Quaternion Adaptive Line Enhancer based on Singular Spectrum Analysis

S. Sanei^{*}, C. C. Took[†], and S. Enshaeifar[†]

*School of Science and Technology, Nottingham Trent University, UK, [†]Department of Computer Science, University of Surrey, UK

Abstract-Quaternion adaptive line enhancer (QALE) has been proposed recently for the recovery of two- (2-D) or threedimensional (3-D) periodic signals from their noisy mixtures [1] with the help of quaternion-valued adaptive filtering theory. Similar to the traditional 1-D [2] version, QALE, relies mainly on the second order similarity between the signal and its delayed version and is more effective when the signal is narrowband. Here, quaternion-valued singular spectrum analysis (QSSA) [3] is used to develop a robust 3-D ALE system where in the reconstruction stage of QSSA the eigentriples are adaptively selected (filtered) using the delayed version of the data. Unlike the QALE where (second) order statistics are taken into account, in the proposed QSSA-QALE the full eigen-spectrum of the embedding matrix is exploited. Consequently, the system works for non-Gaussian noise and wideband 3-D periodic signals. The two systems have been implemented and their results compared. It is shown that the QSSA-QALE significantly outperformed QALE when the noise is not Gaussian.

Index Terms—Quaternion singular spectrum analysis, quaternion adaptive line enhancer, ALE, QSVD

I. INTRODUCTION

Most of natural signals and time series, which are often noisy in nature, contain periodic or cyclostationary components. Among them, those representing a kind of free movement are multidimensional. As an example, the gait movement trajectory recorded using a triaxial accelerometer is periodic or quasiperiodic in 3-D space [4]. These signals are often buried in noise or mixed with other periodic or aperiodic signals. Extraction of such cyclic signal components is very important for clinical or industrial assessment and monitoring.

The traditional adaptive line enhancer (ALE) was introduced by Widrow et al. [2] and widely used for the separation of a generally weak sinusoid, periodic, or narrowband signal from strong broad-band noise. On the other hand, in [5] the ALE has been improved using singular spectrum analysis SSA to enable the use of ALE for the recovery of wideband periodic signals corrupted with non-Gaussian noise.

In an ALE the input s(n) is assumed to be the sum of a narrow-band signal x(n) and a broad-band signal v(n)which is considered as noise. e(n) is the error signal between s(n) and the estimated signal $\hat{x}(n)$. The vector of prediction filter parameters **w** are iteratively and automatically adjusted based on e(n) so that the statistical mean squared error (MSE), $E[e^2(n)]$ is minimized, where $E[\cdot]$ stands for statistical expectation.

In the case where the periodic data is 3-D, such as the one in Fig. 1, successive application of 1-D ALE for each



Fig. 1: The noise free 3-D sinusoid; (left) its 3-D illustration and (right) its variation along each axis.



Fig. 2: Block diagram of the QALE proposed in [1] .

dimension in 3-D space is not effective in general for the recovery of such signals from their noisy versions as the signal may not exhibit periodicity in any one of the three dimensions. This can be seen in Fig. 1, where none of the components of the 3-D signal in any one of the three dimensions is periodic. The proposed algorithm in this article therefore, aims at recovering a 3-D periodic wave by developing an ALE which can operate in 3-D. For this purpose, recently, a quaternion based ALE (QALE) algorithm was proposed to restore narrowband 3-D periodic signals buried in Gaussian noise [1]. The block diagram of a QALE is depicted in Fig. 2, where $\mathbf{x}^{a}(n) = [\mathbf{x}^{T}(n) \quad \mathbf{x}^{H}(n)]^{T}$.

The QALE optimization is based on quaternion least mean square (QLMS) stochastic gradient adaptive filtering designed in [6] for filtering of hyper-complex processes. Such a system can be applied to both circular and noncircular signals and therefore, exploits the correlation between the real and complex components of a quaternion-valued signal. Their analysis has shown that for circular data in the quaternion (Hamiltonian, \mathbb{H}) domain the pseudocovariance $E\{\mathbf{x}\mathbf{x}^T\}$ does not vanish as it does in the complex domain \mathbb{C} .

Quaternions, used for more than 150 years (conceived by Hamilton in 1843), can be regarded as a noncommutative extension of complex numbers, and comprise of at most four variables [7]. A quaternion variable $q \in \mathbb{H}$ which has a real/scalar part $\Re(q)$ (here, denoted by subscript *a*), and a

vector part $\Im(q)$ comprising of three imaginary parts (denoted by subscripts *b*, *c*, and *d*), can be expressed as

$$q = [\Re(q), \Im(q)] = [q_a, \mathbf{q}] \in \mathbb{H}$$

= $[q_a, (q_b, q_c, q_d)]$
= $q_a + iq_b + jq_c + \kappa q_d$ $\{q_a, q_b, q_c, q_d\} \in \mathbb{R}$ (1)

where i, j, and κ are the orthogonal unit vectors and have the properties $ij = \kappa$, $j\kappa = i$, $\kappa i = j$, and $ij\kappa = i^2 = j^2 = \kappa^2 = -1$. Quaternions have found applications in computer graphics, for the modelling of three-dimensional (3-D) rotations [8], in robotics [9], molecular modelling [10], processing colour images [11], hyper-complex digital filters [12], texture segmentation [13], source separation [14], watermarking [15], spectrum estimation [16] quaternion singular value decomposition and in the MUSIC algorithm to process polarized waves [17], [18], quaternion least squares [10], [19], quaternion singular spectrum analysis (QSSA) [3] and very recently QALE [1]. In [6] the formulation for a quaternion LMS adaptive filtering has also been provided and used for the processing of quaternion valued signals.

The QALE however is most effective for restoration of narrowband 3-D periodic signals (such as sinusoids) from white Gaussian noise, as for the ALE of 1-D signals. To extend this to the detection of wideband 3-D signals in a process similar to that in [5], we combine QALE with QSSA [3], both relying on augmented statistics. Original SSA is a subspace based method for decomposition or prediction of 1-D time series. QSSA extends the SSA to multidimensional applications by exploiting the quaternion-valued statistics.

II. METHODOLOGY

A. QLMS and QALE

The conventional LMS algorithm minimises $E[ee^*]$ where $e(n) = d(n) - \mathbf{w}^T(n)\mathbf{x}(n)$, d(n) is the desired or target signal, $\mathbf{x}(n)$ is the input signal, $\mathbf{w}(n)$ is the vector of filter parameters, and $(\cdot)^*$, $(\cdot)^H$, and $(\cdot)^T$ refer to conjugate, conjugate transpose, and transpose operations for a vector respectively.

In an ALE however, $d(n) = x(n - \Delta)$ as mentioned in Section I and x(n) is a periodic noisy signal where the 1-D time delay $\Delta = mP$, P is the signal period and m is an integer. When the noise is white, m can be as small as unity.

In 3-D applications there is need for a quaternion delay along the signal base-line trajectory. This is naturally a shift equivalent to an integer multiple of the signal cycle period in the 3-D space.

In our application the quaternion input signal is defined as

$$x_q(n) = x_a(n) + ix_b(n) + jx_c(n) + \kappa x_d(n)$$
(2)

where $x_a(n)$, $\mathbf{x}_b(n)$, $x_c(n)$, and $x_d(n)$ are the four signals in four orthogonal directions. For a 3-D case, an example can be the hand movement in the x-y-z coordinates.

In the augmented quaternion least mean square (QLMS) proposed in [6] similar to original LMS, we have:

$$\mathbf{J}(n) = e(n)e(n)^* = e_a^2(n) + e_b^2(n) + e_c^2(n) + e_d^2(n)$$
(3)

In order for the QLMS to cater for general quaternion processes, a quaternion-valued semi-widely linear model can be employed [20];

$$\mathbf{y}(n) = \mathbf{w}^{T}(n)\mathbf{x}(n) + \mathbf{g}^{T}(n)\mathbf{x}(n)$$
(4)

This model incorporates the information contained in both the covariance, $\mathbf{C}_{xx} = E[\mathbf{x}\mathbf{x}^H]$, and pseudocovariance, $\mathbf{P}_{xx} = E[\mathbf{x}\mathbf{x}^T]$. According to [6], using QLMS the unified update equation is derived as:

$$\mathbf{h}^{a}(n+1) = \mathbf{h}^{a}(n) + \mu[2e^{a}(n)\mathbf{x}^{a*}(n) - \mathbf{x}^{a*}(n)e^{a*}(n)]$$
(5)

where

$$\mathbf{h}^{a}(n) = \begin{bmatrix} \mathbf{w}^{T}(n) & \mathbf{g}^{T}(n) \end{bmatrix}^{T}$$
(6)

where the augmented error is given by

$$e^{a}(n) = d(n) - \mathbf{h}^{aT}(n)\mathbf{x}^{a}(n)$$
(7)

and

$$\mathbf{x}^{a}(n) = \mathbf{x}^{a}_{s}(n) + \mathbf{v}^{a}(n)$$
(8)

is the augmented input noisy signal and the target signal for the QLMS filter, is a quaternion shift of the input signal i.e.

$$d(n) = \mathbf{r}^{a}(n) = \mathbf{x}^{a}(n - \Delta_{q})$$
(9)

vice versa. Therefore, the output $\mathbf{y}^a(n) = \hat{\mathbf{x}}^a_s(n)$ is an estimation of the noise free signal $\mathbf{x}^a_s(n)$.

Although in some cases such a 3-D shift, Δ_q , for at least one signal cycle is practically easy to obtain, such as those for a prescribed hand movement trajectory in an action research arm test (ARAT) [21], [22], in general, the problem is solved by estimating the 3-D shift Δ_q through the following simple optimization:

$$\hat{\Delta}_q = \max_{\Delta_q} \left\langle x^a(n) x^a(n - \Delta_q) \right\rangle \tag{10}$$

where $\langle . \rangle$ refers to temporal cross-correlation.

B. Augmented QSSA

The basic quaternion trajectory matrix $\mathbf{X} \in \mathbb{H}^{L \times K}$ can be generated through exploitation of augmented statistics as shown in Algorithm 1. Unlike real-valued SSA in which the covariance is generated as $E\{\mathbf{X}\mathbf{X}^T\}$, for AQSSA the basic trajectory matrix incorporates information augmented by all three quaternion involutions to generate the augmented trajectory matrix $\mathbf{X}^a \in \mathbb{H}^{4L \times K}$ as:

$$\mathbf{X}^{a} = [\mathbf{X}^{T}, \mathbf{X}^{iT}, \mathbf{X}^{jT}, \mathbf{X}^{kT}]^{T}$$
(11)

where $\mathbf{X}^{\alpha T}$, $\alpha \in \{i, j, k\}$, is the transpose of α -involution operation of the trajectory matrix \mathbf{W} . The generated \mathbf{X}^{a} is then used to compute the new augmented covariance matrix $\mathbf{C}^{a} \in \mathbb{H}^{4L \times 4L}$.

$$\mathbf{C}^{a} = E\{\mathbf{X}^{a}\mathbf{X}^{aH}\}$$

$$= \begin{bmatrix} \mathbf{C}_{\mathbf{X}\mathbf{X}} & \mathbf{C}_{\mathbf{X}^{i}} & \mathbf{C}_{\mathbf{X}^{j}} & \mathbf{C}_{\mathbf{X}^{k}} \\ \mathbf{C}_{\mathbf{X}^{i}}^{H} & \mathbf{C}_{\mathbf{X}^{i}\mathbf{X}^{i}} & \mathbf{C}_{\mathbf{X}^{i}\mathbf{X}^{j}} & \mathbf{C}_{\mathbf{X}^{i}\mathbf{X}^{k}} \\ \mathbf{C}_{\mathbf{X}^{j}}^{H} & \mathbf{C}_{\mathbf{X}^{j}\mathbf{X}^{i}} & \mathbf{C}_{\mathbf{X}^{j}\mathbf{X}^{j}} & \mathbf{C}_{\mathbf{X}^{j}\mathbf{X}^{k}} \\ \mathbf{C}_{\mathbf{X}^{k}}^{H} & \mathbf{C}_{\mathbf{X}^{k}\mathbf{X}^{i}} & \mathbf{C}_{\mathbf{X}^{k}\mathbf{X}^{j}} & \mathbf{C}_{\mathbf{X}^{k}\mathbf{X}^{k}} \end{bmatrix}$$

$$(12)$$

Algorithm 1 AQSSA algorithm **Decomposition**

$$\mathbf{x} = [x_1, \dots, x_N] \in \mathbb{H}$$
$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_K \\ x_2 & x_3 & x_4 & \dots & x_{K+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_L & x_{L+1} & x_{L+2} & \dots & x_N \end{bmatrix} \in \mathbb{H}^{L \times K}$$
$$\mathbf{X}^a = [\mathbf{X}^T, \mathbf{X}^{iT}, \mathbf{X}^{jT}, \mathbf{X}^{kT}]^T \in \mathbb{H}^{4L \times K}$$
$$\mathbf{C}^a = E\{\mathbf{X}^a \mathbf{X}^{aH}\} \text{ as in equation (12)}$$
$$\downarrow \text{ Quaternion SVD}$$
$$\mathbf{X}^a = \mathbf{U} \mathbf{\Lambda}^{1/2} \mathbf{V}^H$$
$$\mathbf{X}^a = \sum_{j=1}^r \mathbf{X}_j^a = \sum_{j=1}^r \sqrt{\lambda_j} \mathbf{u}_j \mathbf{v}_j^H$$
$$\text{ where } r = \max\{j : \lambda_j > 0\}$$

Reconstruction

$$\begin{split} \hat{\mathbf{X}}_{g}^{a} &= \sum_{j \in S_{g}} \mathbf{X}_{j}^{a} = \begin{bmatrix} \hat{f}_{11} & \hat{f}_{12} & \dots & \hat{f}_{1,K} \\ \hat{f}_{21} & \hat{f}_{22} & \dots & \hat{f}_{2,K+1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{f}_{L,1} & \hat{f}_{L,L+1} & \dots & \hat{f}_{L,N} \end{bmatrix} \\ \begin{cases} \hat{x}_{1} &= \hat{f}_{11} \\ \hat{x}_{2} &= (\hat{f}_{12} + \hat{f}_{21})/2 \\ \hat{x}_{3} &= (\hat{f}_{13} + \hat{f}_{22} + \hat{f}_{31})/3 \\ \dots \end{cases} \end{split}$$

where C_{XX} is the standard covariance matrix and the matrices C_{X^i} , C_{X^j} , and C_{X^k} are the complementary matrices. Thus, C^a captures the complete second order information. The steps of AQSSA are summarised in Algorithm 1 [3].

In the reconstruction stage, similar to the real-valued SSA, one of the main challenges is to find S_g , the group of the eigentriples for reconstructing the component of interest. The subspaces are generally characterised by various statistical or physical constraints based on some desired properties. In this work we aim to extract and restore the desired signal components assuming that the original source is periodic and its frequency components fall within a limited range.

III. QSSA-BASED QALE

The QSSA described in Algorithm 1 is used to develop a new QALE. The new QSSA-based QALE is depicted in Fig. 3. In this system the eigentriples of the QSSA are adaptively selected in a way to minimise the error between the reconstructed signal using QSSA and the delayed version of the original signal.

Unlike the QALE recently introduced in [1] in which the least mean square (LMS) error between the original signal and its delayed version is minimized, here the QSSA allows for filtering the signal before comparison and therefore makes the



Fig. 3: Block diagram of the prposed QSSA based QALE.

operation less sensitive to noise and its type. In this method the augmented diagonal matrix of parameters \mathbf{W}^a is applied to the augmented eigenvalue matrix $\mathbf{\Lambda}^a$ in order to select the correct eigenvalues adaptively:

$$\mathfrak{J}(\mathbf{W}^a) = ||\mathbf{R}^a - \mathbf{U}^a \mathbf{W}^a \mathbf{\Lambda}^{a\frac{1}{2}} \mathbf{V}^{aH}||_F^2$$
(13)

where \mathbf{R}^{a} is the augmented covariance matrix of the signal of interest and $||.||_{F}$ denotes Frobenius norm. \mathbf{U}^{a} , $\mathbf{\Lambda}^{a}$, (is a matrix of eigenvalues) and \mathbf{V}^{a} are the augmented quaternion SVD factors [17], as shown in Algorithm 1, and \mathbf{W}^{a} is a $4l \times 4l$ diagonal matrix of adaptive weights w_{ij} and has the same size as $\mathbf{\Lambda}^{a}$. In principle, an augmented gradient descent optimization approach similar to the augmented quaternion LMS as described before, can be followed to iteratively estimate \mathbf{W}^{a} ;

$$\mathbf{W}_{k+1}^{a} = \mathbf{W}_{k}^{a} - \mu \mathbf{U}^{a} \mathbf{\Lambda}^{a\frac{1}{2}} \mathbf{V}^{aH} (\mathbf{R}^{a} - \mathbf{U}^{a} \mathbf{W}_{k}^{a} \mathbf{\Lambda}^{a\frac{1}{2}} \mathbf{V}^{aH})^{H}$$
(14)

where μ is the iteration step size (which is usually set manually but can be reduced or adapted after each iteration). In the reconstruction process, \mathbf{W}^a is multiplied by $\Lambda^{a\frac{1}{2}}$ and the desired signal is recovered during the SSA reconstruction process.

IV. EXPERIMENT

The performance of QSSA-QALE was assessed for the synthetic signal shown in Fig. 4. This signal is constructed by adding white Gaussian 3-D noise to the signal in Fig. 1. This signal was created through applying the following equations:

$$\begin{aligned} x &= \sin(\alpha n)\cos(6\beta n) + \Gamma_x(n) \\ y &= \sin(\alpha n)\sin(6\beta n) + \Gamma_y(n) \\ z &= \gamma[n + \sin(\frac{n}{3})] + \Gamma_z(n) \end{aligned}$$

where $\Gamma(n)$ is the white Gaussian noise with different noise levels. We also examined both systems for non-Gaussian (heavy-tailed Laplacian) noise to enable comparison between QALE and SSA-QALE. The constants α , β , and γ can be changed; in this application they are set respectively to 3, 0.02, and 1. The target signal is also another later segment of the same signal with an interval Δ_q (equivalent to an integer number of signal cycles) which has been shifted forward along the 3-D direction. It is evident that with the added noise the signal in 3-D is not recognisable. In Figs. 7 and 8 the results of



Fig. 4: The noisy 3-D sinusoid; (left) its 3-D illustration and (right) its variation along each axis; Gaussian noise with SNR=5 dB.



Fig. 5: The result of applying the proposed QSSA-QALE to the 3-D signal of Fig. 4 and its comparison with the noise-free signals of Fig. 2; (left) 3-D illustration and (right) variation along each axis.

QALE and QSSA-QALE are demonstrated for Non-Gaussian noises. In our attempt, we considered that the baseline of movement and the sample directions were known a priori, so, the 3-D shift could be performed accurately. Observe that the noise effect in obscuring the signals is clearer in the 3-D cases than their 1-D counterparts. Certainly, by applying the traditional ALE successively to each of the x, y, and z dimensions, no conclusive result is expected. This is because, in general, the 3-D periodic signals are not necessarily periodic in any of the above directions.



Fig. 6: The original noisy sinusoid with non-Gaussian (Laplacian) noise and SNR=5 dB.



0.02 0.01 -0.01 -0.02 0.02 0.02 0.04 0.02 0.04 0.02 0.04 0.02 0.04

Fig. 8: The 3-D result of applying the proposed QSSA-based QALE to the simulated signals. Here, non-Gaussian noise with SNR=5 dB has been used.

By decreasing the signal-to-noise ratio (SNR), the performance of the algorithm deteriorates. The performance was evaluated in terms of mean square error (MSE) defined as:

$$MSE = \frac{\|\mathbf{x}_{s}^{a}(n) - \hat{\mathbf{x}}_{s}^{a}(n)\|^{2}}{\|\mathbf{x}_{s}^{a}(n)\|^{2}}$$
(15)

where $\|\cdot\|^2$ refers to Euclidean norm and each term can be expanded to sum square of its quaternion components, e.g.

$$\|\mathbf{x}_{s}^{a}(n)\|^{2} = \|\mathbf{x}_{s_{a}}^{a}(n)\|^{2} + \|\mathbf{x}_{s_{b}}^{a}(n)\|^{2} + \|\mathbf{x}_{s_{c}}^{a}(n)\|^{2}$$
(16)

The results of both QSSA-based QALE and QALEare depicted in Fig. 9. Evidently, the new approach in this paper significantly outperforms QALE when the noise is non-Gaussian.

V. CONCLUSIONS

A novel quaternion-valued adaptive line enhancer based on augmented QSSA has been proposed to cater for the recovery of 3-D periodic signals from their noisy counterparts particularly for non-Gaussian noise scenario. The results demonstrate that the proposed QSSA-QALE is more effective for 3-D signals compared to QALE recently proposed. In the design of proposed QALE we used the QLMS algorithm. For rigour, the performance of the algorithm has been evaluated in terms of MSE and compared with that of the original QALE, which uses QLMS and follows the traditional 1-D ALE design. There are many applications in nature for this technique. One example can be the recovery of hand tremor moving freely in an unconstrained 3-D motion.



Fig. 7: The 3-D result of applying the QALE to the simulated signals. Here, non-Gaussian (Laplacian) noise with SNR=5 dB has been used.

Fig. 9: Comparison between the performances of QALE and QSSA-QALE both *w.r.t* Gaussian as well as non-Gaussian (Laplacian) noises.

REFERENCES

- S. Sanei, C. C. Took, E. Shirin, and T. K. M. Lee, "Quaternion adaptive line enhancer," in *European Signal Processing Conference, EUSIPCO*, vol. 1, 2017, pp. 2694–2698.
- [2] B. Widrow, J. R. Glover Jr, J. M. McCool, J. Kaunitz, C. S. Williams, R. H. Hearn, J. R. Zeidler, E. Dong Jr, and R. C. Goodlin, "Adaptive noise cancelling: Principles and applications," *Proceedings of the IEEE*, vol. 63, no. 12, pp. 1692–1716, 1975.
- [3] S. Enshaeifar, S. Kouchaki, C. C. Took, and S. Sanei, "Quaternion singular spectrum analysis of electroencephalogram with application in sleep analysis," *IEEE Transactions on Neural Systems & Rehabilitation Engineering*, 2015.
- [4] S. Sanei and H. Hassani, Singular spectrum analysis of biomedical signals. CRC Press, 2015.
- [5] S. Sanei, T. K. Lee, and V. Abolghasemi, "A new adaptive line enhancer based on singular spectrum analysis," *IEEE Transactions on Biomedical Engineering*, vol. 59, no. 2, pp. 428–434, 2012.
- [6] C. C. Took and D. P. Mandic, "The quaternion lms algorithm for adaptive filtering of hypercomplex processes," *IEEE Transactions on Signal Processing*, vol. 57, no. 4, pp. 1316–1327, 2009.
- [7] W. R. Hamilton, *Elements of quaternions*. 2nd ed. London, U.K.: Longmans, Green, & Company, 1899.
- [8] S. B. Choe and J. J. Faraway, "Modeling head and hand orientation during motion using quaternions," SAE Technical Paper, Tech. Rep., 2004.
- [9] D. Biamino, G. Cannata, M. Maggiali, and A. Piazza, "Mac-eye: A tendon driven fully embedded robot eye," in *IEEE-RAS International Conference on Humanoid Robots*. IEEE, 2005, pp. 62–67.
- [10] C. F. Karney, "Quaternions in molecular modeling," Journal of Molecular Graphics and Modelling, vol. 25, no. 5, pp. 595–604, 2007.
- [11] S.-C. Pei and C.-M. Cheng, "Color image processing by using binary quaternion-moment-preserving thresholding technique," *IEEE Transactions on Image Processing*, vol. 8, no. 5, pp. 614–628, 1999.
- [12] H. Toyoshima, "Computationally efficient implementation of hypercomplex digital filters," in *IEEE International Conference on Acoustics*, *Speech and Signal Processing (ICASSP)*, vol. 3. IEEE, 1998, pp. 1761– 1764.
- [13] T. Bulow and G. Sommer, "Hypercomplex signals-a novel extension of the analytic signal to the multidimensional case," *IEEE Transactions on Signal Processing*, vol. 49, no. 11, pp. 2844–2852, 2001.
- [14] V. Zarzoso and A. K. Nandi, "Closed-form semi-blind separation of three sources from three real-valued instantaneous linear mixtures via quaternions," in *IEEE Symposium on Signal Processing and its Applications*, vol. 1. IEEE, 2001, pp. 1–4.
- [15] P. Bas, N. Le Bihan, and J.-M. Chassery, "Color image watermarking using quaternion fourier transform," in *IEEE International Conference* on Acoustics, Speech, and Signal Processing (ICASSP), vol. 3. IEEE, 2003, pp. III–521.
- [16] S. Said, N. Le Bihan, and S. J. Sangwine, "Fast complexified quaternion fourier transform," *IEEE Transactions on Signal Processing*, vol. 56, no. 4, pp. 1522–1531, 2008.
- [17] N. Le Bihan and J. Mars, "Singular value decomposition of quaternion matrices: a new tool for vector-sensor signal processing," *Signal processing*, vol. 84, no. 7, pp. 1177–1199, 2004.
- [18] S. Miron, N. Le Bihan, and J. Mars, "Quaternion-music for vector-sensor array processing," *IEEE Transactions on Signal Processing*, vol. 54, no. 4, pp. 1218–1229, 2006.
- [19] T. Jiang and L. Chen, "Algebraic algorithms for least squares problem in quaternionic quantum theory," *Computer Physics Communications*, vol. 176, no. 7, pp. 481–485, 2007.
- [20] B. Picinbono and P. Bondon, "Second-order statistics of complex signals," *IEEE Transactions on Signal Processing*, vol. 45, no. 2, pp. 411–420, 1997.
- [21] T. K. Lee, M. Belkhatir, and S. Sanei, "A comprehensive review of past and present vision-based techniques for gait recognition," *Multimedia Tools and Applications*, vol. 72, no. 3, pp. 2833–2869, 2014.
- [22] T. K. Lee, J. S. Lim, S. Sanei, and S. W. Gan, "Advances on singular spectrum analysis of rehabilitative assessment data," *Medical Imaging* and *Health Informatics*, vol. 5, no. 2, pp. 352–358, 2015.