ROBUST DENOISING OF PIECE-WISE SMOOTH MANIFOLDS

Shay Deutsch[1], *Antonio Ortega*[2] *and Gérard Medioni*[3]

[1]Department of Mathematics [2] Department of Computer Science, [3] Department of Electrical Engineering
[1] University of California, Los Angeles, [2,3] University of Southern California

ABSTRACT

A common smoothness model used in graph based regularization approaches is to require the energy of signals to be small with respect to the graph Laplacian of the graph. In this paper, we suggest an alternative approach which can effectively incorporate the high frequency information of the graph for unsupervised piece-wise smooth manifold denoising using Spectral Graph Wavelets. Our approach is based on a novel technique to remove noise from SGW coefficients estimated from a local tangent space based graph, which allows us to effectively regularize manifolds with singularities, such as for example intersecting manifolds. Experimental results on synthetic and real datasets in computer vision applications show that our proposed approach outperforms the state of the art, and is an effective tool to remove noise from manifolds with complex structures without oversmoothing at discontinuities.

Index Terms— Manifold Denoising, Graph Signal Processing, Spectral Graph Wavelets

1. INTRODUCTION

Many important machine learning problems require processing data on irregular, unstructured domains. Manifold learning methods [1, 2, 3, 4] address this problem by assuming that the data lives on a low dimensional manifold, and develop efficient algorithms to learn manifold global structure. However, the assumption that real-world data lies strictly or very close to a manifold does not often hold in practice. Manifold denoising methods [5] [6] were proposed in order to address this limitation, yet most of them over-fit or under-fit either the local or the global manifold structure for a non-trivial amount of noise in the data.

Similar to the works mentioned above, our ultimate goal, given a set of points, is to learn a manifold from them, but we focus on the first step of denoising the positions of the points. In our recent work [7] we showed that some of these limitations can be overcome using the spectral and vertex domain localization power of Spectral Graph Wavelets (SGW) [8]. To apply Spectral Graph Wavelets on high dimensional, unstructured data, we proposed to build a graph in which each vertex corresponds to one of the noisy observations, with edge weights between two vertices a function of the Euclidean distance between the corresponding observations in the ambient space. Then we applied Spectral Graph Wavelets to several graph signals, each representing one of the coordinates of a vertex in one of the dimensions of the ambient space. Denoising was then performed independently for each of the graph signals by discarding high frequency SGW transform coefficients, an approach that was justified by showing that for manifolds with sufficient smoothness properties, the coordinate dimensions of the signals are also smooth. The suggested approach was shown to be effective under a smoothness model in which a signal $\mathbf{f}$ is smooth if its graph Laplacian regularizer $\mathbf{f}^T\mathbf{L}\mathbf{f}$ is small. However, such an approach is not well-equipped to process manifolds with complex geometric structure such as manifolds with singularities, which may contain valuable information in the high frequency bands. While this is a common smoothness model in manifold learning and manifold denoising, we typically observe that for complex manifolds some important information may be lost when high frequency content is discarded [9], [10].

In this paper, we propose a new approach for unsupervised denoising of piecewise-smooth manifolds that generalizes our prior work to data with complex structure. Our regularization framework again uses Spectral Graph Wavelets, but we employ a fundamentally different approach than used in [7]. First, we propose a novel graph construction for SGW processing, which encodes the local geometric structure on the graph using a local tangent space affinity graph based on Tensor Voting [11], thus incorporating higher order information to the estimated spectral graph wavelets coefficients. In our prior work [7] a simple nearest neighbor graph in the ambient space was used. Second, we denoise by solving a diffusion process on the graph through the application Tikonov regularization in each estimated frequency band for each of the manifold coordinates. This is in contrast with our prior work, where we simply set to zero the high SGW frequencies before reconstructing the estimated denoised set of points. Note that the SGW transform is not critically sampled, and thus a coefficients in each band is associated to each node of the graph. In order to denoise the SGW coefficients we use a second graph, derived from the original graph, and such that the spectral energy is more tightly localized, which allows us to better handle complex manifolds at finer scales. Experimental results on both synthetic data-set and computer vision applications demonstrate that our approach significantly outperforms state of the art manifold denoising algorithms in estimating noisy manifolds with complex geometric structure.

This paper is organized as follows: in Section 2 we summarize the related work. Section 3 provides an overview of Spectral Graph Wavelets and Tensor Voting and the Tensor Voting Graph which are used for constructing the proposed graph. In Section 4 we describes our new approach for denoising manifolds with singularities. The experimental results are provided in Section 5 and in Section 6 we conclude our work.

2. RELATED WORK

Existing manifold denoising methods suffer from at least one of the following limitations: i) They tend to over-penalize either the local or global manifold structure, thus discarding significant features of the manifold; ii) They are sensitive to parameter selection on the graph, such as the choice of parameter k in a k nearest neighbor sparsification of the original similarity graph; iii) They make restrictive model assumptions, for example about manifold smoothness, which do not take into consideration the possible presence of discontinuities [9].

In our previous work, MFD [7], we proposed an effective way to perform manifold regularization on noisy manifolds that can better handle over-fitting or under-fitting by using Spectral Graph Wavelets as a tool that provides localization in the vertex and spectral domains. However, the suggested approach does not take into account more general classes of manifolds with singularities, for which critical information may manifest itself in the higher frequencies of the graph. For instance, in numerous computer vision application, (e.g motion segmentation), input features are observed to lie on multiple manifolds which may be intersecting [10], [12]. As another example, in molecular structure analysis in biology the data points have been observed to be piecewise smooth or even with a non-manifold structure [13]. Thus all the methods discussed above overlook a large class of piece-wise smooth manifolds that can occur in many applications.

3. PRELIMINARIES

We start with preliminary mathematical notations and definitions of Graph Signal Processing. Consider a set of points $\mathbf{x} = \{\mathbf{x}_i\}$, $i = 1, ...N, \mathbf{x}_i \in \mathbb{R}^D$, which are sampled from an unknown manifold M. For each $\mathbf{x}_i \in M$, let $T_{\mathbf{x}_i}M$ denote its tangent space, and let O_i be the subspace that corresponds to the local tangent space estimate of $T_{\mathbf{x_i}}M$. An undirected, weighted graph $G = (V, W)$ is constructed over $\mathbf{x}$, where V corresponds to the nodes and $\mathbf{W}$ to the set of edges on the graph. In this work, the adjacency matrix $\mathbf{W} = (w_{ij})$, consisting of the weights w_{ij} between node i and node j, is constructed based on local tangent space similarity filtered using the Gaussian kernel function as follows:

$$w_{ij} = \begin{cases} \tilde{w}_{ij} F(O_i, O_j) & \text{if } \mathbf{x}_j \in \text{kNN}(\mathbf{x}_i) \\ 0 & \text{else} \end{cases} \qquad (1)$$

where

$$\tilde{w}_{ij} = \begin{cases} \exp\left(\frac{-||\mathbf{x}_i - \mathbf{x}_j||_2^2}{2\sigma_d^2}\right) & \text{if } \mathbf{x}_j \in \text{kNN}(\mathbf{x}_i) \\ 0 & \text{else} \end{cases} \qquad (2)$$

kNN$(\mathbf{x}_i)$ denotes the k nearest Euclidean neighbors of $\mathbf{x}_i$, $||\;||$ denotes the $L2$ distance between the points $\mathbf{x}_i, \mathbf{x}_j$, and σ_d^2 is the RBF parameter. The local tangent space affinity function $F(O_i, O_j)$ is chosen based on the maximal principal angle between the sub-spaces O_i, O_j as follows [14]:

$$F(O_i, O_j) = \min_{u \in O_i} \max_{\tilde{v} \in O_j} \langle u, \tilde{v} \rangle \qquad (3)$$

where the estimate for the local tangent space is provided by Tensor Voting [15], and the affinity is estimated using the Tensor Voting Graph [16]. In order to characterize the global smoothness of a function $\mathbf{f}_r \in \mathbb{R}^N$, we define its graph Laplacian quadratic form with respect to the graph as:

$$|| \bigtriangledown \mathbf{f}_r||^2 = \sum_{V(i,j)} w_{ij} (f_r(i) - f_r(j))^2 = \mathbf{f}_r^T \mathbf{L} \mathbf{f}_r, \qquad (4)$$

where $\mathbf{L}$ denotes the combinatorial graph Laplacian, defined as $\mathbf{L} = \mathbf{D} - \mathbf{W}$, with $\mathbf{D}$ the diagonal degree matrix with entries $d_{ii} = d(i)$, where $d(i)$ is the degree of the node i. The eigenvalues and eigenvectors of $\mathbf{L}$ are $\lambda_1, \ldots, \lambda_N$ and $\phi_1, \ldots, \phi_N$, respectively. The graph Fourier transform (GFT) $\hat{f}_r$ of the function f_r (which is a function over the vertices of the graph G), is defined as the expansion of f_r in terms of the eigenvectors ϕ of the graph Laplacian, so that for frequency λ_l we have: $\hat{f}_r(\lambda_l) = \sum_i f_r(i)\phi_l^*(i)$. Note that in practice, instead of $\mathbf{x}$, we observe a set of noisy points $\tilde{\mathbf{x}}$ that do not lie strictly on the manifold. In order to learn the manifold M we wish to denoise the coordinates of $\tilde{\mathbf{x}}$ to obtain an approximation $\hat{\mathbf{x}}$ of the noise-free coordinates.

3.1. Spectral Graph Wavelets

Spectral graph Wavelets (SGW) [8] define a scaling operator in the Graph Fourier domain introduced in the previous section. SGWs are constructed using a kernel function operator $T_g = g(\mathbf{L})$ which acts on a function f_r by modulating each of its Fourier modes: $\widehat{T_g f_r(\lambda_l)} = g(\lambda_l)\widehat{f}_r(\lambda_l)$. Given a function f_r, the wavelet coefficients take the form:

$$\Psi_{f_r}(s, n) = (T_g^s f_r(n)) = \sum_{l=1}^{N} g(s\lambda_l)\widehat{f}_r(\lambda_l)\phi_l(n). \qquad (5)$$

The SGW can be computed with a fast algorithm based on approximating the scaled generating kernels by low order polynomials. The wavelet coefficients at each scale can then be computed as a polynomial of $\mathbf{L}$ applied to the input data. When the graph is sparse, which is typically the case under the manifold learning model, the computational complexity scales linearly with the number of points, leading to a computational complexity of $O(N)$ [8] for an input signal $\mathbf{f}_r \in \mathbb{R}^N$. Including a scaling function corresponding to a low pass filter operation, SGWs map an input graph signal, a vector of dimension N, to $N(J+1)$ scaling and wavelet coefficients, which are computed efficiently using the Chebyshev polynomial approximation.

We will also use the following definitions

Definition 1 *Let $\mathcal{N}(n, K)$ denote the set of vertex n's neighbors on the graph that are within K hops away from n.*

Definition 2 *Let $\mathbf{W}_{\mathcal{N}(K)}$ and $\mathbf{L}_{\mathcal{N}(K)}$ denote the affinity matrix and its corresponding Laplacian, which are obtained using affinity function (1) and connecting all vertices i on the graph that are $N(n, K)$ hops apart on G from n for each $n \in V$.*

As an example for the definitions above, for $K = 1$ we have that $\mathbf{W}_{\mathcal{N}(K=1)} = \mathbf{W}$ and $\mathbf{L}_{\mathcal{N}(K=1)} = \mathbf{L}$.

3.2. Local Tangent Space Estimation

For the local tangent space estimation, we use Tensor Voting [15], and the Tensor Voting Graph [16] for the local tangent space affinity $F(O_i, O_j)$. Given two point $\mathbf{x}_i, \mathbf{x}_j$ the Tensor vote from $\mathbf{x}_j$ to $\mathbf{x}_i$ is given by:

$$\mathbf{T}_{ij} = \exp^{\frac{-||\mathbf{x}_i - \mathbf{x}_j||_2^2}{\sigma}} \left(\mathbf{I} - \frac{(\mathbf{x}_i - \mathbf{x}_j)(\mathbf{x}_i - \mathbf{x}_j)^T}{||\mathbf{x}_i - \mathbf{x}_j||_2^2} \right) \qquad (6)$$

and the local information accumulated at each point is

$$\mathbf{T}_i = \sum_{\mathbf{x}_j \in B(\mathbf{x}_i, \sigma))} \exp^{\frac{-||\mathbf{x}_i - \mathbf{x}_j||_2^2}{\sigma}} \left(\mathbf{I} - \frac{(\mathbf{x}_i - \mathbf{x}_j)(\mathbf{x}_i - \mathbf{x}_j)^T}{||\mathbf{x}_i - \mathbf{x}_j||_2^2} \right) \qquad (7)$$

where $\mathbf{I}$ is the $D \times D$ identity matrix, $\mathbf{T}_i$ is a Semi-Positive matrix, and $B(\mathbf{x}_i, \sigma)$ is a ball of radius σ around $\mathbf{x}_i$ which contain all points within distance equal or smaller σ from $\mathbf{x}_i$. for clarity, we modified the sentence as follows: From the Eigendecomposition of each $\mathbf{T}_i$ we obtain a set of eigenvalues $\{\alpha_{ik}\}_{k=1}^{D}$, $\alpha_{i1} \geq \alpha_{i2} ... \geq \alpha_{iD}$ and their corresponding eigenvectors $\{\hat{e}_{ik}\}_{k=1}^{D}$. Note that across different samples, the tensor $\mathbf{T}_i$ quantifies the potential information at each point $\mathbf{x}_i$ as the samples move along the normal directions of the manifold. Using the eigendecomposition of each tensor, the tensor voting graph can be constructed. The tensor voting graph employs a graph construction in which the affinity between points on the graph corresponds to the contribution that was made to the tangent space point estimation by the neighboring points that participated in the voting process. For more details we refer to [15], [16].

4. PROPOSED APPROACH

We now present our denoising approach for denoising manifolds with singularities. In our graph construction, each vertex corresponds to one of the noisy observations of the manifold, and edge weights that are based on local tangent space distance affinity (defined in (1)). After constructing the local tangent space based graph, we take the SGW transform using low order polynomials to ensure tight vertex localization of the SGW coefficients themselves. Denoising the Spectral Graph wavelet coefficients $\tilde{\Psi}_{\tilde{\mathbf{f}}_{\mathbf{r}}}(s(j), n)$, $n = 1..N$ is performed independently for each of the SGW bands $\mathbf{\Psi}_{\tilde{\mathbf{f}}_r}(s(j))$, $s(j)$, $2 \leq j \leq J$, and for each r, $1 \leq r \leq D$ corresponding to each noisy graph signal $\tilde{f}_r()$, $\tilde{f}_r(n) = f_r(n) + \epsilon_r(n)$, i.e., the noisy values of all sampled points in dimension r. At the final stage, we take the inverse spectral wavelet transform of the denoised spectral graph wavelet coefficients $\mathbf{\Psi}^*_{\tilde{\mathbf{f}}_{\mathbf{r}}}(s(j))$ to obtain the denoised graph signal $\hat{f}_r()$, which corresponds to the manifold coordinate values of dimension r. The full reconstructed manifold points are provided by $\hat{\mathbf{x}}_n = \Big(\hat{f}_1(n), \hat{f}_2(n)...\hat{f}_D(n)\Big)$, for each point $\hat{\mathbf{x}}_n$. A brief description of the proposed algorithm and a pseudo code is provided next.

4.1. Description of the Proposed Algorithm

The main idea of Algorithm 1 is to denoise the SGW coefficients $\tilde{\Psi}_{\tilde{\mathbf{f}}_{\mathbf{r}}}(s(j))$ that correspond to the vertex and spectral localization encoded in the Laplacian $\mathbf{L}_{\mathcal{N}(j)}$ obtained form $\mathbf{W}_{\mathcal{N}(j)}$. For each coordinate dimension r and for each SGW bands $\mathbf{\Psi}_{\tilde{\mathbf{f}}_r}(s(j))$, j, $2 \leq j \leq J$ we apply the Tikhonov regularization directly to the SGW coefficients

$$\min_{\mathbf{\Psi}_{\mathbf{f}_r}(s)} \Big\{||\mathbf{\Psi}_{\mathbf{f}_r}(s) - \mathbf{\Psi}_{\tilde{\mathbf{f}}_r}(s)||_2^2 + \gamma \mathbf{\Psi}^T{}_{\mathbf{f}_r}(s) \mathbf{L}_{\mathcal{N}(j)} \mathbf{\Psi}_{\mathbf{f}_r}(s)\Big\} \tag{8}$$

Using equality (19) in [17] and replacing the graph signal $\tilde{\mathbf{f}}_r$ with the SGW band coefficients $\mathbf{\Psi}_{\tilde{\mathbf{f}}_r}(s)$, it can be shown that the optimal solution to this problem is

$$\Psi^*{}_{\tilde{\mathbf{f}}_r}(s, n) = \sum_{l=1}^{N} [\frac{1}{1 + \gamma \lambda_l^j}] \hat{\Psi}_{\tilde{\mathbf{f}}_r}(s, \lambda_l) \phi_l(n) \tag{9}$$

where $\hat{\Psi}_{\tilde{\mathbf{f}}_r}(s, \lambda_l)$ is the Graph Fourier transform of $\mathbf{\Psi}_{\tilde{\mathbf{f}}_r}(s)$. To solve this problem efficiently, we use a few steps of a diffusion process on the fixed graph $\mathbf{W}_{\mathcal{N}(j)}$, by solving:

$$\mathbf{\Psi}^*{}_{\tilde{\mathbf{f}}_r}(s(j)) = (\mathbb{I} + \gamma \mathbf{L}_{\mathcal{N}(j)})^{-1} \mathbf{\Psi}_{\tilde{\mathbf{f}}_r}(s(j)) \tag{10}$$

Note that one step of a diffusion process on the graph is equivalent to solving Tikhonov regularization [6]. Thus, in the denoising approach we treat the SGW coefficients as graph signals themselves, simultaneously providing localized vertex and spectral domain information of the graph signal at different bandwidth, to which denoising is applied (to all spectral bands covering the spectral information of the graph, i.e the graph eigenvalues). This strategy allows us to efficiently regularize the manifold in fine scales. We note that the proposed graph construction allows us to apply denoising using a novel piece-wise smooth model rather than the standard smoothness model requiring a small energy of the graph Laplacian with respect to the graph signal defined on it. We note that Tikonov regularization was used in this in this case due to its well known connection with diffusion processes and its effectiveness in smoothing the SGW coefficients. However for certain classification tasks, $l1$ type regularization based methods may be considered.

Algorithm 1: Denoising manifolds with singularities

Data: The data set $\tilde{\mathbf{x}}$, σ is the Tensor Voting scale, k nearest neighbors on the local tangent space, m - order of Chebyshev polynomial approximation

```
Construct W based on local tangent space distance as in
  Equation 1. Construct L from W, and L_N(j) from W_N(j)
  for each j = 2..J
for r ← 1 to D do
    Assign the corresponding coordinate values f̃_r to its
      corresponding vertex on the graph. ;
    Transform the noisy graph signals f̃_r using SGW.
    for j ← 2 to J do
        Solve (10) for SGW coefficients Ψ_f̃_r(s(j)) with
          respect to L_N(j).
    Take the inverse spectral wavelet transform.
```

Result: The reconstructed manifold points $\hat{\mathbf{x}}$.

5. EXPERIMENTAL RESULTS

We show experimental results on manifolds with singularities including synthetic and real datasets. We compare our method to state of the art denoising methods, and evaluate the effect of denoising in terms of its effect on the local geometric structure, as well as in clustering applications. In our experiments we $m = 5$ as the order of the Chebyshev polynomial approximation and $J = 5$ as the number of scales the SGW transform. For the choice of k nearest neighbor graph we used $k = 20$ in all the synthetic datasets experiments. An appealing property of our approach, also similar to [7], is its robustness to k nearest neighbor selection on the graph, leveraging the vertex and frequency localization properties of SGW. To solve (10) we used the implicit Euler with $\gamma = \mu \delta t$ where $\mu = 1$ is the diffusion constant and $\delta t = 0.25$ is the time step.

5.1. Experimental results on intersecting manifolds

For the synthetic datasets, all manifolds were sampled using a uniform distribution, and all dimensions were contaminated with isotropic Gaussian noise. The effect of denoising in the local intersection area can be seen in Figures 1 and 2. Table 1 compares the local tangent space error for all points before and after denoising, showing a significant reduction in the local tangent space estimation error using our denoising approach. We also show the effect of denoising in terms of clustering accuracy before and after denoising of two intersecting spheres in Table 2, showing significant improvement after denoising.

Data	Noisy Data	Denoised data
Intersecting Circles	10.81	**5.8**

Table 1. Average error of local tangent space estimation before and after denoising.

Data	Noisy Data	Denoised data
Clustering accuracy	86%	96 %

Table 2. Clustering accuracy of two intersecting spheres before and after denoising

5.2. Denoising Motion Capture Data

We experiment with denoising real data from the CMU motion capture dataset, which is a dataset of human motion sequences. We choose sequences from subject 86, and contaminate the data using Gaussian noise of variance 0.2 in all dimensions. The chosen sequences are mixed and correspond to different motions, therefore can be possibly viewed as a manifold with singularities [10]. Table 3 shows a comparison of the denoising results of our method in terms of RMSE error to other existing manifold denoising methods. Note that our approach is competitive with MFD, while in our proposed approach we do not rely on thresholding the number of bands as in [7], which makes our proposed method relatively less sensitive to parameter selection.

5.3. Application to Motion Segmentation

We also test our framework on a noisy set of feature points in the problem of motion segmentation. In this problem, we are given a set of feature points that are tracked through a sequence of video frames. We evaluate our regularization method on the Hopkins 155 motion database, where the goal is to segment a video sequence into multiple spatiotemporal regions corresponding to different rigid-body motions. We test the robustness of our method on the three motion data, which is more challenging, since it contains more intersections between the manifolds that represent different objects. In practice, the feature trajectories would be almost always corrupted, and thus we contaminated the feature trajectories using a large amount of Gaussian noise in all dimensions with variance 0.2. After performing regularization using our method we construct a new affinity graph using the denoised features and perform Spectral Clustering [19]. We compare our method to SSC [18] which is the state of the art method for clustering multiple linear intersecting manifolds. The comparison of our method in Table 4 shows that our regularization achieves significantly better classification accuracy than the state of the art.

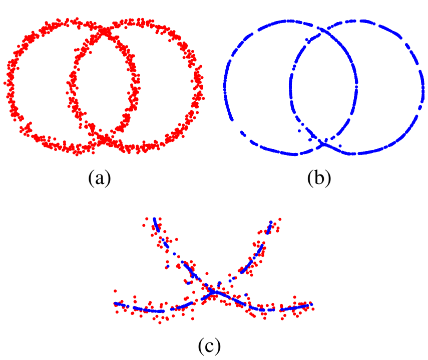

Fig. 1. Experimental results on two noisy intersecting circles (a) Noisy circles (b) Results with our new denoising method (c) Zooming into the local intersection area.

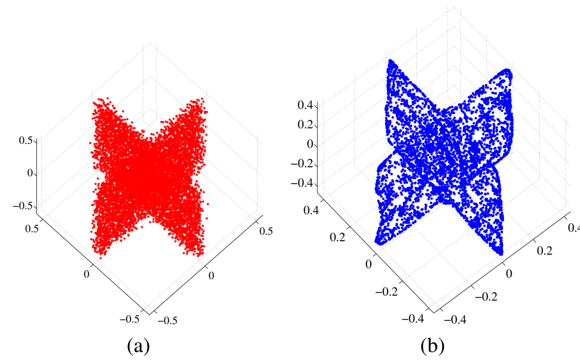

Fig. 2. Experimental results on noisy intersecting planes (a) Noisy intersecting planes (b) The denoised planes using our method.

Method/RMSE	Mocap data
Proposed	**6.5**
MFD [7]	7.2
MD [6]	32.5
LLD [5]	14.5

Table 3. RMSE error on Mocap data

6. DISCUSSION

We presented a new approach for unsupervised denoising of non-linear piecewise-smooth manifolds. Our approach significantly extends previous work in unsupervised manifold denoising, by using a novel graph construction that effectively encodes local geometric structure in the computed SGW coefficients, leading to an effective denoising framework on a large class of complex manifolds and moreover is not restricted to signals where the energy of the graph Laplacian is small. Experimental results on a range of both synthetic and real datasets showed that our suggested method significantly outperforms the state of the art manifold denoising methods. Future work includes an explicit handling of boundary conditions [20].

Method	Accuracy	Three Motions
SSC [18]	Mean	74.1%
	Median	81.2%
Proposed	Mean	**80.4%**
	Median	**85.2%**

Table 4. Clustering accuracy (%) of different methods on the Hopkins 155 motion segmentation database distorted with severe amount of noise.

7. REFERENCES

[1] J. Tenenbaum, V. de Silva, and J. Langford, "A Global Geometric Framework for Nonlinear Dimensionality Reduction," *Science*, vol. 290, no. 5500, pp. 2319–2323, 2000.

[2] D.L. Donoho and C. Grimes, "Hessian eigenmaps: Locally linear embedding techniques for high dimensional data," in *Proceedings of the National Academy of Sciences of the United States of America*, 2003, vol. 100, pp. 5591–5596.

[3] S. Roweis and L. Saul, "Nonlinear dimensionality reduction by locally linear embedding," *SCIENCE*, vol. 290, pp. 2323–2326, 2000.

[4] M. Belkin and P. Niyogi, "Laplacian eigenmaps for dimensionality reduction and data representation," *Neural Computation*, vol. 15, no. 6, pp. 1373–1396, 2003.

[5] Dian Gong, Fei Sha, and Gérard G. Medioni, "Locally linear denoising on image manifolds.," in *AISTATS*. 2010, JMLR Proceedings, pp. 265–272, JMLR.org.

[6] M. Hein and M. Maier, "Manifold denoising," in *Advances in Neural Information Processing Systems 19*. Max-Planck-Gesellschaft, 2007, MIT Press.

[7] Shay Deutsch, Antonio Ortega, and Gerard Medioni, "Manifold denoising based on spectral graph wavelets," *IEEE International Conference on Acoustics, Speech and Signal Processing ICASSP*, 2016.

[8] David K. Hammond, Pierre Vandergheynst, and Rémi Gribonval, "Wavelets on graphs via spectral graph theory," *Applied and Computational Harmonic Analysis*, 2011.

[9] Mikhail Belkin, Qichao Que, Yusu Wang, and Xueyuan Zhou, "Graph laplacians on singular manifolds: Toward understanding complex spaces: graph laplacians on manifolds with singularities and boundaries," *CoRR*, vol. abs/1211.6727, 2012.

[10] Shay Deutsch and Gérard G. Medioni, "Intersecting manifolds: Detection, segmentation, and labeling," in *Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence, IJCAI*, 2015.

[11] P. Mordohai and G. Medioni, *Tensor Voting: A Perceptual Organization Approach to Computer Vision and Machine Learning*, Morgan & Claypool Publishers, 2006.

[12] Shay Deutsch and Gérard G. Medioni, "Learning the geometric structure of manifolds with singularities using the tensor voting graph," *Journal of Mathematical Imaging and Vision*, vol. 57, no. 3, pp. 402–422, 2017.

[13] Martin S. Pollock S.N. Coutsias E.A. Watson J.P. Brown, W.M., "Algorithmic dimensionality reduction for molecular structure analysis," *Journal of Chemical Physics*, 2008.

[14] Partha Niyogi, Stephen Smale, and Shmuel Weinberger, "Finding the homology of submanifolds with high confidence from random samples," *Discrete & Computational Geometry*, vol. 39, no. 1, pp. 419–441, 2008.

[15] P. Mordohai and G. Medioni, "Dimensionality estimation, manifold learning and function approximation using tensor voting," *Journal of Machine Learning Research*, vol. 11, pp. 411–450, 2010.

[16] Shay Deutsch and Gérard G. Medioni, "Unsupervised learning using the tensor voting graph," in *Scale Space and Variational Methods in Computer Vision - 5th International Conference, SSVM 2015, Lège-Cap Ferret, France, May 31 - June 4, 2015, Proceedings*, 2015, pp. 282–293.

[17] D. I Shuman, P. Vandergheynst, and P. Frossard, "Chebyshev polynomial approximation for distributed signal processing," in *Proceedings of the IEEE International Conference on Distributed Computing in Sensor Systems*, Barcelona, Spain, June 2011.

[18] E. Elhamifar and R. Vidal, "Sparse manifold clustering and embedding," in *Advances in Neural Information Processing Systems*, pp. 55–63. 2011.

[19] L. Van der Maaten and G. Hinton, "Visualizing Data using t-SNE," *Journal of Machine Learning Research*, vol. 9, pp. 2579–2605, 2008.

[20] Zuoqiang Shi, Jian Sun, and Minghao Tian, "Harmonic extension," *CoRR*, vol. abs/1509.06458, 2015.