AN IMPROVED DOA ESTIMATOR BASED ON PARTIAL RELAXATION APPROACH

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ABSTRACT

In the partial relaxation approach, at each desired direction, the manifold structure of the remaining interfering signals impinging on the sensor array is relaxed, which results in closed form estimates for the interference parameters. By adopting this approach, in this paper, a new estimator based on the unconstrained covariance fitting problem is proposed. To obtain the null-spectra efficiently, an iterative rooting scheme based on the rational function approximation is applied. Simulation results show that the performance of the proposed estimator is superior to the classical and other partial relaxation methods, especially in the case of low number of snapshots, irrespectively of any specific structure of the sensor array while maintaining a reasonable computational cost.

Index Terms— DOA Estimation, Partial Relaxation Approach, Covariance Fitting, Eigenvalue Decomposition, Rank-one Modification Problem

1. INTRODUCTION

Direction-of-Arrival (DOA) estimation has been a fundamental and long-established application in sensor array processing. The application of DOA estimation spans multiple fields of research, including wireless communication, radio astronomy, sonar, etc. [1, 2, 3, 4].

Recently, the partial relaxation approach [5] has emerged as a promising approach for DOA estimation. Similar to the conventional maximum likelihood estimators, the partial relaxation methods take both the effect of the signal in the direction of interest and the interfering directions into account. However, the manifold structure from the array geometry of the received signals from interfering directions is relaxed to make the problem computationally tractable, hence the name partial relaxation. In comparison to the corresponding conventional multidimensional fitting methods, the partial relaxation methods have a lower computational complexity while obtaining superior error performance as compared to the conventional spectral search algorithms.

Under the partial relaxation approach, the covariance fitting estimator with the positive semidefinite (PSD) constraint, referred to as the constrained covariance fitting estimator, poses several open questions despite obtaining the best threshold performance under the investigated scenarios [5]. First, the PSD constraint that has been added to the covariance fitting problem appears to be arbitrary and has merely been introduced to make the problem computationally tractable. Second, if the sample covariance matrix is singular, which occurs if the number of snapshots is smaller than the number of antennas, or theoretically in the noiseless case, the inverse of the sample covariance matrix does not exist. As a consequence, the constrained covariance fitting is not applicable. Therefore, in this paper, a new covariance fitting estimator which does not consider the PSD constraint is proposed and efficiently solved. It can be shown that this estimator outperforms other estimators in the partial relaxation family and also the conventional subspace methods.

The paper is organized as follows. The signal model is introduced in Section 2. A short overview of the partial relaxation approach and the formulation of the unconstrained covariance fitting estimator, are presented in Section 3. An computationally efficient method, which can be applied to all methods of the partial relaxation family, is presented in Section 4. To illustrate the performance gain of the proposed methods, simulation results are presented in Section 5. Lastly in Section 6, some remarks and extensions to further research are discussed.

2. SIGNAL MODEL

Consider an array of M sensors receiving N narrowband signals emitted from the sources with the corresponding unknown DOAs $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_N]^T$. Furthermore, assume that $N < M$. The measurement vector at the sensor array $\boldsymbol{x}(t) = [x_1(t), \ldots, x_M(t)]^T \in \mathbb{C}^{M\times 1}$ in the baseband at the time instance t is modeled as:

$$\boldsymbol{x}(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}(t) + \boldsymbol{n}(t) \text{ with } t = 1, \ldots, T, \qquad (1)$$

where $\boldsymbol{s}(t) = [s_1(t), \ldots, s_N(t)]^T \in \mathbb{C}^{N\times 1}$ is the baseband source signal vector from N sources and $\boldsymbol{n}(t) \in \mathbb{C}^{M\times 1}$ denotes represents the additive circularly complex noise vector at the sensor array with the noise covariance matrix $\mathbb{E}\{\boldsymbol{n}(t)\boldsymbol{n}(t)^H\} = \sigma_n^2 \boldsymbol{I}_M$. The steering matrix $\boldsymbol{A}(\boldsymbol{\theta})$ in (1), which is assumed to have full column rank, is given by:

$$\boldsymbol{A}(\boldsymbol{\theta}) = [\boldsymbol{a}(\theta_1), \ldots, \boldsymbol{a}(\theta_N)], \qquad (2)$$

where $\boldsymbol{a}(\theta_n) \in \mathbb{C}^{M\times 1}$ denotes the sensor array response for the DOA θ_n. The equation in (1) can be rewritten for multiple snapshots $t = 1, \ldots, T$ in a compact notation as:

$$\boldsymbol{X} = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{S} + \boldsymbol{N}, \tag{3}$$

where $\boldsymbol{X} = [\boldsymbol{x}(1), \ldots, \boldsymbol{x}(T)] \in \mathbb{C}^{M\times T}$ is the received baseband signal matrix. In a similar manner, we define the source signal matrix $\boldsymbol{S} \in \mathbb{C}^{N\times T}$ and sensor noise matrix $\boldsymbol{N} \in \mathbb{C}^{M\times T}$ as $\boldsymbol{S} = [\boldsymbol{s}(1), \ldots, \boldsymbol{s}(T)]$ and $\boldsymbol{N} = [\boldsymbol{n}(1), \ldots, \boldsymbol{n}(T)]$, respectively.

Assume that the source signals and the noise are uncorrelated, then the true covariance matrix $\boldsymbol{R}$ of the received signal $\boldsymbol{y}(t)$ is given by $\boldsymbol{R} = \mathbb{E}\{\boldsymbol{x}(t)\boldsymbol{x}(t)^H\} = \boldsymbol{A}\boldsymbol{R}_s\boldsymbol{A}^H + \sigma_n^2\boldsymbol{I}_M$, where $\boldsymbol{R}_s = \mathbb{E}\{\boldsymbol{s}(t)\boldsymbol{s}(t)^H\}$ is the covariance matrix of the transmitted signal $\boldsymbol{s}(t)$. In practice, the true covariance matrix $\boldsymbol{R}$ is not available and the sample covariance matrix $\hat{\boldsymbol{R}}$ is used:

$$\hat{\boldsymbol{R}} = \frac{1}{T}\boldsymbol{X}\boldsymbol{X}^H. \tag{4}$$

Subspace techniques rely on the properties of the eigenspaces of the sample covariance matrix $\hat{\boldsymbol{R}}$, which is decomposed as:

$$\hat{\boldsymbol{R}} = \hat{\boldsymbol{U}}\hat{\boldsymbol{\Lambda}}\hat{\boldsymbol{U}}^H \tag{5}$$

$$= \hat{\boldsymbol{U}}_s\hat{\boldsymbol{\Lambda}}_s\hat{\boldsymbol{U}}_s^H + \hat{\boldsymbol{U}}_n\hat{\boldsymbol{\Lambda}}_n\hat{\boldsymbol{U}}_n^H. \tag{6}$$

The diagonal matrix $\hat{\boldsymbol{\Lambda}}_s \in \mathbb{C}^{N\times N}$ in (6) contains the N-largest eigenvalues $\hat{\lambda}_1, \ldots, \hat{\lambda}_N$, and $\hat{\boldsymbol{U}}_s \in \mathbb{C}^{M\times N}$ contains the corresponding principal eigenvectors of the sample covariance matrix $\hat{\boldsymbol{R}}$. Similarly, $\hat{\boldsymbol{\Lambda}}_n \in \mathbb{C}^{(M-N)\times(M-N)}$ and $\hat{\boldsymbol{U}}_n \in \mathbb{C}^{M\times(M-N)}$ contain the $(M-N)$-noise eigenvalues $\hat{\lambda}_{N+1}, \ldots, \hat{\lambda}_M$ and the associated noise eigenvectors, respectively.

3. PARTIAL RELAXATION APPROACH

3.1. General Concept

In the family of the maximum likelihood estimators, the DOA estimation problem is formulated as:

$$\left\{\tilde{\boldsymbol{\theta}}\right\} = \underset{\boldsymbol{A}(\boldsymbol{\theta})\in\mathcal{A}_N}{\arg\min}\; f\left(\boldsymbol{A}\left(\boldsymbol{\theta}\right)\right). \tag{7}$$

where the array manifold $\mathcal{A}_N$ is parameterized as:

$$\mathcal{A}_N = \{\boldsymbol{A} = [\boldsymbol{a}(\vartheta_1), \ldots, \boldsymbol{a}(\vartheta_N)]\,|\vartheta_1 < \ldots < \vartheta_N\} \tag{8}$$

and $f(.)$ is an objective function depending on the criteria considered for the DOA estimation task. In the partial relaxation approach [5], however, instead of enforcing the steering matrix $\boldsymbol{A}$ to be an element in the highly structured array manifold $\mathcal{A}_N$, we assume that $\boldsymbol{A} \in \bar{\mathcal{A}}_N$, where the relaxed array manifold $\bar{\mathcal{A}}_N$ is defined as follows:

$$\bar{\mathcal{A}}_N = \left\{\boldsymbol{A} = [\boldsymbol{a}, \boldsymbol{B}]\,|\boldsymbol{a} \in \mathcal{A}_1, \boldsymbol{B} \in \mathbb{C}^{M\times(N-1)}\right\}. \tag{9}$$

To obtain the estimated DOA, the grid search is applied as follows: first we minimize the objective function in (7) with respect to $\boldsymbol{B}$, and then perform a grid search over $\boldsymbol{a} = \boldsymbol{a}(\theta) \in \mathcal{A}_1$ to find N-deepest local minima corresponding to the DOAs. Applying this principle, in which we partition $\boldsymbol{A} = [\boldsymbol{a}, \boldsymbol{B}]$, $\boldsymbol{S} = \left[\boldsymbol{s}, \boldsymbol{J}^T\right]^T$ in the signal model in (3) with $\boldsymbol{s} \in \mathbb{C}^{T\times 1}$ and $\boldsymbol{J} \in \mathbb{C}^{(N-1)\times T}$ and denote $\boldsymbol{D} = \boldsymbol{B}\boldsymbol{J}$, the partially relaxed constrained covariance fitting (PR-CCF) estimator is formulated in [5] for each direction $\boldsymbol{a} = \boldsymbol{a}(\theta)$ as:

$$\begin{aligned} &\min_{\sigma_s^2\geq 0, \boldsymbol{D}} \left|\left|\hat{\boldsymbol{R}} - \sigma_s^2\boldsymbol{a}\boldsymbol{a}^H - \boldsymbol{D}\boldsymbol{D}^H\right|\right|_F^2 \\ &\text{subject to } \hat{\boldsymbol{R}} - \sigma_s^2\boldsymbol{a}\boldsymbol{a}^H - \boldsymbol{D}\boldsymbol{D}^H \succeq \boldsymbol{0} \\ &\text{rank}(\boldsymbol{D}) \leq N-1. \end{aligned} \tag{10}$$

As presented in [5], assuming the invertibility of the sample covariance matrix $\hat{\boldsymbol{R}}$, the PR-CCF estimator outperforms conventional methods in the threshold region. However, if $\hat{\boldsymbol{R}}$ is singular, it can be shown that the minimum of the objective function in (10) is constant regardless of the direction $\boldsymbol{a}$, and hence no grid search can be performed to determine the DOAs. Motivated by this consideration, in the next subsection, we propose a modified method based on unconstrained covariance fitting under the partial relaxation framework that shows improved error performance.

3.2. Partially-Relaxed Unconstrained Covariance Fitting (PR-UCF)

Comparing with the constrained version in (10), the formulation of the PR-UCF omits the PSD constraint to yield the following optimization problem for each direction $\boldsymbol{a} = \boldsymbol{a}(\theta)$:

$$\begin{aligned} &\min_{\sigma_s^2\geq 0, \boldsymbol{D}} \left|\left|\hat{\boldsymbol{R}} - \sigma_s^2\boldsymbol{a}\boldsymbol{a}^H - \boldsymbol{D}\boldsymbol{D}^H\right|\right|_F^2 \\ &\text{subject to rank}(\boldsymbol{D}) \leq N-1. \end{aligned} \tag{11}$$

Keeping σ_s^2 fixed, the minimizer $\tilde{\boldsymbol{D}}$ of the optimization problem in (11) is obtained as the best $(N-1)$-rank approximation of $\hat{\boldsymbol{R}} - \sigma_s^2\boldsymbol{a}\boldsymbol{a}^H$. Hence, the optimization of the PR-UCF estimator in (11) is equivalent to:

$$\min_{\sigma_s^2\geq 0} \sum_{k=N}^{M} \lambda_k^2\left(\hat{\boldsymbol{R}} - \sigma_s^2\boldsymbol{a}\boldsymbol{a}^H\right). \tag{12}$$

If we denote the objective function in (12) as $g\left(\sigma_s^2\right)$, a numerical local minimizer $\tilde{\sigma}_{s,\mathrm{U}}^2$ can be determined by noting that $g(\sigma_s^2)$ is continuously differentiable. Applying the results in [6, 7], the derivative $g'\left(\sigma_s^2\right)$ is given by:

$$g'\left(\sigma_s^2\right) = -\sum_{k=N}^{M} \frac{2\bar{\lambda}_k(\sigma_s^2)}{\sigma_s^4\boldsymbol{a}^H\left(\hat{\boldsymbol{R}} - \bar{\lambda}_k(\sigma_s^2)\boldsymbol{I}\right)^{-2}\boldsymbol{a}}, \tag{13}$$

where we introduce the shorthand notation:

$$\bar{\lambda}_k(\sigma_s^2) = \lambda_k\left(\hat{\boldsymbol{R}} - \sigma_s^2\boldsymbol{a}\boldsymbol{a}^H\right). \tag{14}$$

Note that the denominator in each summand of the expression in (13) is nonnegative, we observe that:

- If $\sigma_s^2 \to 0$ then $\bar{\lambda}_k(\sigma_s^2) \geq 0$ with $k = N, \ldots, M$ and therefore:
$$\lim_{\sigma_s^2\to 0} g'\left(\sigma_s^2\right) < 0. \tag{15}$$
- If $\sigma_s^2 \to \infty$, the rank-one component $-\sigma_s^2\boldsymbol{a}\boldsymbol{a}^H$ is dominant to $\hat{\boldsymbol{R}}$ and thus $\bar{\lambda}_M\left(\sigma_s^2\right) \approx -\sigma_s^2\,||\boldsymbol{a}||_2^2$. The remaining eigenvalues $\bar{\lambda}_k\left(\sigma_s^2\right)$ with $k = N, \ldots, (M-1)$ are bounded thanks to the Weyl's inequality [8]. Combining the two mentioned remarks, there exists a scalar $\sigma_{s,\text{right}}^2 < \infty$ such that
$$g'\left(\sigma_{s,\text{right}}^2\right) > 0. \tag{16}$$

As a consequence, we can choose a sufficiently small $\sigma^2_{s,\text{left}}$ and sufficiently large $\sigma^2_{s,\text{right}}$ which satisfy (15) and (16), respectively. Then, a simple bisection search on $g'\left(\sigma_s^2\right)$ in (13) in the interval $\left[\sigma^2_{s,\text{left}}, \sigma^2_{s,\text{right}}\right]$ can be used to compute a local minimizer [1] $\tilde{\sigma}^2_{s,\text{U}}$ of (12).

4. EFFICIENT IMPLEMENTATION FOR PR-UCF

The proposed estimator PR-UCF requires extensive computation of the eigenvalues over the entire angular field-of-view. The computational complexity of the full eigenvalue decomposition may limit the usage of the proposed partial relaxation approach in practice. Furthermore, from an algorithmic perspective, the expressions in (13) requires the computation of the eigenvalues of a generic matrix form as follows:

$$\boldsymbol{D} - \rho \boldsymbol{z}\boldsymbol{z}^H = \bar{\boldsymbol{U}}\bar{\boldsymbol{D}}\bar{\boldsymbol{U}}^H, \tag{17}$$

where $\boldsymbol{D} = \text{diag}\,(d_1, \ldots, d_K) \in \mathbb{R}^{K\times K}$ is a constant real diagonal matrix, $\rho \in \mathbb{R}$ is an arbitrary positive real scalar and $\boldsymbol{z} = [z_1, \ldots, z_K]^T \in \mathbb{C}^{K\times 1}$ is a direction-dependent complex-valued vector. The relationship between the generic form in (17) and the bisection algorithm in Subsection 3.2 is further detailed in the subsections below.

4.1. Computation Procedure

Initially proposed in [9] to compute the eigenvalue decomposition of symmetric tridiagonal matrices in a parallel manner, the procedure of rank-one update on the eigenvalues of (17) is based on the interlacing theorem as follows [7]:

Theorem 1 *Let $\{d_1, \ldots, d_K\}$ be the elements on the diagonal of the matrix $\boldsymbol{D} \in \mathbb{R}^{K\times K}$ where $\{d_1, \ldots, d_K\}$ are distinct and sorted in descending order. Further assume that $\rho > 0$ and $\boldsymbol{z} \in \mathbb{C}^{K\times 1}$ contains only nonzero entries. If the eigenvalues $\left\{\bar{d}_1, \ldots, \bar{d}_K\right\}$ of the matrix $\boldsymbol{D} - \rho\boldsymbol{z}\boldsymbol{z}^H$ are also sorted in descending order, then:*

- *$\left\{\bar{d}_1, \ldots, \bar{d}_K\right\}$ are the K zeros of the secular function $p(x) = 0$, where $p(x)$ is given by:*

$$p(x) = 1 - \rho \boldsymbol{z}^H \left(\boldsymbol{D} - x\boldsymbol{I}\right)^{-1} \boldsymbol{z} \tag{18}$$

$$= 1 - \rho \sum_{k=1}^{K} \frac{|z_k|^2}{d_k - x}. \tag{19}$$

- *$\left\{\bar{d}_1, \ldots, \bar{d}_K\right\}$ satisfy the interlacing property, i.e.,*

$$d_1 > \bar{d}_1 > d_2 > \bar{d}_2 > \ldots > d_K > \bar{d}_K. \tag{20}$$

Based on Theorem 1, computationally efficient rooting of the secular function in (19) is of great importance for the acceleration of our proposed estimator. Without loss of generality, we consider the k-th root of the secular function $\bar{d}_k$ which lies inside the interval (d_{k+1}, d_k) where $k = 1, \ldots, K$ and $d_{K+1} = -\infty$. By defining the two auxiliary rational functions:

$$\psi_k(x) = -\rho \sum_{j=1}^{k} \frac{|z_j|^2}{d_j - x} \tag{21}$$

[1] From the simulations, the function $g(\sigma_s^2)$ is generally unimodal and therefore the bisection search converges to a global minimum.

Algorithm 1 Determining the k-th root of the secular function

1: **Initialization**: $\tau = 0$, arbitrary $x^{(0)} \neq d_k$, $\epsilon = 10^{-6}$
2: **repeat**
3: Find the parameters p and q such that:
$R_{k-1;p,q}(x^{(\tau)}) = \psi_k(x^{(\tau)})$ and $R'_{k-1;p,q}(x^{(\tau)}) = \psi'_k(x^{(\tau)})$
4: Find the parameters r and s such that:
$R_{k;r,s}(x^{(\tau)}) = \phi_k(x^{(\tau)})$ and $R'_{k;r,s}(x^{(\tau)}) = \phi'_k(x^{(\tau)})$
5: Find $x^{(\tau+1)} \in (d_{k+1}, d_k)$ which satisfies:
$-R_{k-1;p,q}(x^{(\tau+1)}) = 1 + R_{k;r,s}(x^{(\tau+1)})$
6: $\tau \leftarrow \tau + 1$
7: **until** $\left|x^{(\tau+1)} - x^{(\tau)}\right| < \epsilon$
8: **return** $\bar{d}_k = x^{(\tau+1)}$

$$\phi_k(x) = \begin{cases} -\rho \sum\limits_{j=k+1}^{K} \dfrac{|z_j|^2}{d_j - x} & \text{if } 1 \leq k \leq K-1 \\ 0 & \text{if } k = K, \end{cases} \tag{22}$$

the secular function in (19) can be rewritten as:

$$-\psi_k(x) = 1 + \phi_k(x). \tag{23}$$

Since both $\psi_k(x)$ and $\phi_k(x)$ are defined as the sum of multiple rational functions, a straightforward approach to solve (23) iteratively around a given point $x^{(\tau)}$ consists of using rational functions of first degree as approximants. The author in [10] suggest approximants of type:

$$R_{k;p,q}(x) = \begin{cases} p + \dfrac{q}{d_{k+1} - x} & \text{if } 0 \leq k \leq K-1 \\ 0 & \text{if } k = K, \end{cases} \tag{24}$$

and choose the parameters p, q such that the approximants coincide at a given point $x^{(\tau)}$ with the corresponding exact functions in (21) and (22), respectively, up to the first-order derivative. For convenience purposes, the steps for determining the roots of the secular function in (19) are summarized in Algorithm 1. Note that since the approximant in (24) is only of first order, the steps 2-4 in Algorithm 1 can be solved in closed form. Interestingly, this approach is also applicable to all partial relaxation estimators in [5].

4.2. Application to PR-UCF

To reduce the number of required eigenvalues for the derivative, the function $g(\sigma_s^2) = \sum\limits_{k=N}^{M} \lambda_k^2 \left(\hat{\boldsymbol{R}} - \sigma_s^2 \boldsymbol{a}\boldsymbol{a}^H\right)$ is rewritten as follows:

$$g(\sigma_s^2) = \text{tr}\left(\hat{\boldsymbol{R}} - \sigma_s^2 \boldsymbol{a}\boldsymbol{a}^H\right) - \sum_{k=1}^{N-1} \lambda_k^2 \left(\hat{\boldsymbol{R}} - \sigma_s^2 \boldsymbol{a}\boldsymbol{a}^H\right) \tag{25}$$

$$= \text{tr}\left(\hat{\boldsymbol{R}}^2\right) - 2\hat{\sigma}_s^2 \boldsymbol{a}^H \hat{\boldsymbol{R}} \boldsymbol{a} + \hat{\sigma}_s^4 \left|\left|\boldsymbol{a}\right|\right|_2^4 - \sum_{k=1}^{N-1} \lambda_k^2 \left(\hat{\boldsymbol{R}} - \hat{\sigma}_s^2 \boldsymbol{a}\boldsymbol{a}^H\right). \tag{26}$$

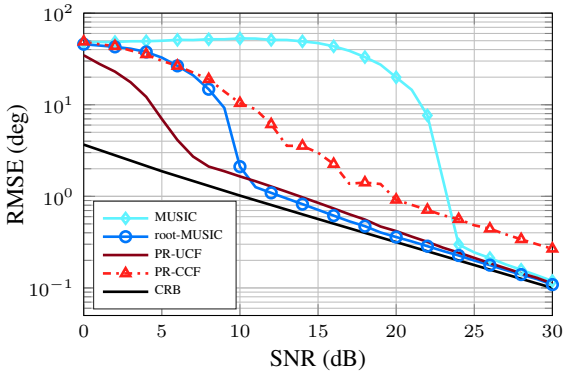

Fig. 1: Uncorrelated sources, number of snapshots $T = 8$

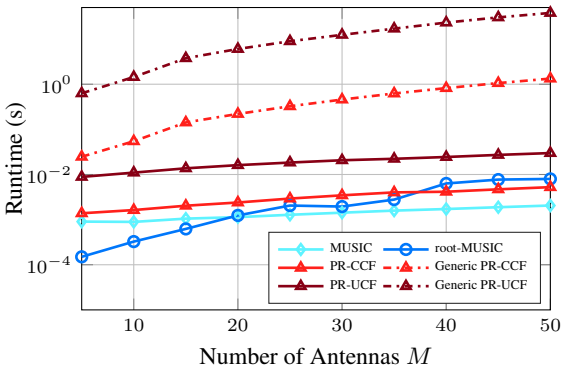

Fig. 2: SNR $= 10$ dB, number of snapshots $T = 100$

The derivative $g'\left(\sigma_s^2\right)$ is calculated as:

$$g'\left(\sigma_s^2\right) = -2\boldsymbol{a}^H\hat{\boldsymbol{R}}\boldsymbol{a} + 2\sigma_s^2\left\|\boldsymbol{a}\right\|_2^4 + \sum_{k=1}^{N-1}\frac{2\bar{\lambda}_k(\sigma_s^2)}{\sigma_s^4\boldsymbol{a}^H\left(\hat{\boldsymbol{R}} - \bar{\lambda}_k(\sigma_s^2)\boldsymbol{I}\right)^{-2}\boldsymbol{a}} \tag{27}$$

where $\bar{\lambda}_k(\sigma_s^2)$ is defined in (14). By substituting $\boldsymbol{z} = \hat{\boldsymbol{U}}^H\boldsymbol{a}$, we obtain:

$$\bar{\lambda}_k(\sigma_s^2) = \lambda_k\left(\hat{\boldsymbol{\Lambda}} - \sigma_s^2\boldsymbol{z}\boldsymbol{z}^H\right), \tag{28}$$

$$g'\left(\sigma_s^2\right) = -2\boldsymbol{z}^H\hat{\boldsymbol{\Lambda}}\boldsymbol{z} + 2\sigma_s^2\left\|\boldsymbol{z}\right\|_2^4 + \sum_{k=1}^{N-1}\frac{2\bar{\lambda}_k\left(\sigma_s^2\right)}{\sigma_s^4\sum\limits_{j=1}^{M}\frac{\left|z_j\right|^2}{\left(\hat{\lambda}_j - \bar{\lambda}_k(\sigma_s^2)\right)^2}}. \tag{29}$$

The expressions in (28) and (29) suggest to apply the procedure in Subsection 4.1 with $\boldsymbol{D} = \hat{\boldsymbol{\Lambda}}$, $\rho = \sigma_s^2$ and $\boldsymbol{z} = \hat{\boldsymbol{U}}^H\boldsymbol{a}$. For faster convergence of Algorithm 1, the eigenvalues from the previous iteration in the bisection search can be reused as initialization values for the current iteration.

5. SIMULATION RESULTS

In this section, simulation results regarding the performance of different DOA estimators are presented and compared with the stochastic Cramer-Rao Bound (CRB) [11]. The number of Monte-Carlo runs is 4000. The Root-Mean-Squared-Error (RMSE) is calculated as:

$$\text{RMSE} = \sqrt{\frac{\sum\limits_{n=1}^{N}\left(\tilde{\theta}_n - \theta_n\right)^2}{N}}, \tag{30}$$

The estimated DOAs $\tilde{\boldsymbol{\theta}} = [\tilde{\theta}_1, \ldots, \tilde{\theta}_N]^T$ and the true DOAs $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_N]^T$ in (30) are sorted in ascending order. In our simulations we assume two uncorrelated but closely spaced source signals at $\boldsymbol{\theta} = [45°, 50°]^T$ which impinge on a ULA of $M = 10$ antennas with the spacing equal to half of the wavelength. The source signals have the mean value of zero and unit power. The Signal-to-Noise-Ratio (SNR) is calculated as SNR $= \frac{1}{\sigma_n^2}$.

Figure 1 depicts a scenario where the number of snapshots $T = 8$ is smaller than the number of antennas $M = 10$. In this case, the sample covariance matrix calculated in (4) is singular, and therefore the PR-CCF is not applicable. Instead, we apply the diagonal loading technique with the loading factor $\gamma = 10^{-4}$ on the sample covariance matrix [12], [13]. To avoid outliers in RMSE caused by misdetection and to simulate the DOA tracking process [14], 1% of the estimates with the largest error for all investigated algorithms are removed before calculating the RMSE. It can be observed that even in the case of a very low number of snapshots, PR-UCF obtains the best SNR threshold behavior. The performance of PR-CCF is highly degraded even with the diagonal loading technique.

In Figure 2, the running time of the DOA estimation algorithms with respect to the number of antennas M are depicted. The term *Generic* in Figure 2 refers to the naive implementation using the MATLAB command `eig` for the eigenvalue decomposition. The running time of the partial relaxation methods implemented with Algorithm 1 is drastically reduced in comparison with the default MATLAB command and follow a similar trend of MUSIC.

6. CONCLUSIONS AND OUTLOOK

In this paper, a new DOA estimator under the partial relaxation approach based on the covariance fitting problem is introduced. Simulation results show that, even though no particular structure of the sensor array, e.g., Vandermonde structure from a uniform linear array, is required, the proposed DOA estimator exhibits superior threshold performance to root-MUSIC in difficult scenarios. By applying known results regarding the evaluation of the eigenvalues of a Hermitian matrix modified with a rank one component, considerably lower computational time is achieved.

For future work, the theoretical error behavior and consistency of methods is an interesting open problem and requires further investigation. A theoretical optimal weighting should be acquired. Furthermore, the derivation of search-free partially-relaxed estimators based on polynomial rooting is also of great interest.

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