. 5, no. 1, pp. 10–22, Feb. 1998.

[15] G. Raleigh, S. N. Diggavi, A. F. Naguib, and A. Paulraj, "Characterization of fast fading vector channels for multi-antenna communication systems," in *Proceedings of 1994 28th Asilomar Conference on Signals, Systems and Computers*, vol. 2, Oct. 1994, pp. 853–857 vol.2.

[16] A. Abdi and M. Kaveh, "A space-time correlation model for multielement antenna systems in mobile fading channels," *IEEE Journal on Selected Areas in Communications*, vol. 20, no. 3, pp. 550–560, April 2002.

[17] A. Abdi, J. A. Barger, and M. Kaveh, "A parametric model for the distribution of the angle of arrival and the associated correlation function and power spectrum at the mobile station," *IEEE Transactions on Vehicular Technology*, vol. 51, no. 3, pp. 425–434, May 2002.

[18] S. Shahbazpanahi, A. B. Gershman, Z.-Q. Luo, and K. M. Wong, "Robust adaptive beamforming for general-rank signal models," *IEEE Transactions on Signal Processing*, vol. 51, no. 9, pp. 2257–2269, Sept. 2003.

[19] R. M. Gray, "Toeplitz and circulant matrices: A review," *Foundations and Trends in Communications and Information Theory*, vol. 2, no. 3, pp. 155–239, 2006.

[20] W. C. Y. Lee, *Mobile Communications Engineering: Theory and Applications*. New York, NY, USA: McGraw-Hill, Inc., 1997.

[21] X. Wang, M. Amin, and X. Cao, "Analysis and design of optimum sparse array configurations for adaptive beamforming," *IEEE Transactions on Signal Processing*, vol. 66, no. 2, pp. 340–351, Jan. 2018.

[22] X. Wang, M. G. Amin, and X. Cao, "Optimum adaptive beamformer design with controlled quiescent pattern by antenna selection," in *2017 IEEE Radar Conference (RadarConf)*, May 2017, pp. 0749–0754.

[23] S. Boyd and L. Vandenberghe, *Convex Optimization*. New York, NY, USA: Cambridge University Press, 2004.