Recovering signals from their FROG trace

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***Abstract*—The problem of recovering a signal from its power spectrum is called phase retrieval. This problem appears in a variety of scientific applications, such as ultra-short laser pulse characterization and diffraction imaging. However, the problem for one-dimensional signals is ill-posed as there is no one–to–one mapping between a one-dimensional signal and its power spectrum. In the field of ultra-short laser pulse characterization, it is common to overcome this ill-posedness by using a technique called Frequency-Resolved Optical Gating (FROG). In FROG, the measured data, referred to as FROG trace, is the Fourier magnitude of the product of the underlying signal with several translated versions of itself. Therefore, in order to recover a signal from its FROG trace, one needs to invert a system of phaseless quartic equations. In this paper, we explore the symmetries and uniqueness of the FROG mapping. Our main result states that a signal bandlimited to B is determined uniquely, up to symmetries, by only $3B$ FROG measurements.**

***Index Terms*—phase retrieval, phaseless quartic system of equations, ultra-short laser pulse characterization, FROG**

I. Introduction

In many scientific and engineering applications, one aims to estimate a signal from its power spectrum, or equivalently, from its auto-correlation. This problem is called *phase retrieval* and it appears in many applications, such as X-ray crystallography, speech recognition, blind channel estimation, alignment and astronomy [1], [2], [3], [4], [5], [6], [7].

Almost all one-dimensional signals cannot be determined uniquely from their power spectra. Two exceptions are minimum phase signals and sparse signals with non-periodic support [8], [9]. In order to overcome the fundamental ill-posedness for general signals, one needs to acquire additional information on the sought signal. For instance, this can be done by taking measurements with multiple known masks [10], [11]. As an important special case, the masks can be translated versions of a single reference mask. In this case, the acquired data is the phaseless short-time Fourier transform (STFT) of the sought signal. It has been shown that under different conditions and scenarios, this information is sufficient for efficient and stable recovery [12], [13], [14], [15], [16], [17], [18]. For a recent survey of phase retrieval from a signal processing point–of–view; see [19].

D. Edidin acknowledges support from Simons Collaboration Grant 315460. Y.C. Eldar acknowledges support from the European Union's Horizon 2020 research and innovation program under grant agreement No. 646804–ERCCOG–BNYQ, and from the Israel Science Foundation under Grant no. 335/14.

Here, we consider an extension of the standard phase retrieval problem in which the measurements are a phaseless *quartic* function of the underlying signal. This quartic problem arises in an ultra-short laser pulse characterization method, called Frequency-Resolved Optical Gating (FROG). FROG serves as a simple, commonly-used, technique for full characterization of ultra-short laser pulses and enjoys good experimental performance [20]. In order to characterize the signal, the FROG device measures the Fourier magnitude of the product of the signal with a translated version of itself, for several different translations. The acquired data is called *FROG trace*. We refer to the inverse problem of recovering a signal from its FROG trace as the *quartic phase retrieval problem*. An illustration of the FROG setup is presented in Figure I.1.

In this paper we provide sufficient conditions on the number of samples required to determine a bandlimited signal uniquely, up to trivial ambiguities, from its FROG trace. Particularly, we show that it is sufficient to consider only three translations of the signal to determine almost all bandlimited signals. If one can also measure the power spectrum of the signal, then it is sufficient to consider only two translations. Surprisingly, the required number of measurements is almost the same as in the STFT phase retrieval problem, although the FROG method does not require a reference signal.

The outline of this paper is as follows. In Section II we formulate the FROG problem and discuss its symmetries. Section III presents our main result. The outline of the proof is given in Section IV. Section V concludes the paper.

Throughout the paper we use the following notation. We denote the Fourier transform of a signal $z \in \mathbb{C}^N$ by $\hat{z}_k = \sum_{n=0}^{N-1} z_n e^{-2\pi \iota kn/N}$, where $\iota := \sqrt{-1}$. We further use $\overline{z}$ for its conjugate. We reserve $x \in \mathbb{C}^N$ to be the underlying signal. In the sequel, all signals are assumed to be periodic with period N and all indices should be considered as modulo N, i.e., $z_n = z_{n+N\ell}$ for any integer $\ell \in \mathbb{Z}$.

II. Mathematical formulation of the FROG problem and its symmetries

In this section, we introduce the FROG problem and identify its symmetries. We note, however, that FROG includes several techniques that manipulate the underlying signal and its delayed versions in several ways [20]. In this paper we focus

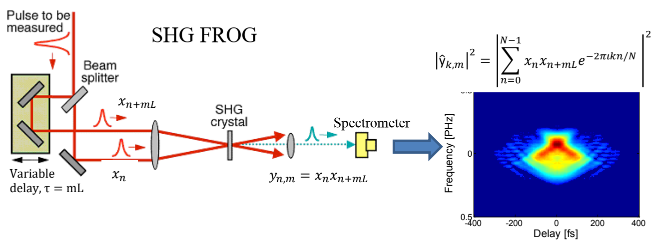

Fig. I.1. Illustration of the SHG FROG technique (courtesy of [21]).

only on the ubiquitous model of Second–Harmonic Generation (SHG) FROG.

Let us define the bivariate signal

$$y_{n,m} = x_n x_{n+mL}, \tag{II.1}$$

where L is a fixed positive integer. The FROG trace is equal to the one-dimensional Fourier magnitude of $y_{n,m}$ for each fixed m, namely,

$$|\hat{y}_{k,m}|^2 = \left|\sum_{n=0}^{N-1} x_n x_{n+mL} e^{-2\pi\iota nk/N}\right|^2. \tag{II.2}$$

To ease notation, we assume hereinafter that L divides N.

Our analysis holds for bandlimited signals. Formally, we define a bandlimited signal as follows:

Definition II.1. We say that $x \in \mathbb{C}^N$ is a *B-bandlimited signal* if its Fourier transform $\widehat{x}$ contains $N - B$ consecutive zeros.

The FROG trace (II.2) is a quartic intensity map $\mathbb{C}^N \mapsto \mathbb{R}^{N\times\frac{N}{L}}$ that has three symmetries. In phase retrieval, they are commonly referred to as *trivial ambiguities*. These symmetries are the set of operations whose action on the signal does not change the intensity map. In other words, these are the invariants of the FROG mapping. Indeed, the FROG trace is invariant under global rotation, global translation and reflection [21]. While the first symmetry is continuous, the latter two are discrete. These symmetries are similar to analog results in phase retrieval, see for instance [22], [19]. The next proposition summarizes the symmetries and shows that for bandlimited signals—which are the main interest of this paper—the global translation ambiguity is also continuous.

Proposition II.2 ([23])**.** *Let $x \in \mathbb{C}^N$ be the underlying signal and let $\widehat{x} \in \mathbb{C}^N$ be its Fourier transform. Let $|\hat{y}_{k,m}|^2$ be the FROG trace of x as defined in* (II.2) *for some fixed L. Then, the following signals have the same FROG trace as x:*

1) *the rotated signal $xe^{\iota\psi}$ for some $\psi \in \mathbb{R}$;*
2) *the translated signal x^ℓ obeying $x_n^\ell = x_{n-\ell}$ for some $\ell \in \mathbb{Z}$ (equivalently, a signal with Fourier transform $\widehat{x}^\ell$ obeying $\widehat{x}_k^\ell = \widehat{x}_k e^{-2\pi\iota\ell k/N}$ for some $\ell \in \mathbb{Z}$);*
3) *the reflected signal $\tilde{x}$ obeying $\tilde{x}_n = \overline{x_{-n}}$.*

For B-bandlimited signals as in Definition II.1 with $B \leq N/2$, the translation ambiguity is continuous. Namely, any signal with a Fourier transform obeying $\widehat{x}_k^\psi = \widehat{x}_k e^{\iota\psi k}$ for some $\psi \in \mathbb{R}$ has the same FROG trace as x.

Figure II.1 illustrates a 5-bandlimited real signal of length $N = 11$ with Fourier transform given by $\widehat{x} = (1, \iota, -\iota, 0, 0, 0, 0, 0, 0, \iota, -\iota)^T$. The second signal is x shifted by three entries. A third signal is a “translated” version of x by 1.5 entries. Namely, the kth entry of its Fourier transform is given by $\widehat{x}_k e^{-2\pi\iota(1.5)k/N}$. Clearly, the third signal is not equal to x, up to global translation. Nonetheless, since x is bandlimited, all three signals have the same FROG trace.

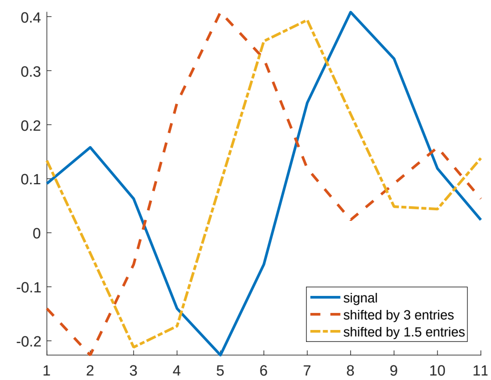

Fig. II.1. This figure presents three 5-bandlimited signals. Two of them are shifted versions of each other. The third one is “shifted” by 1.5 entries, namely, the kth entry of its Fourier transform is modulated by $e^{-2\pi\iota(1.5)k/N}$. All signals have the same FROG trace (II.2) (courtesy of [19]).

From an algebraic perspective, the symmetries of the FROG trace form a group. Particularly, the FROG intensity map $\mathbb{C}^N \to \mathbb{R}^{N\times N/L}$ is invariant under the action of the group $G = S^1 \times \mu_N \ltimes \mu_2$, where S^1 corresponds to the continuous rotation ambiguity on the circle and $\ltimes$ denotes a semi-direct product. Here, μ_2 and μ_N correspond to the discrete reflection and translation symmetries, respectively. Observe that we use a semi-direct product for the last symmetry since μ_2 and μ_N do not commute; if one reflects the signal and then translates it, it is not the same as translating and then reflecting. Interestingly, $\mu_N \ltimes \mu_2$ is the dihedral group D_{2N} of symmetries of the regular N-gon. If we consider bandlimited signals, then the translation ambiguity is continuous and the FROG trace is invariant under the action of the group $G = S^1 \times S^1 \ltimes \mu_2$.

III. Main Result

We are now ready to present the main result of this paper. Our result states that almost any B-bandlimited signal is determined by its FROG trace, up to trivial ambiguities, as long as $L \leq N/4$ and $B \leq N/2$. Particularly, we show that we need to take into account only three translations. Consequently, $3B$ measurements are enough to determine the underlying signal. For instance, if $L = N/4$ then the measurements corresponding to $m = 0, 1, 2$ determine the signal. The bandlimited assumption is met in standard ultrashort pulse characterization experiments [24]. If in addition we have access to the signal's power spectrum, then it suffices to choose $L \leq N/3$. In this case, one may consider only two translations. For example, if $L = N/3$, then one can choose $m = 0, 1$. Indeed, the power spectrum of the sought pulse is often available, or it can be measured by a spectrometer, which is integrated into a typical FROG device.

Theorem III.1 ([23])**.** *Let $x \in \mathbb{C}^N$ be a B-bandlimited signal as defined in Definition II.1 for some $B \leq N/2$. If $N/L \geq 4$, then generic signals are determined uniquely from their FROG trace as in* (II.2)*, modulo the trivial ambiguities of Proposition II.2, from $3B$ measurements. If in addition we have access to the signal's power spectrum and $N/L \geq 3$, then $2B$ measurements are sufficient.*

The notion of *generic* signal that appears in Theorem III.1 means that the set of signals which cannot be uniquely determined, up to trivial ambiguities, is contained in the zero set of a non-zero polynomial. This implies that we can reconstruct almost all signals under the stated conditions.

This result significantly improves upon earlier work on the uniqueness of the FROG method. In [25], it was shown that a continuous signal is determined by its full continuous FROG trace and its continuous power spectrum. The uniqueness of the discrete case, as the problem often appears in practice, was first considered in [21]. It was proven that a discrete bandlimited signal is determined from its entire FROG trace (i.e., $L = 1$) and its power spectrum. Our result requires only $2B$ FROG measurements if the signal's power spectrum is available, where B is the signal's bandlimit. Furthermore, this is the first result showing that the FROG trace is sufficient to determine the signal without the power spectrum information.

It is interesting to view our results in the broader perspective of phaseless systems of equations. In [26], it was shown that $4N - 4$ quadratic equations arising from random frame measurements are sufficient to uniquely determine all signals. Our result requires the number of measurements to be only three times the bandwidth of the signal. Nonetheless, it holds "only" for almost all signals. Another related setup are phaseless STFT measurements. This case resembles the FROG setup, where a known reference window replaces the unknown delayed signal. Several works derived uniqueness results for this case under different conditions [12], [14], [15], [16][19, Section 3.4]. In [13], it was shown that it is sufficient to set $L < N/2$ to determine almost all non-vanishing signals. Comparing to Theorem III.1, we conclude that the FROG case is not significantly harder than the phaseless STFT setup.

Before moving forward to outline the proof of Theorem III.1, we mention that several algorithms exist to estimate a signal from its FROG trace [27], [28]. One way to try estimating the signal is by minimizing a least-squares objective

$$\min_{z\in\mathbb{C}^N} \frac{1}{2}\sum_{k=0}^{N-1}\sum_{m=0}^{N/L-1}\left(|\hat{y}_{k,m}|^2 - \left|\sum_{n=0}^{N-1} z_n z_{n+mL} e^{-2\pi\iota nk/N}\right|^2\right)^2. \tag{III.1}$$

This problem is non-convex (a polynomial of degree 8) and empirical evidence suggests that it suffers from multiple local minima [23]. One popular iterative algorithm is the Principal Components Generalized Projections (PCGP) [29]. The PCGP algorithm alternates between imposing the known intensities (the measured data) and the non-linear relation (II.1). The latter step is performed by employing principal components analysis (PCA) on a data matrix constructed from the previous signal's estimation.

IV. Outline of the proof of Theorem III.1

In this section we present the outline of the proof of Theorem III.1. The detailed proof is given in [23].

We begin the proof by reformulating the measurement model to a more convenient structure. Particularly, it can be shown that

$$\hat{y}_{k,m} = \frac{1}{N}\sum_{\ell=0}^{N-1} \widehat{x}_\ell \widehat{x}_{k-\ell}\omega^{\ell m}, \tag{IV.1}$$

where

$$\omega := e^{2\pi\iota/r}, \quad r := N/L, \tag{IV.2}$$

and we assume that N/L is an integer. Equation (IV.1) implies that, for each fixed k, $\hat{y}_{k,m}$ provides $r = N/L$ samples from the (inverse) Fourier transform of $\widehat{x}_\ell \widehat{x}_{k-\ell}$. Note that $\overline{\widehat{y}}_{k,-m} = \sum_{\ell=0}^{N-1} \overline{\widehat{x}}_\ell \overline{\widehat{x}}_{k-\ell}\omega^{\ell m}$. Because of the reflection ambiguity in Proposition II.2, it implies that the FROG trace is invariant to sign flip of m. For instance, for $r = 3$, the equations for $m = 1$ and $m = 2$ are the same since $m = 2$ is equivalent to $m = -1$.

To ease notation, we assume that N is even, that $\widehat{x}_k \neq 0$ for $k = 0, \ldots, N/2 - 1$, and that $\widehat{x}_k = 0$ for $k = N/2, \ldots, N - 1$. If the signal's non-zero Fourier coefficients are not in the interval $0 \ldots, N/2 - 1$, then we can cyclically reindex the signal without affecting the proof. If N is odd, one should replace $N/2$ by $\lfloor N/2 \rfloor$ everywhere in the sequel.

Considering (IV.1), our bandlimit assumption forms a "pyramid" structure. Here, each row represents fixed k and varying ℓ of $\widehat{x}_\ell \widehat{x}_{k-\ell}$:

$$\begin{array}{c}
\widehat{x}_0^2, 0, \ldots, 0 \\
\widehat{x}_0\widehat{x}_1, \widehat{x}_1\widehat{x}_0, 0, \ldots, 0 \\
\widehat{x}_0\widehat{x}_2, \widehat{x}_1^2, \widehat{x}_2\widehat{x}_0, \ldots, 0 \ldots \\
\vdots \\
\widehat{x}_{N/2-1}\widehat{x}_0, \widehat{x}_{N/2-2}\widehat{x}_1, \ldots, \widehat{x}_{N/2-1}\widehat{x}_0, 0, \ldots, 0 \\
0, \widehat{x}_1\widehat{x}_{N/2-1}, \widehat{x}_2\widehat{x}_{N/2-2}, \ldots, \widehat{x}_{N/2-1}\widehat{x}_1, 0, \ldots, 0 \\
\vdots \\
0, 0, \ldots \widehat{x}_{N/2-1}\widehat{x}_{N/2-1}, \ldots, 0, \ldots, 0.
\end{array} \tag{IV.3}$$

Then, $\widehat{y}_{k,m}$ as in (IV.1) is a subsample of the Fourier transform of each one of the pyramid's rows. It also implies that the FROG trace is a subsample of the power spectrum taken along the rows.

From the first row of (IV.3), we see that

$$|\widehat{y}_{0,0}| = \frac{1}{N}|\widehat{x}_0^2|.$$

Because of the rotation ambiguity, we set $\widehat{x}_0$ to be real and, without loss of generality, normalize it so that $\widehat{x}_0 = 1$. From the second row of (IV.3), we conclude that

$$|\widehat{y}_{1,0}| = \frac{1}{N}|\widehat{x}_0\widehat{x}_1 + \widehat{x}_1\widehat{x}_0| = \frac{2}{N}|\widehat{x}_1|.$$

Therefore, we can determine $|\widehat{x}_1|$. Because the translation symmetry for bandlimited signals, as presented in Proposition II.2, is continuous, we set arbitrarily $\widehat{x}_1 = |\widehat{x}_1|$. Note that this is not true for general signals for which the translation symmetry is discrete.

In order to proceed, we present a key lemma which is the main pillar of the proof. The lemma ensures that given three distinct equations, quadratic phaseless systems of the form (IV.4) have no unique solution in general. The proof of the lemma is provided in [23].

Lemma IV.1. *Consider the system of equations*

$$|z+v_1| = n_1, \quad |z+v_2| = n_2, \quad |z+v_3| = n_3, \tag{IV.4}$$

for non-negative scalars $n_1, n_2, n_3 \in \mathbb{R}^N$.

1) *Let $v_1, v_2, v_3 \in \mathbb{C}^N$ be distinct and suppose that the imaginary part of the ratio $\frac{v_1-v_2}{v_1-v_3}$ is not zero. If the system* (IV.4) *has a solution, then it is unique. Moreover, if n_1, n_2, n_3 are fixed for generic $v_1, v_2, v_3 \in \mathbb{C}^N$, then the system will have no solution.*
2) *Let $v_1, v_2, v_3 \in \mathbb{R}^N$. If $z = a + \iota b$ is a solution, then $\overline{z} = a - \iota b$ is a solution as well. Hence, if the system has a solution, then it has two solutions. Moreover, if n_1, n_2, n_3 are fixed for generic $v_1, v_2, v_3 \in \mathbb{R}^N$, then the system will have no solution.*

Lemma IV.1 can be extended to systems of $s \geq 3$ equations, namely,

$$|z+v_1| = n_1, \dots, |z+v_s| = n_s.$$

If one of the ratios $\frac{v_1-v_p}{v_1-v_q}$ for $p, q = 2, \dots, s$, $p \neq q$, is not real, then there is at most one solution to the system.

Based on the second part of Lemma IV.1, one can show that $\widehat{x}_2$ is determined up to the reflection symmetry. More involved arguments, based on the first part of Lemma IV.1, show that $\widehat{x}_3$ and $\widehat{x}_4$ are also determined uniquely; see [23] for the technical details.

The final step of the proof is to show that given $\widehat{x}_0, \widehat{x}_1, \dots, \widehat{x}_k$ for some $k \geq 4$, we can determine $\widehat{x}_{k+1}$ up to symmetries. Using (IV.3), for an even $k = 2s$, we get the system of equations for $m = 0, \dots, r-1$,

$$\begin{aligned} &\left|\frac{\widehat{y}_{k+1,m}}{1+\omega^{m(k+1)}}\right| = \\ &\frac{1}{N}\left|z + \frac{\omega^m}{1+\omega^{m(k+1)}}\widehat{x}_1\widehat{x}_k + \dots + \frac{\omega^{ms}}{1+\omega^{m(k+1)}}\widehat{x}_s\widehat{x}_{s+1}\right|, \end{aligned} \tag{IV.5}$$

where we omit the values of m for which $\omega^{m(k+1)} = -1$. Under the assumptions of Theorem III.1, we get at least three distinct equations. Therefore, applying Lemma IV.1 implies directly that $\widehat{x}_{k+1}$ is determined uniquely. For an odd $k = 2s+1$, we get a similar system of equations. When $r = 3$, the system provides only two distinct equations. If in addition we assume the knowledge of $|\widehat{x}|$, then we have an additional third equation $|z| = |\widehat{x}_{k+1}|$ and therefore we can invoke Lemma IV.1. This concludes the proof.

V. Conclusion and perspective

The problem of phase retrieval arises naturally in the field of ultra-short laser pulse characterization. In order to overcome the ill-posedness of the problem, it is common to use the FROG method. Recovering the signal from its FROG trace involves inverting a system of phaseless quartic equations. In this manner, the FROG problem differs significantly from standard phase retrieval problems that involve quadratic equations.

In this work, we analyzed the uniqueness of the FROG method. We have shown that it is sufficient to take only 3B FROG measurements in order to determine a generic B-bandlimited signal uniquely, up to symmetries. If the power spectrum of the sought signal is also available, then $2B$ measurements suffice.

The FROG model considered in this paper has a natural extension, called blind FROG. In blind FROG, we aim to characterize two signals simultaneously. Particularly, for two signals $u, v \in \mathbb{C}^N$, the blind FROG trace is given by [20], [30]

$$|\hat{y}_{k,m}|^2 = \left|\sum_{n=0}^{N-1} u_n v_{n+mL} e^{-2\pi\iota nk/N}\right|^2. \tag{V.1}$$

The goal is then to estimate both u and v simultaneously from their phaseless measurements. Interestingly, blind FROG has an additional continuous symmetry that does not appear in FROG. Specifically, for any $\phi \in [0, 2\pi)$, the blind FROG trace is invariant under the mapping

$$(u_n, v_n) \to (u_n e^{\iota n\phi}, v_n e^{-\iota n\phi}), \quad n = 0, \dots, N-1.$$

A previous paper showed that a pair of signals can be determined uniquely if the power spectrum of each signal is known and $L = 1$ [21]. However, based on the methodology presented in this paper, we believe that this result can be greatly improved.

As aforementioned in Section III, there exist a variety of algorithms for estimating a signal from its FROG trace. However, the problem is inherently non-convex and therefore it is not clear under what conditions these algorithms provide a reliable estimation of the sought signal, even in the absence of noise. In recent years, several non-convex algorithms for phase retrieval were proposed and analyzed in a variety of settings, see for instance [12], [31], [32], [33], [34], [35]. An analysis of FROG algorithms is a desired future research direction.

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