EFFICIENT MULTIPLIER-LESS STRUCTURES FOR RAMANUJAN FILTER BANKS

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ABSTRACT

Ramanujan filter banks (RFB) are useful to generate *time-period plane* plots which allow one to localize multiple periodic components in the time domain. For such applications, the RFB produces more satisfactory results compared to short time Fourier transforms and other conventional methods, as demonstrated in recent years. This paper introduces a novel multiplier-less, hence computationally very efficient, structure to implement Ramanujan filter banks, based on a new result connecting Ramanujan sums and natural periodic bases.

Index Terms— Ramanujan sums, Ramanujan filter banks, time-period plane, hidden periodicities, period localization.

1. INTRODUCTION

Ramanujan filter banks (RFB) were recently introduced in [11] and further studied in [19]. They find application in extracting periodicity information in discrete time signals. In particular they generate a *time-period plane* plot which allows one to localize periodicities in the time domain. Since the problem of period estimation and localization is different from traditional power spectrum estimation using time frequency plots [8], [9], [20], [21], the RFB produces more satisfactory results compared to short time Fourier transforms and other conventional methods [5], [14], [18] as demonstrated in [11], [19]. RFBs are based on the Ramanujan-sum [7] introduced by Ramanujan in 1918. Ramanujan-sums were studied extensively in the context of period estimation in [15], [16], [12], [13], and are especially useful when data-lengths are small, or when periodic components are localized or time-varying.

In this paper we introduce a novel multiplier-less structure to implement Ramanujan filter banks. The structure is computationally more efficient compared to the direct implementations [11], [19] or other DFT based methods. First we review Ramanujan filter banks briefly in Sec. 2. Then in Sec. 3 we derive a new result that connects Ramanujan sums to the natual periodic basis. Based on this we derive the multiplier-less RFB structure in Sec. 4. Numerical results are then presented showing that the performance of the new structure is identical to that of the more expensive structure in [11, 19].

Notations. $x(n)$ has period P if P is the *smallest positive integer* such that $x(n) = x(n+P)$ for all n. (k, q) denotes the greatest common divisor (gcd) of k and q, so $(k, q) = 1$ means that they are *coprime.* $\mathrm{lcm}(a, b)$ stands for the least common multiple. The notation $q|N$ means that q is a divisor (or factor) of N. $W_q = e^{-j2\pi/q}$, and $\phi(q) =$ Euler totient function [2] (# of integers in $1 \leq n \leq q$ coprime to q).

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2. REVIEW OF RAMANUJAN FILTER BANKS

The qth Ramanujan sum ($q \geq 1$) is a sequence in n defined as

$$c_q(n) = \sum_{\substack{k=1\\(k,q)=1}}^{q} e^{j2\pi kn/q} = \sum_{\substack{k=1\\(k,q)=1}}^{q} W_q^{-kn} \qquad (1)$$

where $-\infty \leq n \leq \infty$. Thus the summation runs over only those k that are coprime to q; $c_q(n)$ has period q, and its use in identifying periodicities in signals is well known [6], [10], [16], [12]. The DFT of one period of $c_q(n)$ is nonzero (and equals q) only at the *coprime frequencies* k. Regarded as a filter, its frequency response is made of Dirac functions at the coprime frequencies $2\pi k/q$ (where $(k, q) = 1$). In practice the causal FIR version

$$C_q^{(l)}(z) = \sum_{n=0}^{lq-1} c_q(n) z^{-n} \qquad (2)$$

is used and has frequency response demonstrated in Fig. 1 for $C_9^{(l)}(z)$. Since $\phi(9) = 6$ there are six coprime frequencies (center frequencies of the passbands). Each passband has width approximately $2\pi/ql$, which gets narrower as l increases. Fig. 2 shows a Ramanujan filter bank with N filters. Each filter $C_q^{(l)}(z)$ extracts a subspace of the space of all period-q components of $x(n)$. By analyzing the outputs of these filters, multiple periodic components of $x(n)$ (periods $\leq N$) can be estimated, based on the following result proved in [19]:

Theorem 1. *The lcm property of Ramanujan filter banks*: In Fig. 2, let $x(n)$ be a period-P input signal with $1 \leq P \leq N$. Let nonzero outputs be produced by the subset of filters $c_{q_i}(n)$ with periods $q_1, q_2, \cdots, q_K$. Then the period P is given by $P = \mathrm{lcm}\{q_1, q_2, \cdots, q_K\}$. $\diamondsuit$

Since the RFB works on real-time signals, its output can be used to produce a time vs period plot, from which various periodic components and their localizations can be gleaned. By contrast, it is shown in [19] that a conventional "comb filter bank" [4], [1] creates ambiguities in period estimation.

3. RAMANUJAN-SUMS AND NATURAL BASES

It is known that $c_q(n)$ is *integer valued* [7], [15] in spite of the trigonometric functions in its definition. For example,

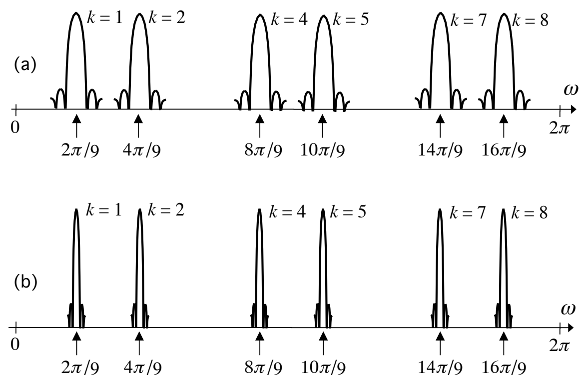

Fig. 1. Frequency response magnitudes of FIR Ramanujan filters $C_q^{(l)}(e^{j\omega})$, demonstrated for $q=9$. (a) Small l, and (b) large l.

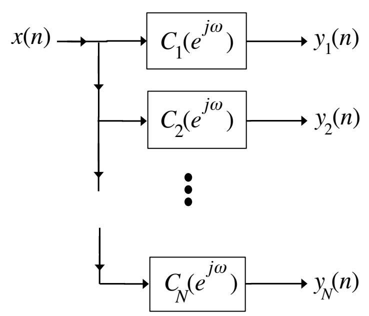

Fig. 2. The Ramanujan analysis filter bank.

$c_6(n) = \{2, 1, -1, -2, -1, 1\}$, where one period is shown. To some extent this helps to reduce the multiplier complexity in the implementations. More significantly, a completely multiplierless implementation of the filter bank is possible, and is based on a fundamental result to be derived in this section. It was shown in [15] that Ramanujan sums satisfy the beautiful recursion

$$c_q(n) = q\delta_q(n) - \sum_{\substack{q_k|q \\ q_k<q}} c_{q_k}(n) \tag{3}$$

Here $\delta_q(n)$ is the periodic impulse, where the "1" occurs at locations that are multiples of q, e.g., $\delta_3(n) = \cdots 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1 \cdots$ Suppose we repeat (3) for all the divisors of q. For example with $q=10$ we have $c_{10}(n)+c_5(n)+c_2(n)+c_1(n) = 10\delta_{10}(n),\ c_5(n)+c_1(n) = 5\delta_5(n),\ c_2(n)+c_1(n) = 2\delta_2(n),\ c_1(n) = \delta_1(n) = 1$. This can be written in matrix form as

$$[\,\mathbf{c}_{10} \quad \mathbf{c}_5 \quad \mathbf{c}_2 \quad \mathbf{c}_1\,] \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{\mathbf{A}_{10}} = \begin{bmatrix} 10 & 5 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 5 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here $\mathbf{A}_{10}$ (more generally $\mathbf{A}_q$) is a $K \times K$ matrix ($K =$ number of divisors of q). It is always lower triangular with all elements equal to "1" or "0", and with diagonal elements =1. So it has unit determinant, and $\mathbf{A}_q^{-1}$ is also a *lower triangular integer matrix.* Thus, in our example we have

$$[\,\mathbf{c}_{10} \quad \mathbf{c}_5 \quad \mathbf{c}_2 \quad \mathbf{c}_1\,] = \begin{bmatrix} 10 & 5 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 5 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}}_{\mathbf{A}_{10}^{-1}} \tag{4}$$

Each column $\mathbf{d}_m$ of the tall matrix on the right can be regarded as arising from a periodic signal with period equal to one of the divisors of q. In fact $\mathbf{d}_m$ represents the simplest periodic signal of the form $m\delta_m(n)$ which we call *natural periodic signals* or periodic impulses. We will show that for any q, the matrix $\mathbf{A}_q^{-1}$ has elements $0, 1$, and -1 only. Thus, $c_q(n)$ is an integer linear combination of $q_k\delta_{q_k}(n)$ where $q_k \leq q$ are divisors of q. Moreover in this linear combination all the coefficients are $0, 1$, or -1. For example,

$$\begin{aligned} c_{10}(n) &= 10\delta_{10}(n) - 5\delta_5(n) - 2\delta_2(n) + 1 \\ c_5(n) &= 5\delta_5(n) - 1 \\ c_2(n) &= 2\delta_2(n) - 1 \end{aligned} \tag{5}$$

and so on. So we have the following:

Theorem 2: *Connection between Ramanujan-sum and natural periodic basis.* The Ramanujan-sum can be expressed in the form

$$c_q(n) = \sum_{q_k|q} \alpha_{q_k} \times q_k\delta_{q_k}(n) \tag{6}$$

where $\alpha_{q_k} \in \{0, 1, -1\}$. ◇

Remark. It only remains to prove that $\mathbf{A}_q^{-1}$ has elements $\in \{0, 1, -1\}$. Note that $\mathbf{A}_q$ was defined as follows: first the divisors of q are arranged in decreasing order

$$q_K > q_{K-1} > \ldots > q_1 \tag{7}$$

where $q_K = q$ and $q_1 = 1$. Then the (i,j)th element of $\mathbf{A}_q$ was defined from the divisors q_i and q_j as follows:

$$[\mathbf{A}_q]_{i,j} = \begin{cases} 1 & \text{if } q_i|q_j \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

The order (7) ensures that $\mathbf{A}_q$ is lower triangular with diagonal elements $= 1$. If divisors had arbitrary ordering, $\mathbf{A}_q$ would not be lower triangular, but it is immaterial for the claim that the inverse has elements $\in \{0, 1, -1\}$. So ordering of divisors is not fundamental, although convenient.

Proof of Theorem 2. First we show that $\mathbf{A}_q^{-1}$ has all elements $\in \{0, 1, -1\}$ when q is a power of a prime, i.e., $q = p^\alpha$

($p =$ prime and $\alpha =$ positive integer). This will then be used as the basis for an induction argument. When $q = p^\alpha$, its divisors are the powers $1, p, p^2 \cdots, p^\alpha$. So $\mathbf{A}_q$ is a lower triangular Toeplitz matrix with all elements on and below the diagonal equal to unity, as demonstrated below for $\alpha = 3$:

$$\mathbf{A}_q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{9}$$

Its inverse is therefore the lower triangular Toeplitz matrix

$$\mathbf{A}_q^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{10}$$

Since lower triangular matrices represent causal LTI filtering, the above inverse represents the well-known fact that the inverse of $(1 + z^{-1} + z^{-2} + \ldots)$ is $1 - z^{-1}$. Thus the inverse of $\mathbf{A}_q$ has elements $\in \{0, 1, -1\}$ whenever $q = p^\alpha$. Next, we know that any arbitrary q can be written in the form

$$q = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_L^{\alpha_L} \tag{11}$$

where p_i are primes. Our induction will be on the number of prime factors L. Thus assuming that $\mathbf{A}_q^{-1}$ has elements $\in \{0, 1, -1\}$ we will show that the same is true for

$$\widehat{q} = q \times p^\alpha \tag{12}$$

where p is a prime different from $p_i, i \leq L$. Since $(p_i, p) = 1$, it follows that the set of divisors of $\widehat{q}$ are the integers dp^i where d is any divisor of q and $0 \leq i \leq \alpha$. For example, let $\widehat{q} = qp$. With the divisors of q as in (7), the divisors of $\widehat{q}$ are

$$\{pq_K,\ pq_{K-1},\ \ldots, pq_1\} \quad \text{and} \quad \{q_K,\ q_{K-1},\ \ldots, q_1\} \tag{13}$$

Now consider the matrix $\mathbf{A}_{\widehat{q}}$. For simplicity assume the divisors are ordered as in (13). Then some thought shows that the matrix $\mathbf{A}_{\widehat{q}}$ is as follows:

$$\mathbf{A}_{\widehat{q}} = \begin{bmatrix} \mathbf{A}_q & \mathbf{0} \\ \mathbf{A}_q & \mathbf{A}_q \end{bmatrix} \tag{14}$$

Similarly when $\widehat{q} = q \times p^\alpha$, the matrix $\mathbf{A}_{\widehat{q}}$ is a block triangular matrix as demonstrated below for $\alpha = 3$:

$$\mathbf{A}_{\widehat{q}} = \begin{bmatrix} \mathbf{A}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_q & \mathbf{A}_q & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_q & \mathbf{A}_q & \mathbf{A}_q & \mathbf{0} \\ \mathbf{A}_q & \mathbf{A}_q & \mathbf{A}_q & \mathbf{A}_q \end{bmatrix} \tag{15}$$

In general there are $\alpha + 1$ blocks, vertically and horizontally. The inverse of this matrix is readily verified to be

$$\mathbf{B}_{\widehat{q}} = \begin{bmatrix} \mathbf{B}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{B}_q & \mathbf{B}_q & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{B}_q & \mathbf{B}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{B}_q & \mathbf{B}_q \end{bmatrix} \tag{16}$$

Here $\mathbf{B}_q \stackrel{\Delta}{=} \mathbf{A}_q^{-1}$ has elements $\in \{0, 1, -1\}$. So $\mathbf{B}_{\widehat{q}}$ has elements $\in \{0, 1, -1\}$. This easily generalizes to any α. $\triangledown \triangledown \triangledown$

4. MULTIPLIER-LESS RAMANUJAN FILTER BANK

Consider again the truncated FIR Ramanujan filters (2) which have impulse response

$$c_q^{(l)}(n) = \begin{cases} c_q(n) & 0 \leq n \leq ql - 1 \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

The filters include l periods of $c_q(n)$, and the total duration is lq. Thus the duration is proportional to the period q which is being analyzed by the analysis bank. This makes the time resolution of the filters proportional to the period q. Smaller periods are analyzed with finer resolution.

We shall now derive a structure for the FIR Ramanujan filter bank $C_q^l(z), 1 \leq q \leq N$ which is especially attractive because of its low complexity. We showed that $c_q(n)$ can be expressed in terms of the periodic delta functions $\delta_{q_k}(n)$ as in (6) where $q_k | q$, and $\alpha_{q_k} \in \{0, 1, -1\}$. Since the FIR filter $c_q^{(l)}(n)$ has support $0 \leq n \leq ql - 1$, the periodic delta $\delta_{q_k}(n)$ has unit values at $n = 0,\ q_k,\ 2q_k,\ \ldots,\ ql - q_k$ that is, at

$$n = q_k i, \quad 0 \leq i \leq \frac{ql}{q_k} - 1 \tag{18}$$

(Fig. 3), where $r_k = q/q_k$ is an integer. Thus the truncated periodic delta function has z-transform

$$\sum_{i=0}^{lr_k - 1} z^{-iq_k} = \frac{1 - z^{-r_k q_k l}}{1 - z^{-q_k}} = \frac{1 - z^{-ql}}{1 - z^{-q_k}} \tag{19}$$

So the FIR Ramanujan filter for period-q is

$$C_q^{(l)}(z) = \sum_{q_k | q} \alpha_{q_k} q_k \times \left(\frac{1 - z^{-ql}}{1 - z^{-q_k}} \right) \tag{20}$$

The coefficients α_{q_k} are just the integers in the first column of $\mathbf{A}_q^{-1}$, and we know that $\alpha_{q_k} \in \{0, 1, -1\}$.

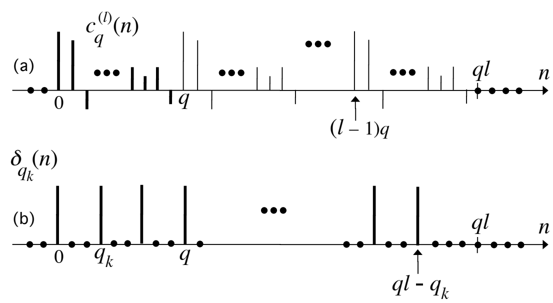

Fig. 3. The q-periodic FIR filter $c_q^{(l)}(n)$ in the Ramanujan filter bank, and the associated periodic delta function $\delta_{q_k}(n)$.

Now consider the analysis filter bank $C_q^{(l)}(z), 1 \leq q \leq N$. The set of all divisors of all the periods q is contained in the set of all integers in the range $1 \leq i \leq N$. We can therefore implement the filter bank $\{C_q^{(l)}(z)\}$ by implementing a pre-filter bank or *divisor-filter-bank*

$$D_i(z) = \left(\frac{1}{1 - z^{-i}} \right), \quad 1 \leq i \leq N \tag{21}$$

and a post-filter bank or *period-filter-bank*,

$$F_q(z) = 1 - z^{-ql}, \quad 1 \le q \le N \tag{22}$$

and combining them as shown in Fig. 4. In this figure, the input to $F_q(z)$ is obtained by taking the outputs $v_{q_k}(n)$ of $D_{q_k}(z)$ where $q_k|q$, and forming the linear combination $\sum_{q_k|q} \alpha_{q_k} v_{q_k}(n)$. So the matrix $\mathbf{T}$ has the element "1" on the diagonals, and elements $\{0, 1, -1\}$ elsewhere.

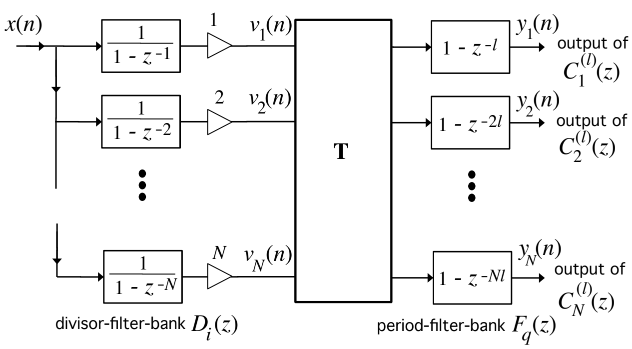

Fig. 4. Implementing an FIR Ramanujan filter bank $C_q^{(l)}(z)$ using a divisor-filter-bank $\{D_i(z)\}$ and a period-filter-bank $\{F_q(z)\}$. $\mathbf{T}$ is lower triangular, and $[\mathbf{T}]_{i,j} \in \{0, 1, -1\}$.

A number of properties of this structure have to be noticed:

1. The filters $D_i(z)$ and $F_q(z)$ are multiplierless. So is the matrix $\mathbf{T}$ which has elements $\{0, 1, -1\}$. The only multipliers are the triangles $1, 2, \cdots, N$ in the figure. So the multiplier complexity is *one simple integer multiplier per filter.*

2. The prefilters $D_i(z)$ are *unstable* IIR filters because they have poles on the unit circle. These poles are eventually cancelled by the zeros in the FIR post filters $F_q(z)$ (because the poles came by writing FIR filters in the economic rational form (19)). In practice, due to roundoff errors, the IIR filters can still create instablity. This can be handled in one of several standard ways known to the signal processing community. In our simulations we replace the filters with

$$D_i(z/\rho) = \left(\frac{1}{1 - \rho^i z^{-i}}\right), \; F_q(z/\rho) = 1 - \rho^{ql} z^{-ql} \tag{23}$$

where $\rho < 1$. This moves the unit-circle poles to points inside the unit circle. Choice of ρ offers a tradeoff. As $\rho \to 1$, the filter bank approximates the original Ramanujan filter bank more accurately, but the roundoff noise gains created by the pre-filters can be large. The introduction of ρ adds $2N$ new multipliers. The total complexity *per filter* is 3 multipliers.

3. The integer l in the post-filter part is the localization parameter. A large l means the filter passbands are narrow and more accurately approximate the impulses. A small l means better localization in the time domain, for locating periodicities. Since the duration of $C_q^{(l)}(z)$ is ql, the localization nicely adjusts according to the period q analyzed by a filter.

4. The structure is *scalable*: if we have to increase N, we can do so without changing any of the existing parts: we just add more prefilters and post filters, and more rows and columns to $\mathbf{T}$ without changing existing element values in $\mathbf{T}$.

Fig. 5 shows an input $x(n)$ which has a period-5 component at $50 \le n \le 100$ and a superposition of period 11 and 14 components at $150 \le n \le 300$, buried in noise with SNR = 5 dB. Fig. 6(a) shows the time-period plane plot obtained using the conventional implementation of the Ramanujan filter bank [11, 19], and Fig. 6(b) shows the results with the multiplierless filter bank of Fig. 4, adjusted for stability using $\rho = 0.999$ (Eq. (23)). Here $l = 10$. The multiplierless filter bank works very well indeed. In both plots, the periodicities $5, 11,$ and 14, and their locations can be seen clearly. The plots show the average of $|y_q(n)|^2$, over a symmetric sliding window of size $2q + 1$. Filter outputs were normalized by $\|c_q^{(l)}(n)\|$ for uniformity, and thresholded appropriately for contrast. Filtering delays were compensated by shifts.

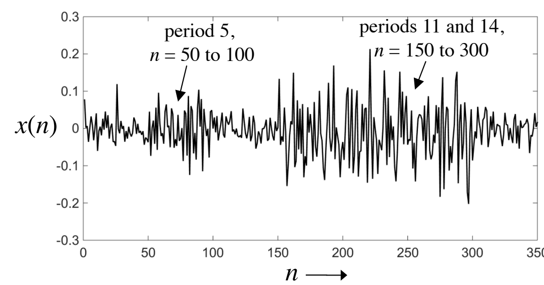

Fig. 5. A signal with periodic components in noise (SNR 5 dB).

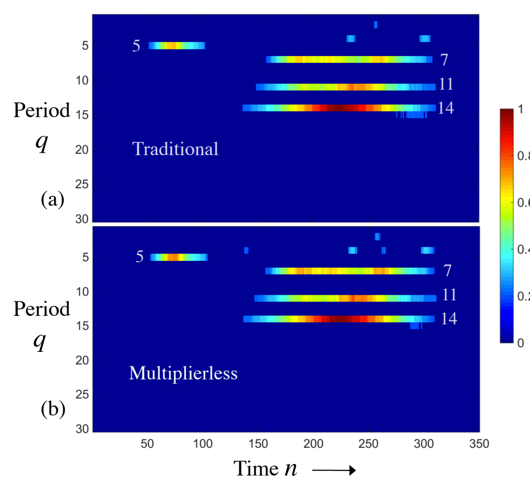

Fig. 6. Time-period plane plots obtained using (a) traditional Ramanujan filter bank [11, 19], and (b) the proposed multiplierless version (Fig. 4).

5. CONCLUDING REMARKS

We introduced an efficient multiplierless structure for the implementation of Ramanjuan filter banks. The parameters l and ρ in the structure are crucial to the tradeoff between time localization, and the accuracy of period estimation. The optimal choice of these parameters, especially when the input $x(n)$ is noisy, remains to be studied. It will be of considerable interest to extend these results to the 2D case where periodicity patterns (lattices) are quite challenging to identify.

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