# A UNIFIED ESTIMATOR FOR SOURCE POSITIONING AND DOA ESTIMATION USING AOA

Yue Wang and K. C. Ho

University of Missouri Columbia, MO 65211, USA yw67d@mail.missouri.edu, hod@missouri.edu

## ABSTRACT

Angles of Arrival (AOAs) are popular measurements to locate a signal source. They can yield a position if the source is near the sensors or a DOA if it is far away. Point positioning and DOA localization require different estimators and the knowledge if the source is near or far is needed. Nevertheless, such knowledge about the source range is often not available in practice. This paper first analyzes the consequences of point positioning of a distant source and DOA estimation of a near one. It next proposes a unified estimator that provides a position estimate if it is near or a DOA estimate if it is far, without requiring any prior knowledge about the range region of where it lies. The estimator is derived using the Maximum Likelihood criterion with the Gauss Newton implementation, using the modified polar coordinates to represent the source location. A preliminary solution using the semi-definite relaxation is developed to initialize the estimator. Simulations validate the performance of the proposed estimator in reaching the CRLB performance.

Index Terms- DOA Estimation, Localization, Position, AOA

### 1. INTRODUCTION

AOA has received much attention recently for the localization of a signal source [1, 2, 3, 4, 5, 6, 7, 8]. Compared to the range based measurement such as Time of Arrival (TOA) [9, 10, 11, 12] or Time Difference of Arrival (TDOA) [13, 14, 15, 16, 17], AOA offers the advantage without the need of time stamping the signal or synchronization among the receivers. It also minimizes the transmission resources as AOA can be determined locally within a sensor, avoiding sensor signal transmission for cross-correlation as in TDOA. Obtaining AOA, however, requires a more sophisticated sensor with multiple receiving elements and higher computation ability [18, 19, 20].

An AOA defines a straightline originating from the sensor that passes through the source. The AOA lines from the sensors intersect together and yield a source position estimate. When the source is far away, however, the AOA lines would be nearly in parallel and fail to intersect properly to form a reasonable position estimate. In such a situation, we can only be able to determine the DOA of the source by combining the AOA measurements.

The knowledge that the source is near or far is seldom available in practice before localization. Applying point positioning estimator for a far away source would yield unreliable result, and this is reflected by the thresholding behavior in the Hybrid Bhattacharyya-Barankin bound [21], called Abel Bound here, as the source range increases. Using DOA estimation for a near source would not cause thresholding effect, but rather produces a large amount of bias. This paper proposes an estimator that does not require such a knowledge. Gang Wang

Ningbo University Ningbo 315211, China wanggang@nbu.edu.cn

It gives the position if the source happens to be near and the DOA if it is distant.

The proposed estimator applies the modified polar coordinates representation (MPR) [22] of the source location, rather than the Cartesian coordinates. MPR uses inverse-range instead of range. If the source is near, the inverse-range is meaningful and gives the source range. When the source is far away, the inverse-range will approach zero, avoiding the damage caused by large range or coordinate estimate that would lead to unreliable angle estimate.

The proposed estimator is derived based on the Maximum Likelihood Estimator (MLE) using AOA. One popular implementation of the MLE is the Gauss-Newton (GN) iteration. It requires an initial solution guess that is close to the actual solution. We propose an initial solution by using the semi-definite relaxation (SDR) technique.

Relatively few previous works on this research topic appear in the literature. Perhaps two related works are [23, 24], and they are on DOA estimation for a mix of near-field and far-field sources. The recent work [22] addresses a similar problem, it is, however, for TDOA measurements only. This paper provides the analysis of AOA localization for point positioning and DOA estimation, develops a new GN MLE localization algorithm regardless of the source that is near or far, and derives an initial solution for MLE using SDR. The performance of the proposed estimator is illustrated to reach the Cramer-Rao Lower Bound (CRLB) or the Abel Bound performance.

## 2. OBJECTIVE AND ANALYSIS

The objective is to determine the position  $\mathbf{u}^o = [x^o, y^o]^T$  of a signal source or its DOA  $\theta^o$  if it is far away, using the AOA measurements obtained from M receivers at known positions  $\mathbf{s}_i = [x_i, y_i]^T$ ,  $i = 1, \ldots, M$ . The AOA measurement from the *i*-th sensor is modeled as

$$\theta_i = \tan^{-1} \left( \frac{y^o - y_i}{x^o - x_i} \right) + n_i. \tag{1}$$

where  $\tan^{-1}$  is the arc-tangent operation with the quadrant taken into account. The measurement vector is  $\boldsymbol{\theta} = [\theta_1, \dots, \theta_M]^T$ .  $\mathbf{n} = [n_1, \dots, n_m]^T$  is the additive zero-mean Gaussian noise with covariance matrix  $\mathbf{Q}$ . The true measurement vector without noise is denoted by  $\boldsymbol{\theta}^o$ .

We shall next examine the effect when the source is away from or near to the sensors, using point positioning and DOA estimation.

#### 2.1. Point Positioning

Point positioning assumes the source is not far away from the sensors so that the AOA lines intersect properly to yield a source location



Fig. 1. (a) Point positioning model. (b) DOA localization model.

estimate as illustrated in Fig. 1 (a). The cost function for localization in the commonly used Cartesian coordinates is

$$J_n(\mathbf{u}) = (\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}(\mathbf{u}))^T \mathbf{Q}^{-1} (\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}(\mathbf{u}))$$
(2)

where  $\bar{\theta}(\mathbf{u})$  is the AOA vector parameterized in terms of the unknown source position  $\mathbf{u}$ , using the model (1) with the noise set to zero. The ML cost function is not quadratic in the unknown and does not yield an analytic solution for  $\mathbf{u}^{\circ}$ . Iterative implementation is typically applied to find the ML solution with an appropriate initial solution guess.

The performance of a positioning estimator is often examined by comparing with the CRLB, assuming the bias is relatively small compared to the variance. A limitation of the CRLB is that it is unable to illustrate the thresholding phenomenon when the noise level or the source range increases. Perhaps a better bound to use is the Abel Bound [21]. It merges the Bhattacharyya Bound [25], which is a generalized version of the CRLB, and the HCR [26, 27] bound, which is a simplified Barankin-style bound, to form a new bound that is able to predict the thresholding phenomenon. We have evaluated the Abel Bound for AOA positioning and the details are provided in Appendix A.

The limitation of point positioning is illustrated in Fig. 2. It shows the mean-square error of the source location estimate obtained by minimizing the ML cost  $J_n(\mathbf{u})$  as the source range increases. The number of sensors is 6 and their positions are generated randomly and tabulated in Table 1. The source is kept at an arbitrary selected angle  $\theta^o = 101.31^o$  when increasing its range  $r^o$ . Each sensor has a different measurement noise power that is generated randomly and the average noise power is  $\sigma^2 = 0.001 \text{rad}^2$ . The noise from different sensors are independent. The position estimate follows the CRLB well as shown in Fig. 2, until the range becomes  $r^o = 45$  at which sudden deviation from the CRLB appears. Also shown is the Abel Bound. It predicts the thresholding occurrence of the MLE well.

#### 2.2. DOA Estimation

Fig. 1 (b) illustrates the scenario where the source is far from the sensors and the AOA lines are close to parallel. In such a situation, we are able to obtain the DOA of the source only. The AOA measurements are modeled as the DOA of the source corrupted by measurement noise. The ML cost function is

$$J_f(\theta) = (\boldsymbol{\theta} - \mathbf{1}\theta)^T \mathbf{Q}^{-1} (\boldsymbol{\theta} - \mathbf{1}\theta)$$
(3)



**Fig. 2.** Thresholding behavior of the source position estimate from MLE in the Cartesian coordinates.

where  $\theta$  denotes the source DOA and 1 is an  $M \times 1$  vector with all of its elements equal to 1. Let  $\mathbf{W} = \mathbf{Q}^{-1}$ . The solution that minimizes (3) is

$$\boldsymbol{\theta} = (\mathbf{1}^T \mathbf{W} \mathbf{1})^{-1} \mathbf{1}^T \mathbf{W} \boldsymbol{\theta} \,. \tag{4}$$

If the source is not sufficiently far away, applying this model for DOA estimation will yield a significant amount of bias with respect to the true DOA  $\theta^o$ . The theoretical bias of the estimator (4) is

$$E\left[\theta - \theta^{o}\right] = (\mathbf{1}^{T}\mathbf{W}\mathbf{1})^{-1}\mathbf{1}^{T}\mathbf{W}(E[\boldsymbol{\theta}] - \mathbf{1}\theta^{o}).$$
 (5)

Fig. 3 illustrates the amount of bias as the source range varies. The settings are the same as in Fig. 2. When the source range is not large enough, the bias is very significant and dominates the performance. The bias is nearly 10 dB higher than the variance at  $r^o = 40$ .



Fig. 3. Bias behaviors of DOA estimation in the far-field model.

## 3. PROPOSED ESTIMATOR

The proposed estimator uses MPR to represent source position in terms of the DOA angle  $\theta^o$  and inverse-range  $g^o = 1/r^o$ ,

$$\tilde{\mathbf{u}}^o = \begin{bmatrix} \theta^o & g^o \end{bmatrix}^T.$$
(6)

If the source is not far from the sensors, the inverse-range is meaningful and the corresponding source position in the Cartesian coordinates is

$$\mathbf{u}^{o} = (1/g^{o}) \begin{bmatrix} \cos \theta^{o} & \sin \theta^{o} \end{bmatrix}^{T}.$$
 (7)

When the source is far way, the inverse-range approaches zero and will not affect much the estimation of the DOA. As a result, the angle estimate remains accurate.

Under the MPR model, the Abel Bound for  $\tilde{\mathbf{u}}^o$  can be obtained by setting in (34)  $\boldsymbol{\xi}(\mathbf{u}^o) = \tilde{\mathbf{u}}^o$ . For m = 1,

$$\Psi_m = \frac{\partial \boldsymbol{\xi}(\mathbf{u}^o)}{\partial \mathbf{u}^{oT}} = -\begin{bmatrix} g^o \sin \theta^o & -g^o \cos \theta^o \\ g^{o2} \cos \theta^o & g^{o2} \sin \theta^o \end{bmatrix}.$$
 (8)

As shown in Fig. 4 which will be elaborated in details in Section 4, the Abel Bounds for the elements of  $\tilde{u}^{o}$ , which coincide with the CRLBs in this case, do not exhibit the thresholding effect as for  $u^{o}$  in Fig. 2, when the source range increases. The Abel Bound confirms that using MPR can yield a meaningful source range for point positioning when the source is near and an accurate DOA regardless of the source range.

We shall next develop the GN iterative MLE in MPR and derive the initial solution guess.

### 3.1. MLE in MPR

From (2), the cost function in terms of  $\tilde{\mathbf{u}}$  is

$$J_n(\tilde{\mathbf{u}}) = \left(\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}(\tilde{\mathbf{u}})\right)^T \mathbf{Q}^{-1} \left(\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}(\tilde{\mathbf{u}})\right) .$$
(9)

 $\bar{\theta}(\tilde{\mathbf{u}})$  represents the AOA vector constructed from  $\tilde{\mathbf{u}}$  directly. Substituting (7) into (1), the true measurement  $\theta_i^o$  can be parameterized in terms of  $\tilde{\mathbf{u}}^o$  as

$$\theta_i^o = \tan^{-1} \left( \frac{\sin \theta^o - g^o y_i}{\cos \theta^o - g^o x_i} \right) = \tan^{-1} \left( \frac{\gamma^o(2)}{\gamma^o(1)} \right) \tag{10}$$

where

$$\boldsymbol{\gamma}^{o} = [\cos\theta^{o} - g^{o}x_{i}, \sin\theta^{o} - g^{o}y_{i}]^{T}.$$
 (11)

The cost function  $J_n(\tilde{\mathbf{u}})$  is highly non-linear with respect to the unknown  $\tilde{\mathbf{u}}$ . The GN approach for optimization is to replace  $\tilde{\mathbf{u}}$  by its linear approximation at some initial value  $\bar{\boldsymbol{\theta}}(\tilde{\mathbf{u}}^{(k)})$  so that  $J_n(\tilde{\mathbf{u}})$  becomes quadratic in the unknown. Taking derivative of the approximated  $J_n(\tilde{\mathbf{u}})$  with respect to  $\tilde{\mathbf{u}}$  yields the iterative equation

$$\tilde{\mathbf{u}}^{(k+1)} = \tilde{\mathbf{u}}^{(k)} + (\mathbf{G}^{(k)T}\mathbf{W}\mathbf{G}^{(k)})^{-1}\mathbf{G}^{(k)T}\mathbf{W}(\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}(\tilde{\mathbf{u}}^{(k)})), \qquad (12)$$
  
$$k = 0, 1, \dots$$

The superscript k denotes the iteration count and 
$$\mathbf{W} = \mathbf{Q}^{-1}$$
.  $\mathbf{G}^{(k)}$  is the  $M \times 2$  gradient matrix, whose *i*-th row is given by

$$\mathbf{G}^{(k)}(i,:) = \left[ \left. \frac{\partial \theta_i^o}{\partial \theta^o} \right. , \left. \frac{\partial \theta_i^o}{\partial g^o} \right] \right|_{\tilde{\mathbf{u}}^{(k)}} \,. \tag{13}$$

From the parametric form (10), the gradients are

$$\frac{\partial \theta_i^o}{\partial \theta^o} = \frac{1}{||\boldsymbol{\gamma}^o||^2} \left[\cos\theta^o, \sin\theta^o\right] \boldsymbol{\gamma}^o \tag{14}$$

and

$$\frac{\partial \theta_i^o}{\partial g^o} = \frac{1}{||\boldsymbol{\gamma}^o||^2} \left[ -y_i \,, \, x_i \right] \boldsymbol{\gamma}^o \,. \tag{15}$$

 $\bar{\theta}(\tilde{\mathbf{u}}^{(k)})$  in (12) is the reconstructed measurement vector whose elements, according to (10), are

$$\theta_i^{(k)} = \tan^{-1} \left( \frac{\sin(\tilde{u}(1)^{(k)}) - y_i \tilde{u}(2)^{(k)}}{\cos(\tilde{u}(1)^{(k)}) - x_i \, \tilde{u}(2)^{(k)}} \right) \,. \tag{16}$$

Applying (12) requires an initial guess  $\tilde{\mathbf{u}}^{(0)}$  that is close to the actual solution. We next propose an initial solution guess obtained by using the SDR technique.

#### 3.2. Initial Solution by SDR

We begin by using (7) and expressing (1) as

$$\frac{\sin\left(\theta_{i}-n_{i}\right)}{\cos\left(\theta_{i}-n_{i}\right)} = \frac{\sin\theta^{o}-g^{o}y_{i}}{\cos\theta^{o}-g^{o}x_{i}}.$$
(17)

When the noise  $n_i$  is small,  $\sin(\theta_i - n_i) \approx \sin\theta_i - n_i \cos\theta_i$  and  $\cos(\theta_i - n_i) \approx \cos\theta_i + n_i \sin\theta_i$ . Ignoring the approximation error, putting them to (17) and rearranging give

$$l_i n_i = \sin \theta_i \cos \theta^o - \cos \theta_i \sin \theta^o - g^o \left( x_i \sin \theta_i - y_i \cos \theta_i \right)$$
(18)

where  $l_i = (\cos \theta^o - g^o x_i) \cos \theta_i + (\sin \theta^o - g^o y_i) \sin \theta_i$ . Let us define the unknown vector as

$$\mathbf{v}^{o} = \left[\cos\theta^{o}, \sin\theta^{o}, g^{o}\right]^{T}.$$
(19)

Stacking (18) for i = 1, ..., M yields the matrix equation

$$\mathbf{Bn} = \mathbf{Av}^o. \tag{20}$$

**B** is the  $M \times M$  diagonal matrix

$$\mathbf{B} = \operatorname{diag}\left(\left[l_1, l_2, \dots, l_M\right]\right). \tag{21}$$

The *i*-th row of  $\mathbf{A}$  is

$$\mathbf{A}(i,:) = \begin{bmatrix} \sin \theta_i & -\cos \theta_i & -(x_i \sin \theta_i - y_i \cos \theta_i) \end{bmatrix}.$$
 (22)

There are only two independent unknowns in  $\mathbf{v}^{o}$ . The solution for  $\mathbf{v}^{o}$  is obtained by solving the weighted least squares problem with a quadratic constraint given by

$$\mathbf{v} = \operatorname{argmin} \, \mathbf{v}^T \mathbf{A}^T \mathbf{\Omega} \mathbf{A} \mathbf{v} \tag{23}$$

subject to

$$1 = v(1)^2 + v(2)^2.$$
(24)

The cost function to minimize in (23) is obtained by pre-multiplying (20) with the weighting matrix  $\mathbf{\Omega}$  and then its transpose. We choose the weighting matrix as  $\mathbf{\Omega} = (\mathbf{B}\mathbf{Q}\mathbf{B}^T)^{-1}$ . (23) with (24) is a generalized trust region subproblem (GTRS) and its solution exists. We solve this constrained optimization problem using SDR instead.

Let  $\mathbf{V} = \mathbf{v}\mathbf{v}^T$  and relax its rank to be larger than one. The original optimization problem is approximated by

$$\mathbf{V} = \underset{\mathbf{V} \in S^{3}}{\operatorname{argmin}} \operatorname{tr} \left( \mathbf{A}^{\mathrm{T}} \mathbf{\Omega} \mathbf{A} \mathbf{V} \right)$$
(25)

subject to

$$0 = 1 - \operatorname{tr} (\mathbf{MV})$$
  
$$\mathbf{V} \succeq 0$$
(26)

where **M** is a  $3 \times 3$  matrix of zero except the (1,1) and (2,2) elements are unity. The solution **V** of this approximated problem can be obtained by the CVX toolbox [28]. Let  $\mathbf{v}_{max}$  be the eigenvector of **V** having the largest eigenvalue  $\lambda_{max}$ . The solution for  $\mathbf{v}^{o}$  is

$$\mathbf{v} = \sqrt{\lambda_{max}} \mathbf{v}_{max} \,. \tag{27}$$

It can be shown as the global solution of the optimization problem. In terms of the initial solution for the GN MLE, from the definition of  $\mathbf{v}$  in (19),

$$\tilde{\mathbf{u}}^{(0)} = \begin{bmatrix} \tan^{-1} \left( v(2)/v(1) \right) & v(3) \end{bmatrix}^T.$$
(28)

The matrix  $\mathbf{B}$  depends on the true solution. We shall set  $\mathbf{B}$  to the identity (other non-zero value may also be possible) and obtain a preliminary solution. It can be used to construct an approximation of  $\mathbf{B}$ , from which a better solution is generated as the initial guess for the GN MLE.

**Table 1**. sensor positions for localization.

i	1	2	3	4	5	6
$x_i$	0	-3.43	-0.59	-4.94	-1.29	-3.35
$y_i$	0	2.55	5.31	-2.13	2.14	5.61

### 4. SIMULATION

We shall examine the performance of the proposed MLE-MPR and the initial solution SDR-MPR, and compare them with the MLE in the Cartesian coordinates (MLE-Cartesian) that is implemented using GN iteration with ideal initialization at the true source location. The setting is the same as that for Fig. 2. The number of ensemble runs is 1000. The unit of the measurement noise power and DOA MSE are in rad<sup>2</sup>.

Fig. 4 illustrates the performance of the proposed MLE-MPR solution under the same scenario as in Fig. 2. MLE-MPR is able to yield performance matching well with the Abel Bounds, for both inverse-range and the DOA estimates. Better accuracy in the DOA estimate appears at range 10 when the source is close the sensors. The DOA estimate converted from MLE-Cartesian is not good when the source range is larger than 40, due to the threholding phenomenon as seen in Fig. 2. Although the inverse-range estimate from MLE-MPR becomes meaningless as it approaches zero when the range increases, it remains to provide an accurate DOA estimate. MLE-MPR unifies the near and distant source localization together.



Fig. 4. Performance of the proposed MLE-MPR.

To complete the study, Fig. 5 examines the performance under a near source situation as the average measurement noise power  $\sigma^2$ increases. The source range is fixed at 80. When the noise power is small, MLE-Cartesian and MLE-MPR achieve the CRLB performance, and SDR-MPR cannot which is caused by the solution bias. As the noise power increases, MLE-Cartesian diverges from the CRLB for the angle estimate at  $\sigma^2 = 0.001$ , while the MLE-MPR remains to lie on the CRLB until  $\sigma^2 = 0.01$ . SDR-MPR provides more stable performance than both MLE-MPR and MLE-Cartesian when the noise power is large.

# 5. CONCLUSIONS

Using AOA measurements, the paper first analyzed the thresholding phenomenon for point positioning in 2-D as the source range increases by evaluating the Abel Bound, and the significant bias of DOA estimation if the source range is not sufficiently large. We next



Fig. 5. Performance of the proposed SDR-MPR and MLE-MPR.

developed an estimator for AOA 2-D localization that does not require the knowledge if the source is near or far from the sensors. It yields a point position if the source is near and a DOA estimate if it is far. The proposed estimator expresses the source position in MPR and applies MLE with GN iterative implementation to obtain the solution. An initial solution is derived based on SDR, which acts as the initialization of the GN MLE in MPR. Simulations validate the performance of the proposed estimator in reaching the CRLB performance, regardless of the source that is near or far from the sensors. Some study for the AOA 3-D positioning case can be found in [29].

# Appendix A

The information matrix of the Abel Bound [21] is

$$\mathbf{H}_{m,l} = \begin{bmatrix} \mathbf{K}_m & \mathbf{L} \\ \mathbf{L}^T & \mathbf{J}_l \end{bmatrix}$$
(29)

where m is the order of the Bhattacharyya matrix and l is the number of test points in the HCR bound. In this study, we choose m = 1 and l = 4.

For 2D AOA localization under Gaussian noise,  $K_1$  is the 1-st order Bhattacharyya matrix given by

$$\mathbf{K}_{1} = \frac{\partial \boldsymbol{\theta}^{oT}}{\partial \mathbf{u}^{o}} \mathbf{Q}^{-1} \frac{\partial \boldsymbol{\theta}^{o}}{\partial \mathbf{u}^{oT}} \,. \tag{30}$$

 $\boldsymbol{\theta}^{o}$  here denote the true measurement vector without noise expressed in terms of the actual source position. The (i, j)-th element of  $\mathbf{J}_{l}$  is

$$[\mathbf{J}_{l}]_{i,j} = \exp\left\{\frac{-1}{2}\left[\boldsymbol{\theta}^{[i]T}\mathbf{Q}^{-1}\boldsymbol{\theta}^{[i]} + \boldsymbol{\theta}^{[j]T}\mathbf{Q}^{-1}\boldsymbol{\theta}^{[j]} - \boldsymbol{\theta}^{oT}\mathbf{Q}^{-1}\boldsymbol{\theta}^{o} - (\boldsymbol{\theta}^{[i]} + \boldsymbol{\theta}^{[j]} - \boldsymbol{\theta}^{o})^{T}\mathbf{Q}^{-1}(\boldsymbol{\theta}^{[i]} + \boldsymbol{\theta}^{[j]} - \boldsymbol{\theta}^{o})\right]\right\}$$
(31)

where  $\boldsymbol{\theta}^{[i]}$  is the *i*-th test point. The *k*-th block of **L** for m = 1 is

$$[\mathbf{L}]_{1,k} = \frac{\partial \boldsymbol{\theta}^{oT}}{\partial \mathbf{u}^{o}} \mathbf{Q}^{-1} \left( \boldsymbol{\theta}^{[k]} - \boldsymbol{\theta}^{o} \right) .$$
(32)

Denoting  $\boldsymbol{\xi}(\mathbf{u}^o)$  as a function of  $\mathbf{u}^o$ , the Abel Bound on the variance of  $\boldsymbol{\xi}(\mathbf{u}^o)$  for our setting of m = 1 is

$$\operatorname{cov}(\boldsymbol{\xi}) \ge \boldsymbol{\Gamma}_{m,l} \mathbf{H}_{m,l}^{-1} \boldsymbol{\Gamma}_{m,l}^{T} , \qquad (33)$$

$$\Psi_1 = \begin{bmatrix} \frac{\partial \boldsymbol{\xi}(\mathbf{u}^o)}{\partial \mathbf{u}^{oT}} \end{bmatrix}, \tag{34}$$

$$\mathbf{\Phi}_{l} = \left[ \mathbf{\xi}(\mathbf{u}^{[1]}) - \mathbf{\xi}(\mathbf{u}^{o}) \ \mathbf{\xi}(\mathbf{u}^{[2]}) - \mathbf{\xi}(\mathbf{u}^{o}) \ \dots \ \mathbf{\xi}(\mathbf{u}^{[l]}) - \mathbf{\xi}(\mathbf{u}^{o}) \right], \quad (35)$$

$$\boldsymbol{\Gamma}_{m,l} = \begin{bmatrix} \boldsymbol{\Psi}_1 & \boldsymbol{\Phi}_l \end{bmatrix} \,. \tag{36}$$

### 6. REFERENCES

- J. Bosse, A. Ferrol, and P. Larzabal, "A spatio-temporal array processing for passive localization of radio transmitters," *IEEE Trans. Signal Process.*, vol. 61, no. 22, pp. 5485-5494, Nov. 2013.
- [2] H. J. Shao, X. P. Zhang, and Z. Wang, "Efficient closed-form algorithms for AOA based self-localization of sensor nodes using auxiliary variables," *IEEE Trans. Signal Process.*, vol. 62, no. 10, pp. 2580-2594, May 2014.
- [3] N. Guzey, H. Xu, and S. Jagannathan, "Localization of nearfield radio controlled unintended emitting sources in the presence of multipath fading," *IEEE Trans. Instrum. Meas.*, vol. 63, no. 11, pp. 2696-2703, Nov. 2014.
- [4] Y. Wang and K. C. Ho, "An asymptotically efficient estimator in closed-form for 3-D AOA localization using a sensor network," *IEEE Trans. Wireless Commun.*, vol. 14, no. 12, pp. 6524-6535, Dec. 2015.
- [5] J. Werner, J. Wang, A. Hakkarainen, D. Cabric, and M. Valkama, "Performance and Cramer-Rao bounds for DoA/RSS estimation and transmitter localization using sectorized antennas," *IEEE Trans. Veh. Technol.*, vol. 65, no. 5, pp. 3255-3270, May 2016.
- [6] L. Wan, G. Han, L. Shu, S. Chan, and T. Zhu, "The application of DOA estimation approach in patient tracking systems with high patient density," *IEEE Trans. Ind. Informat.*, vol. 12, no. 6, pp. 2353-2364, Dec. 2016.
- [7] S. Xu and K. Doanay, "Optimal sensor placement for 3-D angleof-arrival target localization," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 53, no. 3, pp. 1196-1211, Jun. 2017.
- [8] N. Garcia, H. Wymeersch, E. G. Larsson, A. M. Haimovich, and M. Coulon, "Direct localization for massive MIMO," *IEEE Trans. Signal Process.*, vol. 65, no. 10, pp. 2475-2487, May 2017.
- [9] A. Moragrega, P. Closas, and C. Ibars, "Supermodular game for power control in TOA-based positioning," *IEEE Trans. Signal Process.*, vol. 61, no. 12, pp. 3246-3259, Jun. 2013.
- [10] O. Jean and A. J. Weiss, "Passive localization and synchronization using arbitrary signals," *IEEE Trans. Signal Process.*, vol. 62, no. 8, pp. 2143-2150, Aug. 2014.
- [11] N. H. Nguyen and K. Doanay, "Optimal geometry analysis for multistatic TOA localization," *IEEE Trans. Signal Process.*, vol. 64, no. 16, pp. 4180-4193, Aug. 2016.
- [12] T. K. Le and N. Ono, "Closed-form and near closed-form solutions for TOA-based joint source and sensor localization," *IEEE Trans. Signal Process.*, vol. 64, no. 18, pp. 4751-4766, Sep. 2016.
- [13] Y. T. Chan and K. C. Ho, "A simple and efficient estimator for hyperbolic location," *IEEE Trans. Signal Process.*, vol. 42, no. 8, pp. 1905-1915, Aug. 1994.
- [14] L. Wang, T. K. Hon, J. D. Reiss, and A. Cavallaro, "Selflocalization of ad-hoc arrays using time difference of arrivals," *IEEE Trans. Signal Process.*, vol. 64, no. 4, pp. 1018-1033, Feb. 2016.
- [15] T. K. Le and N. Ono, "Closed-form and near closed-form solutions for TDOA-based joint source and sensor localization," *IEEE Trans. Signal Process.*, vol. 65, no. 5, pp. 1207-1221, Mar. 2017.

- [16] X. Qu, L. Xie, and W. Tan, "Iterative constrained weighted least squares source localization using TDOA and FDOA measurements," *IEEE Trans. Signal Process.*, vol. 65, no. 15, pp. 3990-4003, Aug. 2017.
- [17] E. Tzoreff and A. J. Weiss, "Path design for best emitter location using two mobile sensors," *IEEE Trans. Signal Process.*, vol. 65, no. 19, pp. 5249-5261, Oct. 2017.
- [18] J. Wang, J. Chen, and D. Cabric, "Cramer-Rao bounds for joint RSS/DoA-based primary-user localization in cognitive radio networks," *IEEE Trans. Wireless Commun.*, vol. 12, no. 3, pp. 1363-1375, Mar. 2013.
- [19] B. Yao, W. Wang, W. Han, and Q. Yin, "Distributed angle estimation by multiple frequencies synthetic array in wireless sensor localization system," *IEEE Trans. Wireless Commun.*, vol. 13, no. 2, pp. 876-887, Feb. 2014.
- [20] F. E. D. Raimondi, R. C. Farias, O. J. Michel, and P. Comon, "Wideband multiple diversity tensor array processing," *IEEE Trans. Signal Process.*, vol. 65, no. 20, pp. 5334-5346, Jul. 2017.
- [21] J. S. Abel, "A bound on mean-square-estimate error," *IEEE Trans. Inf. Theory*, vol. 39, no. 5, pp. 1675-1680, Sep. 1993.
- [22] Y. Wang and K. C. Ho, "TDOA positioning irrespective of source range," *IEEE Trans. Signal Process.*, vol. 65. no. 6, pp. 1447-1460, Mar. 2017.
- [23] J. Liang and D. Liu, "Passive localization of mixed near-field and far-field sources using two-stage MUSIC algorithm," *IEEE Trans. Signal Process.*, vol. 58. no. 1, pp. 108-120, Aug. 2009.
- [24] B. Wang, J. Liu, and X. Sun, "Mixed sources localization based on sparse signal reconstruction," *IEEE Signal Process. Lett.*, vol. 19. no. 8, pp. 487-490, Aug. 2012.
- [25] A. Bhattacharyya, "On some analogues of the amount of information and their use in statistical estimation," *Shunkya*, vol. 8, no. 1, pp. 1-14, 1946; no. 3, pp. 201-218, 1947.
- [26] J. M. Hammersley, "On estimating restricted parameters," J. Roy. Soc. Ser. B, vol. 12, pp. 192-240, 1950.
- [27] D. G. Chapman and H. Robbins, "Minimum variance estimation without regularity assumptions," *Ann. Math. Statist.*, vol. 21, pp. 581-586, 1951.
- [28] M. Grant, S. Boyd, and Y. Ye, CVX: MATLAB software for disciplined convex programming, 2009, [Online]. Available: http://www.stanford.edu/ boyd/cvx
- [29] Y. Wang and K. C. Ho, "Unified near-field and far-field localization for AOA and hybrid AOA-TDOA positionings," *IEEE Trans. Wireless Commun.*, to appear.