REALIZING DIRECTIONAL SOUND SOURCE IN FDTD METHOD BY ESTIMATING INITIAL VALUE

Daiki Takeuchi, Kohei Yatabe, Yasuhiro Oikawa

Department of Intermedia Art and Science, Waseda University, Tokyo, Japan

ABSTRACT

Wave-based acoustic simulation methods are studied actively for predicting acoustical phenomena. Finite-difference time-domain (FDTD) method is one of the most popular methods owing to its straightforwardness of calculating an impulse response. In an FDTD simulation, an omnidirectional sound source is usually adopted, which is not realistic because the real sound sources often have specific directivities. However, there is very little research on imposing a directional sound source into FDTD methods. In this paper, a method of realizing a directional sound source in FDTD methods is proposed. It is formulated as an estimation problem of the initial value so that the estimated result corresponds to the desired directivity. The effectiveness of the proposed method is illustrated through some numerical experiments.

Index Terms— Sound source directivity, wave-based acoustic simulation, finite-difference time-domain (FDTD), least squares method, Krylov subspace method.

1. INTRODUCTION

Wave-based acoustic simulation methods have been studied for predicting and investigating acoustical phenomena. Among many wave-based methods including finite element method (FEM) and boundary element method (BEM), finite-difference time-domain (FDTD) method is one of the most popular methods because of its capability of calculating an impulse response straightforwardly. Therefore, much research on acoustic simulation using FDTD methods and improvement of FDTD methods have been conducted [1–9].

To obtain a realistic result by simulation, the directivity of a sound source is an important factor. Several research articles have shown measured directivities of various sound sources, and its importance has been discussed [10–13]. Moreover, recent advances of measurement technologies enable to capture directivity of various sound sources [14–18]. Modeling such directivities within the numerical simulation methods is necessary to predict acoustical phenomena related to these sound sources.

However, most of FDTD simulations have been performed with an omnidirectional sound source owing to its simplicity in terms of implementation. Sound source directivity is decided by the way how the sound field is excited, and simple time or spatial excitation strategies including the Gaussian pulse method usually result in an omnidirectional sound source. Although some research has considered directional sound sources for each simulation method [19–21], there exist very little research on directional sources for FDTD methods [22, 23].

In this paper, a method of imposing a directional sound source into FDTD methods is proposed. By estimating initial value from a desired directivity pattern, the directivity is approximately realized. The work presented in this paper focuses on the FDTD methods, while the previous works [19–21] focus on other simulation methods. The work in [22, 23] realized directivity by superposition of monopoles, while the proposed method does not consider explicit formula of a sound source.

2. FINITE-DIFFERENCE TIME-DOMAIN METHOD

In a two-dimensional acoustic field, acoustic wave propagation is expressed by the following wave equation,

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \triangle \right) p(\boldsymbol{r}, t) = 0, \tag{1}$$

where p is sound pressure, c is sound speed, $\boldsymbol{r} = (x, y)$ is position, t is time, and $\triangle$ is the Laplace operator.

An FDTD method is a numerical analysis technique for calculating an approximate solution of the differential equations. In FDTD method, derivatives are approximated by finite differences to yield a discrete recursive scheme. A numerical solution to the original differential equation is obtained by iterating this scheme in the time domain. Since an impulse response is directly obtained as a result of the recursion, FDTD methods are suitable for many practical acoustical problems such that impulse responses are important for the evaluation. Various FDTD schemes have been proposed for solving Eq. (1) or the following set of equations equivalent to Eq. (1):

$$\frac{\partial p}{\partial x} + \rho \frac{\partial u_x}{\partial t} = 0, \quad \frac{\partial p}{\partial y} + \rho \frac{\partial u_y}{\partial t} = 0, \tag{2}$$

$$\frac{\partial p}{\partial t} + \kappa \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) = 0, \tag{3}$$

where Eq. (2) is the equations of motion, Eq. (3) is the continuity equation, u_x and u_y are particle velocities in x and y directions, respectively, ρ is density, and κ is bulk modulus.

2.1. Example of FDTD scheme using staggered grid

Although the proposed method is not restricted to a specific scheme, the following scheme, obtained by applying the central finite difference approximation with the staggered grid system, is chosen as an example for the sake of explanations in the next section [1,24]:

$$u_x(n+1,i+1,j) = u_x(n,i+1,j) - \frac{\Delta t}{\rho \Delta h}\left\{p(n,i+1,j) - p(n,i,j)\right\}, \tag{4}$$

$$u_y(n+1,i,j+1) = u_y(n,i,j+1) - \frac{\Delta t}{\rho \Delta h}\left\{p(n,i,j+1) - p(n,i,j)\right\}, \tag{5}$$

$$p(n+1,i,j) = p(n,i,j) - \frac{\kappa \Delta t}{\Delta h}\left[\left\{u_x(n+1,i+1,j) - u_x(n+1,i,j)\right\} + \left\{u_y(n+1,i,j+1) - u_y(n+1,i,j)\right\}\right], \tag{6}$$

where Δt is a discrete time step, Δh is the size of the staggered grid, and n, i and j are indices of the time step, the spatial steps of x and y, respectively. This scheme takes advantage of the staggered grid which enables implementation of the central finite difference in the form of the forward or backward finite differences. In this paper, the size of p, u_x and u_y are set to $N_p \times N_p$, $N_p \times (N_p+1)$ and $(N_p+1) \times N_p$ for each n, respectively (that is, p, u_x and u_y are elements of $\mathbb{R}^{N_p \times N_p}$, $\mathbb{R}^{N_p \times (N_p+1)}$ and $\mathbb{R}^{(N_p+1) \times N_p}$, respectively). Note that $+1$ in the size of u_x and u_y came from the property of the staggered grid. The number of time sequences is chosen so that the initial wave does not reach to the boundary for avoiding the effect of boundary condition where properly choosing it is not easy [25–28].

Since any recursive scheme starts its iteration from an initial value (p, u_x, u_y at $n=0$), it decides the behavior of the wave propagation. That is, the directivity of a sound source is totally decided by the initial value. Thus, in this situation, the approximation of the desired directivity can be formulated as a problem of finding the corresponding initial value. However, manually setting an appropriate initial condition which generates the desired directivity pattern is a challenging task because it should depend on the choice of an FDTD scheme. Therefore, a method of imposing a directional sound source that is independent of choosing a scheme is necessary.

3. PROPOSED METHOD

As discussed in the previous section, a directional sound source can be realized by obtaining the corresponding initial value of an FDTD method. To do so, an initial value estimation method is proposed. The main idea of the proposed method is illustrated in Fig. 1. Since it is formulated as a least squares problem, a matrix representation of the FDTD method is introduced first.

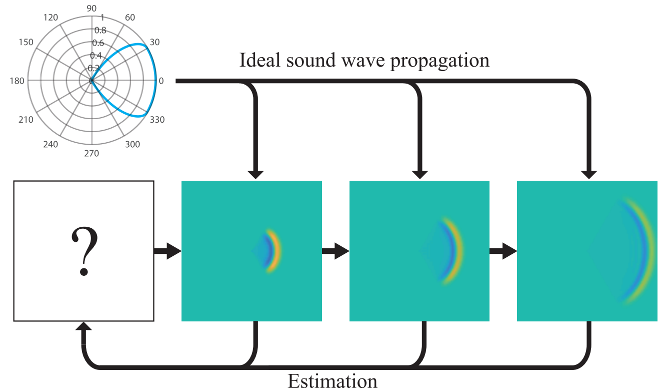

Fig. 1. Estimating initial value from a directivity pattern. The initial value is estimated from a propagated wave with the desired directivity pattern which is ideally created from the given pattern by multiplication.

3.1. Matrix representation of FDTD method

For each time step n, vectors $\boldsymbol{p}^{[n]} \in \mathbb{R}^{N_p^2}$, $\boldsymbol{u}_x^{[n]} \in \mathbb{R}^{N_p(N_p+1)}$ and $\boldsymbol{u}_y^{[n]} \in \mathbb{R}^{(N_p+1)N_p}$ are defined so that all (i,j)-th values of p, u_x and u_y are contained in the corresponding vectors. By these notations, the update rule of the FDTD method in Eqs. (4), (5) and (6) can be interpreted as a linear transformation from $(\boldsymbol{p}^{[n]}, \boldsymbol{u}_x^{[n]}, \boldsymbol{u}_y^{[n]})$ to $(\boldsymbol{p}^{[n+1]}, \boldsymbol{u}_x^{[n+1]}, \boldsymbol{u}_y^{[n+1]})$, which can be explicitly written as

$$\boldsymbol{u}_x^{[n+1]} = \boldsymbol{u}_x^{[n]} + \frac{\Delta t}{\rho \Delta h} D_x^T \boldsymbol{p}^{[n]}, \tag{7}$$

$$\boldsymbol{u}_y^{[n+1]} = \boldsymbol{u}_y^{[n]} + \frac{\Delta t}{\rho \Delta h} D_y^T \boldsymbol{p}^{[n]}, \tag{8}$$

$$\boldsymbol{p}^{[n+1]} = \boldsymbol{p}^{[n]} - \frac{\kappa \Delta t}{\Delta h}\left(D_x \boldsymbol{u}_x^{[n+1]} + D_y \boldsymbol{u}_y^{[n+1]}\right), \tag{9}$$

where $D_x, D_y \in \mathbb{R}^{N_p^2 \times N_p(N_p+1)}$ are the difference operators in x and y directions, respectively, and $\cdot^T$ denotes transpose.

Let a state vector $\boldsymbol{\zeta}^{[n]}$ at time step n be defined as

$$\boldsymbol{\zeta}^{[n]} = [\,\boldsymbol{p}^{[n]T},\, \boldsymbol{u}_x^{[n]T},\, \boldsymbol{u}_y^{[n]T}\,]^T \in \mathbb{R}^{3N_p^2 + 2N_p}, \tag{10}$$

where $[\,\boldsymbol{z}_1,\, \boldsymbol{z}_2\,]$ is horizontal concatenation of $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$. Then, Eqs. (7), (8) and (9) can be compactly represented as

$$\boldsymbol{\zeta}^{[n+1]} = \Phi \boldsymbol{\zeta}^{[n]} = \Phi^{n+1} \boldsymbol{\zeta}^{[0]}, \tag{11}$$

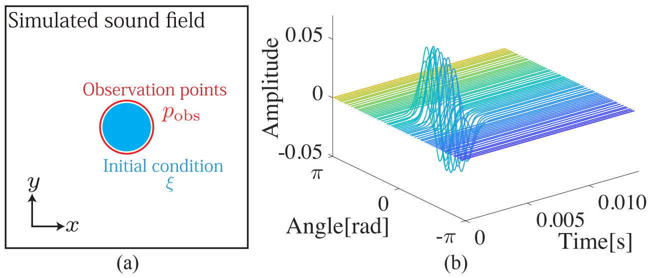

Fig. 2. (a) Region of considering initial value (blue) and observation points for evaluating a directivity pattern (red). (b) Example of sound pressure at the observation points which constructs $\boldsymbol{d}$ in Eq. (20). The desired directivity is ideally imposed by multiplying it to an omnidirectional pulse.

where Φ is a block matrix of the form

$$\Phi = \begin{pmatrix} I - D & -\kappa \Delta t\, D_x & -\kappa \Delta t\, D_y \\ (\Delta t/\rho) D_x^T & I & O \\ (\Delta t/\rho) D_y^T & O & I \end{pmatrix}, \qquad (12)$$

$I \in \mathbb{R}^{(N_p^2+N_p)\times(N_p^2+N_p)}$ and $O \in \mathbb{R}^{(N_p^2+N_p)\times(N_p^2+N_p)}$ denote the identity matrix and the zero matrix, respectively, and

$$D = \frac{\kappa}{\rho}\left(\frac{\Delta t}{\Delta h}\right)^2 (D_x D_x^T + D_y D_y^T). \qquad (13)$$

This matrix representation in Eq. (11) shows that a single time step of the FDTD scheme is equivalent to multiplication of the matrix Φ to the state vector $\boldsymbol{\zeta}^{[n]}$. Using this representation, a least squares problem of the FDTD method can be formulated. Note that any other FDTD scheme can be represented by the same form, Eq. (11), with some appropriate construction of the matrix. Therefore, the proposed formulation explained in the next subsection does not depend on which FDTD scheme is considered.

3.2. Estimating initial value from directivity pattern

For realizing a directional sound source, a method for estimating the initial value corresponding to the desired directivity pattern is proposed here.

In the proposed method, the initial value is assumed to be compactly supported on a small region illustrated as a blue circle in Fig. 2(a). This assumption reflects the size of a directional sound source. After the initial value is propagated by an FDTD scheme, sound pressure around that region is expected to have the desired directivity. Therefore, observation points, illustrated as a red circle in Fig. 2(a), are set around the region, and directivity is evaluated at these points. The aim of the proposed method is to estimate the initial value supported inside the blue region so that the directivity evaluated at the red observation points approximates the desired directivity pattern.

The inverse problem of estimating an initial value is formulated as a least squares problem of the FDTD method. A vector of an initial value is defined as

$$\boldsymbol{\xi} = [\, {\boldsymbol{p}_{\mathrm{in}}}^T,\, {\boldsymbol{u}_{x\mathrm{in}}}^T,\, {\boldsymbol{u}_{y\mathrm{in}}}^T \,]^T \in \mathbb{R}^{N_{\mathrm{in}}}, \qquad (14)$$

where the length of $\boldsymbol{\xi}$ is much smaller than that of $\boldsymbol{\zeta}^{[0]}$ in Eq. (10) because $\boldsymbol{\xi}$ only consists of the small region in Fig. 2(a). The relation between $\boldsymbol{\zeta}^{[0]}$ and $\boldsymbol{\xi}$ will be written as

$$\boldsymbol{\zeta}^{[0]} = E\boldsymbol{\xi}, \qquad (15)$$

where $E : \mathbb{R}^{N_{\mathrm{in}}} \to \mathbb{R}^{3N_p^2+2N_p}$ is a matrix that each column of E contains only single 1 and other elements are 0. That is, E expands the size of the initial value vector $\boldsymbol{\xi}$ to the size of the state vector $\boldsymbol{\zeta}^{[0]}$ by filling 0 to outside of the small region. In a similar manner, a vector of sound pressure at the observation points is denoted by $\boldsymbol{p}_{\mathrm{obs}}^{[n]} \in \mathbb{R}^{N_{\mathrm{obs}}}$ whose length is also much smaller than that of $\boldsymbol{\zeta}^{[n]}$, and the relation is expressed as

$$\boldsymbol{p}_{\mathrm{obs}}^{[n]} = L\boldsymbol{\zeta}^{[n]}, \qquad (16)$$

where $L : \mathbb{R}^{3N_p^2+2N_p} \to \mathbb{R}^{N_{\mathrm{obs}}}$ is a matrix that each row contains only single 1 and other elements are 0. This matrix L picks up the sound pressure at the observation points and discards the other elements in $\boldsymbol{\zeta}^{[n]}$. Using these notations, the relation between the initial value $\boldsymbol{\xi}$ and the sound pressure at the observation points is written as

$$\boldsymbol{p}_{\mathrm{obs}}^{[n]} = L\Phi^n E\boldsymbol{\xi}. \qquad (17)$$

By concatenating each $\boldsymbol{p}_{\mathrm{obs}}^{[n]}$ vertically as

$$\boldsymbol{p}_{\mathrm{obs}} = [\, {\boldsymbol{p}_{\mathrm{obs}}^{[1]}}^T,\, {\boldsymbol{p}_{\mathrm{obs}}^{[2]}}^T,\, \ldots,\, {\boldsymbol{p}_{\mathrm{obs}}^{[n]}}^T \,]^T, \qquad (18)$$

a linear equation considering all time steps is obtained:

$$\boldsymbol{p}_{\mathrm{obs}} = L_{\mathrm{diag}} F E \boldsymbol{\xi}, \qquad (19)$$

where $L_{\mathrm{diag}} = \mathrm{blkdiag}(L, L, \cdots, L)$ is the block diagonal matrix, blkdiag generates a block diagonal matrix from its arguments, and $F = [\, \Phi^T, (\Phi^2)^T, \cdots, (\Phi^n)^T \,]^T$. Then, an estimation problem of the initial value can be formulated as the following least squares problem: Finding

$$\hat{\boldsymbol{\xi}} = \arg\min_{\boldsymbol{\xi}} \| L_{\mathrm{diag}} F E \boldsymbol{\xi} - \boldsymbol{d} \|_2^2, \qquad (20)$$

where $\boldsymbol{d}$ consists of pulse signals at the observation points as illustrated in Fig. 2(b). The desired directivity pattern is ideally imposed to $\boldsymbol{d}$ by multiplying it to received signals obtained by an omnidirectional pulse emitted from the center of the region of the initial value. By solving this problem, the initial value approximating the desired directivity is obtained.

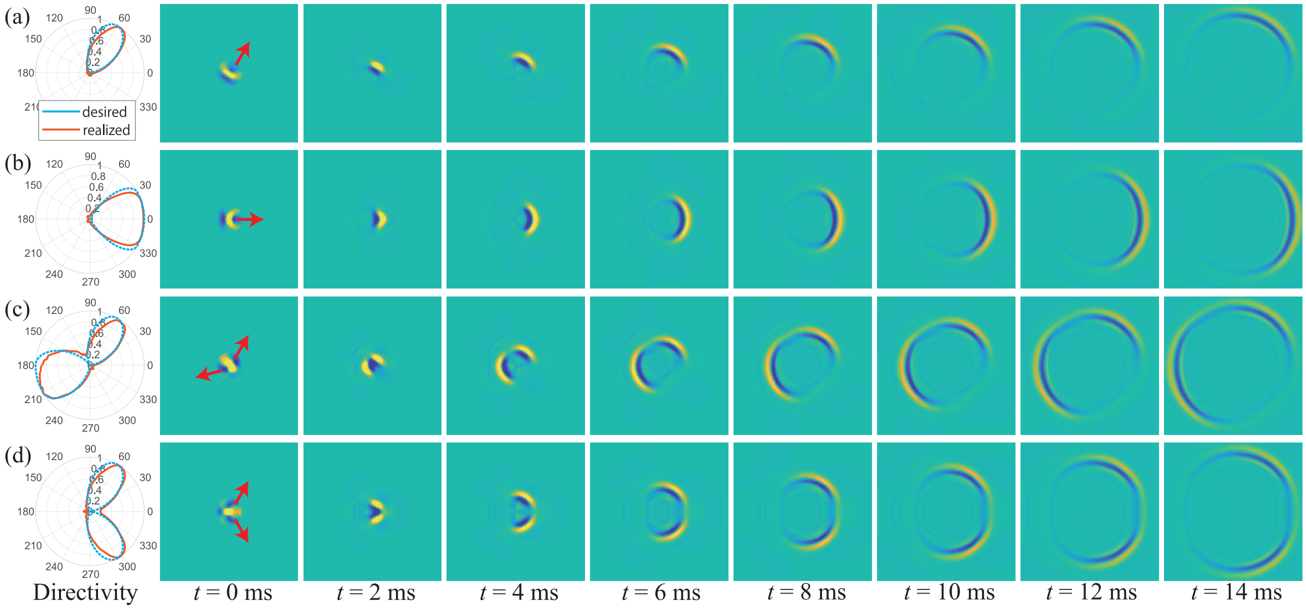

Fig. 3. Propagated waves generated by estimated initial value. Their directivities were evaluated at 2 m away from the center.

Table 1. Simulation condition.

The size of sound field	10 m × 10 m
Shape of initial value	Disk
Radius	1 m
Size of initial value vector $N_{\rm in}$	997
Shape of observation points	Circle
Radius	1.1 m
Number of observation points	60
Sound speed	340 m/s
Density	1.21 kg/m^3
Spatial discretization interval	0.1 m
Time discretization interval	1/48000 s

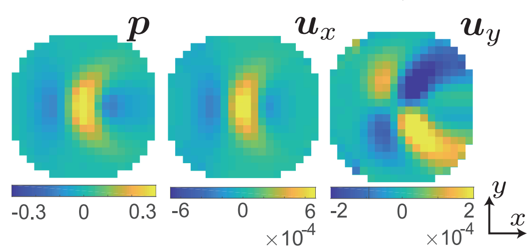

Fig. 4. Estimated initial value corresponding to Fig. 3 (b).

4. NUMERICAL EXPERIMENT

A numerical experiment was conducted in order to confirm the appropriateness of the proposed method. The experimental condition is summarized in Table 1. For solving the least squares problem in Eq. (20), the LSMR solver [29], which is a MINRES (Krylov subspace type method) variant of the well-known LSQR, was utilized. Note that, in such an iterative solver, explicit construction of the matrix is not necessary because only the results of the matrix-vector product are required. That is, one does not have to construct the matrix $L_{\rm diag}FE$ which requires an insane amount of memory, but it can be implemented as a procedure (a function file). The matrix-vector product $\Phi\zeta^{[n]}$, arising from multiplication of F, can be computed as the procedure in Eqs. (4)–(6). Therefore, the least squares problem can be approximately solved by running the FDTD scheme several hundred times inside LSMR.

Fig. 3 shows the results of propagated initial values which were estimated by the proposed method. Each row corresponds to one of four different directivity patterns, and the polar plots were normalized by their maximal values. From the time sequences of sound pressure and the polar plots of directivities, it can be confirmed that directional sound sources were realized by the proposed method. Fig. 4 shows the estimated initial value corresponding to Fig. 3 (b). It can be seen that manually setting such an initial value is not an easy task even when the desired directivity is a simple pattern.

5. CONCLUSIONS

In this paper, the method of realizing directional sound sources by estimating the initial value of an FDTD method was proposed. The proposed method can approximately impose any directivity pattern in an automatic manner even when the corresponding initial value is complicated. In future work, the effect of the setting of the region of the initial value and observation points should be investigated to achieve better performance. Moreover, the property of the matrix $L_{\rm diag}FE$ should be analyzed so that the ability of approximating directivity patterns by the initial value is understood.

6. REFERENCES

[1] D. Botteldooren, “Finite-difference time-domain simulation of low-frequency room acoustic problems,” *J. Acoust. Soc. Am.*, vol. 98, no. 6, pp. 3302–3308, 1995.

[2] S. Bilbao, *Wave and Scattering Methods for Numerical Simulation*, John Wiley & Sons, London, 2004.

[3] K. Kowalczyk and M. van Walstijn, “Room acoustics simulation using 3-D compact explicit FDTD schemes,” *IEEE Trans. Audio, Speech, Lang. Process.*, vol. 19, no. 1, pp. 34–46, Jan. 2011.

[4] S. Bilbao, “Optimized FDTD schemes for 3-D acoustic wave propagation,” *IEEE Trans. Audio, Speech, Lang. Process.*, vol. 20, no. 5, pp. 1658–1663, Jul. 2012.

[5] A. Southern, S. Siltanen, D. T. Murphy, and L. Savioja, “Room impulse response synthesis and validation using a hybrid acoustic model,” *IEEE Trans. Audio, Speech, Lang. Process.*, vol. 21, no. 9, pp. 1940–1952, Sep. 2013.

[6] J. V. Mourik and D. Murphy, “Explicit higher-order FDTD schemes for 3D room acoustic simulation,” *IEEE/ACM Trans. Audio, Speech, Lang. Process.*, vol. 22, no. 12, pp. 2003–2011, Dec. 2014.

[7] J. Sheaffer, M. van Walstijn, and B. Fazenda, “Physical and numerical constraints in source modeling for finite difference simulation of room acoustics,” *J. Acoust. Soc. Am.*, vol. 135, no. 1, pp. 251–261, 2014.

[8] J. Saarelma, J. Botts, B. Hamilton, and L. Savioja, “Audibility of dispersion error in room acoustic finite-difference time-domain simulation as a function of simulation distance,” *J. Acoust. Soc. Am.*, vol. 139, no. 4, pp. 1822–1832, 2016.

[9] B. Hamilton and S. Bilbao, “FDTD Methods for 3-D Room Acoustics Simulation with High-order Accuracy in Space and Time,” *IEEE/ACM Trans. Audio, Speech, Lang. Process.*, vol. 25, no. 11, pp. 2112–2124, Nov. 2017.

[10] F. Otondo and J. H. Rindel, “The influence of the directivity of musical instruments in a room,” *Acta Acust. Acust.*, vol. 90, no. 6, pp. 1178–1184, Nov. 2004.

[11] P. Zhu, F. Mo, J. Kang, and G. Zhu, “Comparisons between simulated and in-situ measured speech intelligibility based on (binaural) room impulse responses,” *Appl. Acoust.*, vol. 97, pp. 65–77, Oct. 2015.

[12] A. Snakowska, J. Jurkiewicz, and L. Gorazd, “A hybrid method for determination of the acoustic impedance of an unflanged cylindrical duct for multimode wave,” *J. Sound Vibr.*, vol. 396, pp. 325–339, May 2017.

[13] R. H. C. Wenmaekers, C. C. J. M. Hak, M. C. J. Hornikx, and A. G. Kohlrausch, “Sensitivity of stage acoustic parameters to source and receiver directivity: Measurements on three stages and in two orchestra pits,” *Appl. Acoust.*, vol. 123, pp. 20–28, Aug. 2017.

[14] Y. Oikawa, K. Yatabe, K. Ishikawa, and Y. Ikeda, “Optical sound field measurement and imaging using laser and high-speed camera,” in *45th Int. Congr. Noise Control Eng. (INTER-NOISE 2016)*, Aug. 2016, pp. 258–266.

[15] K. Yatabe, K. Ishikawa, and Y. Oikawa, “Signal processing for optical sound field measurement and visualization,” *Proc. Meet. Acoust. (POMA)*, vol. 29, no. 1, pp. 020010, 2016.

[16] K. Ishikawa, K. Yatabe, Y. Ikeda, Y. Oikawa, T. Onuma, H. Niwa, and M. Yoshii, “Optical sensing of sound fields: Non-contact, quantitative, and single-shot imaging of sound using high-speed polarization camera,” *Proc. Meet. Acoust. (POMA)*, vol. 29, no. 1, pp. 030005, 2016.

[17] K. Ishikawa, K. Yatabe, N. Chitanont, Y. Ikeda, Y. Oikawa, T. Onuma, H. Niwa, and M. Yoshii, “High-speed imaging of sound using parallel phase-shifting interferometry,” *Opt. Express*, vol. 24, no. 12, pp. 12 922–12 932, June 2016.

[18] K. Ishikawa, K. Yatabe, Y. Ikeda, Y. Oikawa, T. Onuma, H. Niwa, and M. Yoshii, “Optical sound field measurement and imaging using laser and high-speed camera,” in *31st Int. Congr. High-Speed Imaging Photonics (ICHSIP)*, Nov. 2016, pp. 258–266.

[19] H. Hacihabibolu, B. Günel, and A. M. Kondoz, “Time-domain simulation of directive sources in 3-D digital waveguide mesh-based acoustical models,” *IEEE Trans. Audio, Speech, Lang. Process.*, vol. 16, no. 5, pp. 934–946, Jul. 2008.

[20] F. Georgiou and M. Hornikx, “Incorporating directivity in the Fourier pseudospectral time-domain method using spherical harmonics,” *J. Acoust. Soc. Am.*, vol. 140, no. 2, pp. 855–865, 2016.

[21] Y. Tamura, K. Yatabe, and Y. Oikawa, “Least-squares estimation of sound source directivity using convex selector of a better solution,” *Acoust. Sci. Tech.*, vol. 3, pp. 128–136, May 2017.

[22] J. Escolano, J. J. Lopez, and B. Pueo, “Directive sources in acoustic discrete-time domain simulations based on directivity diagrams,” *J. Acoust. Soc. Am.*, vol. 121, no. 6, pp. EL256–EL262, 2007.

[23] D. T. Murphy, A. Southern, and L. Savioja, “Source excitation strategies for obtaining impulse responses in finite difference time domain room acoustics simulation,” *Appl. Acoust.*, vol. 82, pp. 6–14, Aug. 2014.

[24] T. Yokota, S. Sakamoto, and H. Tachibana, “Sound field simulation method by combining finite difference time domain calculation and multi-channel reproduction technique,” *Acoust. Sci. Tech.*, vol. 25, no. 1, pp. 15–23, Jan. 2004.

[25] X. Yuan, D. Borup, J. W. Wiskin, M. Berggren, R. Eidens, and S. A. Johnson, “Formulation and validation of Berenger’s PML absorbing boundary for the FDTD simulation of acoustic scattering,” *IEEE Trans. Ultrason., Ferroelectr., Freq. Control*, vol. 44, no. 4, pp. 816–822, Jul. 1997.

[26] J. Escolano, F. Jacobsen, and J. J. López, “An efficient realization of frequency dependent boundary conditions in an acoustic finite-difference time-domain model,” *J. Sound Vibr.*, vol. 316, no. 1, pp. 234–247, Sep. 2008.

[27] K. Kowalczyk, M. van Walstijn, and D. Murphy, “A phase grating approach to modeling surface diffusion in FDTD room acoustics simulations,” *IEEE Trans. Audio, Speech, Lang. Process.*, vol. 19, no. 3, pp. 528–537, Mar. 2011.

[28] S. Bilbao, “Modeling of complex geometries and boundary conditions in finite difference/finite volume time domain room acoustics simulation,” *IEEE Trans. Speech, Audio, Process.*, vol. 21, no. 7, pp. 1524–1533, Jul. 2013.

[29] D. C. Fong and M. Saunders, “LSMR: An iterative algorithm for sparse least-squares problems,” *SIAM J. Sci. Comput.*, vol. 33, no. 5, pp. 2950–2971, 2011.