ENERGY BLOWUP FOR TRUNCATED STABLE LTI SYSTEMS

Holger Boche and Ullrich J. Mönich

Technische Universität München
Lehrstuhl für Theoretische Informationstechnik

ABSTRACT

In this paper we analyze the convergence behavior of a sampling based system approximation process, where the time variable is in the argument of the signal and not in the argument of the bandlimited impulse response. We consider the Paley–Wiener space $\mathcal{PW}_\pi^2$ of bandlimited signals with finite energy and stable linear time-invariant (LTI) systems, and show that there are signals and systems such that the approximation process diverges in the L^2-norm, i.e., the norm of the signal space. We prove that the sets of signals and systems creating divergence are jointly spaceable, i.e., there exists an infinite dimensional closed subspace of $\mathcal{PW}_\pi^2$ and an infinite dimensional closed subspace of the space of all stable LTI systems, such that the approximation process diverges for any non-zero pair of signal and system from these subspaces.

***Index Terms*—** Paley–Wiener space, linear time-invariant system, approximation, convolution sum, L^2-norm

1. INTRODUCTION

Although sampling theorems are very important on their own [1–6], often the interest is not in a reconstruction of the sampled signal itself, but in the approximation of some processed version of it. This is the situation that is encountered in digital signal processing applications, where the interest is not in the reconstruction of a signal, but rather in the implementation of a system, i.e, the interest is in some transform Tf of the sampled input signal f. Typical transforms are the derivative, the Hilbert transform, or, more generally, the output of any other stable linear time-invariant (LTI) system T.

The approximation of LTI systems by sampling series is a well-studied field [7–12]. A common approach to perform the approximation is to use convolution sum

$$\sum_{k=-\infty}^{\infty} f(k) h_T(t-k), \quad t \in \mathbb{R}, \tag{1}$$

where $\{f(k)\}_{k\in\mathbb{Z}}$ denotes the sequence of equidistant samples of f, and $h_T = T\,\mathrm{sinc}$ the response of the LTI system T to the sinc function. Exactly as in the case of signal reconstruction, the convergence and approximation behavior of (1) is important for practical applications and, therefore, has been studied for various signal spaces [12]. For $\mathcal{PW}_\pi^2$, i.e., bandlimited signals with finite L^2-norm, the series (1) converges in the L^2-norm and uniformly on the real axis for all stable LTI systems and all signals.

Another possible approximation process is the convolution sum where the time variable is in the argument of the signal f, i.e.,

$$\sum_{k=-\infty}^{\infty} f(t-k) h_T(k), \quad t \in \mathbb{R}. \tag{2}$$

While this convolution sum is also uniformly convergent on the whole real axis for all signals in $\mathcal{PW}_\pi^2$ and all stable LTI systems, the L^2-norm, as we will show, can be divergent. That is, for certain signals and systems, the convolution sum (2) does not converge in the norm of the considered space $\mathcal{PW}_\pi^2$. In this work we will prove this result, and, further, will analyze the structure of the sets of signals and systems creating divergence.

2. NOTATION

Let $\hat{f}$ denote the Fourier transform of a function f, where $\hat{f}$ is to be understood in the distributional sense. For $\Omega \subseteq \mathbb{R}$, let $L^p(\Omega)$, $1 \le p < \infty$, be the space of all measurable, pth-power Lebesgue integrable functions on Ω, with the usual norm $\|\cdot\|_p$, and $L^\infty(\Omega)$ the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite. $C(\Omega)$, equipped with the supremum norm, is the space of continuous functions on Ω.

For $1 \le p \le \infty$, $\mathcal{PW}_\pi^p$ denotes the Paley–Wiener space of signals f with a representation $f(z) = 1/(2\pi)\int_{-\pi}^{\pi} g(\omega)\,\mathrm{e}^{iz\omega}\,\mathrm{d}\omega$, $z \in \mathbb{C}$, for some $g \in L^p[-\pi,\pi]$. If $f \in \mathcal{PW}_\pi^p$ then $g(\omega) = \hat{f}(\omega)$. The norm for $\mathcal{PW}_\pi^p$, $1 \le p < \infty$, is given by $\|f\|_{\mathcal{PW}_\pi^p} = (1/(2\pi)\int_{-\pi}^{\pi}|\hat{f}(\omega)|^p\,\mathrm{d}\omega)^{1/p}$. $\mathcal{PW}_\pi^2$ is the frequently used space of bandlimited signals with finite L^2-norm.

We briefly review some basic definitions and facts about stable linear time-invariant (LTI) systems. A linear system $T : \mathcal{PW}_\pi^p \to \mathcal{PW}_\pi^p$, $1 \le p \le \infty$, is called stable if the operator T is bounded, i.e., if $\|T\| = \sup_{\|f\|_{\mathcal{PW}_\pi^p}\le 1}\|Tf\|_{\mathcal{PW}_\pi^p} < \infty$. Furthermore, it is called time-invariant if $(Tf(\,\cdot - a))(t) = (Tf)(t-a)$ for all $f \in \mathcal{PW}_\pi^p$ and $t, a \in \mathbb{R}$.

In this paper we are mainly interested in stable LTI systems operating on the space $\mathcal{PW}_\pi^2$, i.e., in the case $p = 2$. For every stable LTI system $T : \mathcal{PW}_\pi^2 \to \mathcal{PW}_\pi^2$ there exists exactly one function $\hat{h}_T \in L^\infty[-\pi,\pi]$ such that

$$(Tf)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(\omega)\hat{h}_T(\omega)\,\mathrm{e}^{i\omega t}\,\mathrm{d}\omega, \quad t \in \mathbb{R},$$

for all $f \in \mathcal{PW}_\pi^2$. Conversely, every function $\hat{h}_T \in L^\infty[-\pi,\pi]$ defines a stable LTI system $T : \mathcal{PW}_\pi^2 \to \mathcal{PW}_\pi^2$. We have $h_T = T\,\mathrm{sinc}$, where sinc denotes the usual sinc function, which is defined by $\mathrm{sinc}(t) = \sin(\pi t)/(\pi t)$ for $t \ne 0$ and $\mathrm{sinc}(t) = 1$ for $t = 0$. The operator norm of a stable LTI system T is given by $\|T\| = \|\hat{h}\|_{L^\infty[-\pi,\pi]}$. Note that $\hat{h}_T \in L^\infty[-\pi,\pi] \subset L^2[-\pi,\pi]$, and consequently $h_T \in \mathcal{PW}_\pi^2$.

By $\mathcal{T}$ we denote the set of stable LTI systems $T\colon \mathcal{PW}_\pi^2 \to \mathcal{PW}_\pi^2$. Every stable LTI system $T \in \mathcal{T}$ can be identified with a function $\hat{h}_T \in L^\infty[-\pi,\pi]$ and we have $\|T\| = \|T\|_{\mathcal{PW}_\pi^2\to\mathcal{PW}_\pi^2} = \|\hat{h}_T\|_{L^\infty[-\pi,\pi]}$.

3. SPACEABILITY

Before we state the main result, we introduce the concept of spaceability. Spaceability, which has recently been used for example in [13–15], is a concept that describes the structure of some given subset of an ambient normed space or, more generally, topological space. A set S in a linear topological space X is said to be spaceable if $S \cup \{0\}$ contains a closed infinite dimensional subspace of X.

4. BASIC PROPERTIES

It is well-known that the partial sums of the Shannon sampling series

$$(S_N f)(t) = \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}, \quad t \in \mathbb{R},$$

converge to f in the $\mathcal{PW}_\pi^2$-norm, i.e., that we have

$$\lim_{N\to\infty} \|f - S_N f\|_{\mathcal{PW}_\pi^2} = 0 \tag{3}$$

for all $f \in \mathcal{PW}_\pi^2$. However, often the interest is not in the reconstruction of the signal f, but in the approximation of some transformation Tf of f from the samples $\{f(k)\}_{k\in\mathbb{Z}}$. A common class of transform operators are stable linear time-invariant (LTI) systems, and a possible approximation process is given by

$$(T_N f)(t) = \sum_{k=-N}^{N} f(k) h_T(t-k), \quad t \in \mathbb{R}. \tag{4}$$

For $f \in \mathcal{PW}_\pi^2$ and all stable LTI systems $T\colon \mathcal{PW}_\pi^2 \to \mathcal{PW}_\pi^2$ we have

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} |(Tf)(t) - (T_N f)(t)|^2 \, \mathrm{d}t = 0, \tag{5}$$

as can be easily seen: Since $T_N = TS_N$, we have

$$\begin{aligned}\|Tf - T_N f\|_{\mathcal{PW}_\pi^2} &= \|T(f - S_N f)\|_{\mathcal{PW}_\pi^2} \\ &\le \|T\| \|f - S_N f\|_{\mathcal{PW}_\pi^2}.\end{aligned}$$

and (5) follows immediately from (3). Further, it follows from (5) that

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} |(Tf)(t) - (T_N f)(t)| = 0 \tag{6}$$

for all $f \in \mathcal{PW}_\pi^2$. Hence, as shown by (5) and (6), the approximation process (4) converges in the $\mathcal{PW}_\pi^2$-norm and uniformly on the real axis.

Another approximation process is given by

$$(T_N^1 f)(t) = \sum_{k=-N}^{N} f(t-k) h_T(k), \quad t \in \mathbb{R}.$$

Here, the time variable $t \in \mathbb{R}$ is in the argument of f. We will see that this small, seemingly unimportant difference, will make a huge difference in the approximation behavior. As for the global convergence, we have the same uniform convergence as before: For all $f \in \mathcal{PW}_\pi^2$ and all stable LTI systems $T\colon \mathcal{PW}_\pi^2 \to \mathcal{PW}_\pi^2$ we have

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left|(Tf)(t) - (T_N^1 f)(t)\right| = 0.$$

Thus, both series,

$$\sum_{k=-\infty}^{\infty} f(k) h_T(t-k)$$

and

$$\sum_{k=-\infty}^{\infty} f(t-k) h_T(k), \tag{7}$$

are equiconvergent with respect to the maximum norm. However, in general we do not have equiconvergence with respect to the $\mathcal{PW}_\pi^2$-norm. The convergence of (7) in the maximum norm is too weak to make an assertion about the behavior of the $\mathcal{PW}_\pi^2$-norm. Roughly speaking, the energy concentration of $(T_N^1 f)(t)$ on $\mathbb{R}$ is too weak, i.e., for all $\tau > 0$ there exist a input signal $f \in \mathcal{PW}_\pi^2$ and a stable LTI system $T\colon \mathcal{PW}_\pi^2 \to \mathcal{PW}_\pi^2$ such that

$$\limsup_{N\to\infty} \int_{|t|\ge\tau} \left| \sum_{k=-N}^{N} f(t-k) h_T(k) \right|^2 \mathrm{d}t = \infty. \tag{8}$$

We will prove an even stronger statement in the next section. It is stronger, because we can show that there exist two infinite dimensional closed subspaces $D_{\text{sig}} \subset \mathcal{PW}_\pi^2$ and $D_{\text{sys}} \subset \mathcal{T}$ such that we have (8) for all pairs $(f,T) \in D_{\text{sig}} \times D_{\text{sys}}$ with $f \not\equiv 0$ and $T \not\equiv 0$. Hence, the sets of signals and systems creating divergence are jointly spaceable.

5. MAIN RESULT

Theorem 1. *There exist an infinite dimensional closed subspace $D_{sig} \subset \mathcal{PW}_\pi^2$ and an infinite dimensional closed subspace $D_{sys} \subset \mathcal{T}$ such that for all $f \in D_{sig}$, $f \not\equiv 0$, and all $T \in D_{sys}$, $T \not\equiv 0$, we have*

$$\limsup_{N\to\infty} \int_{-\infty}^{\infty} \left| \sum_{k=-N}^{N} f(t-k) h_T(k) \right|^2 \mathrm{d}t = \infty.$$

For the proof of Theorem 1, we need a lemma, the statement of which requires the introduction of some notation. For $M \in \mathbb{Z}$ and $N \in \mathbb{N}$, let

$$q_N^*(k) = \begin{cases} \frac{1}{k}, & -N \le k \le -1, \\ \frac{1}{k}, & 1 \le k \le N, \\ 0, & k = 0 \text{ or } |k| > N, \end{cases}$$

and

$$q_{M,N}(k) = \frac{i}{2} q_N^*(k+M), \quad k \in \mathbb{Z}.$$

It is easy to see that the function

$$\hat{q}_{M,N}(\omega) = \sum_{k=-\infty}^{\infty} q_{M,N}(-k) \, \mathrm{e}^{i\omega k}, \quad \omega \in [-\pi, \pi],$$

i.e., the Fourier series with coefficients $\{q_{M,N}(-k)\}_{k\in\mathbb{Z}}$, is given by

$$\hat{q}_{M,N}(\omega) = \mathrm{e}^{iM\omega} \left(\sum_{k=1}^{N} \frac{1}{k} \sin(k\omega) \right), \quad \omega \in [-\pi, \pi].$$

Note, that we have

$$|\hat{q}_{M,N}(\omega)| = \left| \sum_{k=1}^{N} \frac{1}{k} \sin(k\omega) \right| \le C_1 \tag{9}$$

for all $M \in \mathbb{Z}$, $N \in \mathbb{N}$, and $\omega \in [-\pi, \pi]$ [16, p. 183]. Further, for $K \in \mathbb{N}$, let

$$(F_K \hat{q})(\omega) = \sum_{k=-K}^{K} q(-k) \, \mathrm{e}^{i\omega k}, \quad \omega \in [-\pi, \pi],$$

be the partial sum of the Fourier series of a function $\hat{q} \in L^\infty[-\pi,\pi]$, where

$$q(-k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{q}(\omega)\,\mathrm{e}^{-i\omega k}\ \mathrm{d}\omega$$

denotes the k-th Fourier coefficient of $\hat{q}$.

Lemma 1. *There exists a constant C_2 such that for all $K, M, N \in \mathbb{N}$, $M \geq N$, and all $\omega \in [-\pi,\pi] \setminus \{0\}$ we have*

$$|(F_K \hat{q}_{M,N})(\omega)| \leq \log\left(\frac{1}{|\omega|}\right) + C_2.$$

Lemma 1 can be proved by using standard estimates for trigonometric sums [16, pp. 182–191].

Proof of Theorem 1. The key idea of the proof is to construct an infinite dimensional closed subspace D_{sig} of $\mathcal{PW}_\pi^2$ and an infinite dimensional closed subspace $\hat{D}_{\text{sys}}$ of $C[-\pi,\pi] \subset L^\infty[-\pi,\pi]$, such that we have divergence of (7) in the L^2-norm for any pair $(f, \hat{h}_T) \in D_{\text{sig}} \times \hat{D}_{\text{sys}}$ with $f \not\equiv 0$ and $\hat{h}_T \not\equiv 0$. Due to page constraints, some details of the calculation are omitted in the proof.

We first construct a sequence $\{\hat{\phi}_n\}_{n\in\mathbb{N}}$ of functions in $C[-\pi,\pi]$. For $r \in \mathbb{N}$, let $N_r = 2^{(r+1)^7}$. The construction is done iteratively. In the first iteration, i.e., $s = 1$, we set $M_1(1) = N_1$, $N_1(1) = N_1$, and define $\hat{\phi}_{1,1}(\omega) = \hat{q}_{M_1(1),N_1}(\omega)$. Further, we set $\overline{M}_1 = 2N_1 + 1$.

Now, let $s = 2$. We set $M_2(1) = \overline{M}_1 + N_2$, $M_2(2) = M_2(1) + 2N_2 + 1$, and define $\hat{\phi}_{1,2}(\omega) = \hat{\phi}_{1,1}(\omega) + \frac{1}{2^2}\hat{q}_{M_2(1),N_2}(\omega)$ and $\hat{\phi}_{2,2}(\omega) = \frac{1}{2^2}\hat{q}_{M_2(2),N_2}(\omega)$. Further, we set $\overline{M}_2 = \overline{M}_1 + 2(2N_2+1)$.

Assume that, for some $s \in \mathbb{N}$, and all $k, m \in \mathbb{N}$ with $1 \leq k \leq s$ and $k \leq m \leq s$, we have constructed the functions $\hat{\phi}_{k,m}$ and the numbers $M_m(k)$ and $\overline{M}_k$. We set $M_{s+1}(1) = \overline{M}_s + N_{s+1}$ and $M_{s+1}(n) = M_{s+1}(n-1) + 2N_{s+1} + 1$ for $n = 1, \ldots, s+1$, and define $\hat{\phi}_{n,s+1}(\omega) = \hat{\phi}_{n,s}(\omega) + \frac{1}{(s+1)^2}\hat{q}_{M_{s+1}(n),N_{s+1}}(\omega)$ for $n = 1, \ldots, s$, and $\hat{\phi}_{s+1,s+1}(\omega) = \frac{1}{(s+1)^2}\hat{q}_{M_{s+1}(s+1),N_{s+1}}(\omega)$. Further, we set $\overline{M}_{s+1} = \overline{M}_s + s(2N_{s+1} + 1)$.

For each $n \in \mathbb{N}$ we have inductively constructed a sequence of $C[-\pi,\pi]$ functions $\hat{\phi}_{n,m}$, $m \geq n$. Due to the previous construction it is easy to see that for each $n \in \mathbb{N}$ there exists a function $\hat{\phi}_n \in C[-\pi,\pi]$ such that $\lim_{m\to\infty}\|\hat{\phi}_n - \hat{\phi}_{n,m}\|_{C[-\pi,\pi]} = 0$. This function is given by

$$\hat{\phi}_n(\omega) = \sum_{k=n}^{\infty} \frac{1}{k^2}\hat{q}_{M_k(n),N_k}(\omega), \quad \omega \in [-\pi,\pi],$$

and it follows that

$$\|\hat{\phi}_n\|_{C[-\pi,\pi]} \leq \sum_{k=n}^{\infty}\frac{1}{k^2}\|\hat{q}_{M_k(n),N_k}\|_{C[-\pi,\pi]} \leq \frac{\pi^2 C_1}{6}, \qquad (10)$$

where we used (9) in the second inequality.

By using (9) it is easy to see that

$$|(F_{M_s(m)}\hat{\phi}_n)(\omega)| \leq \frac{\pi^2 C_1}{6}$$

for all $\omega \in [-\pi,\pi]$, $m, n \in \mathbb{N}$ with $m \neq n$, and $s \in \mathbb{N}$ with $s > n$. Further, for $s > n$ and $m = n$, we have

$$(F_{M_s(n)}\hat{\phi}_n)(\omega) = \sum_{k=n}^{s-1}\frac{1}{k^2}\hat{q}_{M_k(n),N_k}(\omega) + \frac{1}{s^2}\sum_{k=M_s(n)-N_s}^{M_s(n)-1}\frac{1}{2i}\frac{1}{k - M_s(n)}\,\mathrm{e}^{ik\omega}.$$

Using basic inequalities and estimates for sums, it can be shown that

$$|(F_{M_s(n)}\hat{\phi}_n)(\omega)| \geq \frac{\log(2)}{2}s^5 - C_3 \qquad (11)$$

for $\omega \in [-\frac{1}{2N_s}, \frac{1}{2N_s}]$ and $s > n$, where C_3 is a constant that does not depend on s.

For $r \in \mathbb{N}$ we consider the numbers $\omega_r = \pi/2^r$ and the intervals $I_r = \left[\omega_r - \frac{\omega_r - \omega_{r+1}}{2}, \omega_r + \frac{\omega_r - \omega_{r+1}}{2}\right]$. We have $I_r \cap I_{r+1} = \emptyset$ for all $r \in \mathbb{N}$. Further, the length of the gap between intervals I_r and I_{r+1} is given by $\Delta_r = \pi/2^{r+3}$. Next, for $n \in \mathbb{N}$, we define the functions

$$\hat{h}_n(\omega) = \sum_{r=1}^{\infty}\frac{1}{r^3}\hat{\phi}_n(\omega - \omega_r), \quad \omega \in [-\pi,\pi]. \qquad (12)$$

Originally, the functions $\hat{\phi}_n$, $n \in \mathbb{N}$, were defined as functions in $C[-\pi,\pi]$. However, as easily can be seen, each $\hat{\phi}_n$ naturally extends to a continuous 2π-periodic function. Whenever we consider shifts of such a function, as for example in (12), we implicitly use the 2π-periodic extension. Since

$$\|\hat{h}_n\|_{C[-\pi,\pi]} \leq \sum_{r=1}^{\infty}\frac{1}{r^3}\|\hat{\phi}_n\|_{C[-\pi,\pi]} \leq \frac{\pi^2 C_1}{6}\sum_{r=1}^{\infty}\frac{1}{r^3} = C_4, \quad (13)$$

where we used (10), it follows that $\hat{h}_n \in C[-\pi,\pi]$, and that the $C[-\pi,\pi]$-norm of $\hat{h}_n$ is bounded independently of n.

Using Lemma 1, it can be shown that

$$\left|(F_K\hat{h}_n)(\omega) - \frac{1}{\hat{r}^3}(F_K\hat{\phi}_n(\,\cdot\, - \omega_{\hat{r}}))(\omega)\right| \leq C_5(\hat{r}), \qquad (14)$$

for all $K, n, \hat{r} \in \mathbb{N}$ and $\omega \in I_{\hat{r}}$, where $C_5(\hat{r})$ is a constant that depends only on $\hat{r}$.

For $r \in \mathbb{N}$ we construct input signals f_r such that $\hat{f}_r \geq 0$ almost everywhere and such that the Fourier transform $\hat{f}_r$ is concentrated on the interval I_r. For δ with $0 < \delta < \pi$ we set

$$\hat{g}_\delta(\omega) = \begin{cases} \sqrt{\frac{1}{\delta}}, & |\omega| \leq \frac{\delta}{2}, \\ 0, & \frac{\delta}{2} < |\omega| \leq \pi. \end{cases}$$

Clearly, we have $g_\delta \in \mathcal{PW}_\pi^2$ with $\|g_\delta\|_{\mathcal{PW}_\pi^2} = 1$, and

$$g_\delta(t) = \sqrt{\frac{1}{\delta}}\frac{\sin(\delta t/2)}{\pi t}, \quad t \in \mathbb{R}.$$

Let $r \in \mathbb{N}$ be arbitrary but fixed. Further, let s_r be the smallest natural number such that $\frac{\omega_r - \omega_{r+1}}{2} \geq \frac{1}{2N_s}$ for all $s \geq s_r$. We set

$$\gamma_r(t) = \sum_{s=s_r}^{\infty}\frac{1}{s^2}g_{\frac{1}{N_s}}(t)\,\mathrm{e}^{i\omega_r t}, \quad t \in \mathbb{R},$$

which corresponds to

$$\hat{\gamma}_r(\omega) = \sum_{s=s_r}^{\infty}\frac{1}{s^2}\hat{g}_{\frac{1}{N_s}}(\omega - \omega_r), \quad \omega \in [-\pi,\pi].$$

We have

$$\|\gamma_r\|_{\mathcal{PW}_\pi^2} \leq \sum_{s=s_r}^{\infty}\frac{1}{s^2}\|g_{\frac{1}{N_s}}\|_{\mathcal{PW}_\pi^2} \leq \sum_{s=1}^{\infty}\frac{1}{s^2} = \frac{\pi^2}{6},$$

and the Fourier transform $\hat{\gamma}_r$ is concentrated on I_r. Moreover, since $I_r \cap I_{r+1} = \emptyset$ for all $r \in \mathbb{N}$, it follows that the functions $\{\gamma_r\}_{r\in\mathbb{Z}}$ are orthogonal.

Using (11) and (14) it can be shown that

$$\lim_{s\to\infty} \int_{-\infty}^{\infty} \left| \sum_{k=-M_s(n)}^{M_s(n)} \gamma_r(t-k) h_n(k) \right|^2 \, \mathrm{d}t = \infty \qquad (15)$$

for all $r, n \in \mathbb{N}$.

Let $f_r = \gamma_r / \|\gamma_r\|_{\mathcal{PW}_\pi^2}$, $r \in \mathbb{N}$. Then $\|f_r\|_{\mathcal{PW}_\pi^2} = 1$ for all $r \in \mathbb{N}$, and $\{f_r\}_{r\in\mathbb{N}}$ is a orthonormal system in $\mathcal{PW}_\pi^2$. It follows that $\{f_r\}_{r\in\mathbb{N}}$ is a basic sequence, i.e., a Schauder basis for

$$D_{\text{sig}} = \overline{\operatorname{span}(\{f_r\}_{r\in\mathbb{N}})}^{\mathcal{PW}_\pi^2}.$$

Further, for $n \in \mathbb{N}$, let $e_n(\omega) = \mathrm{e}^{i2^n\omega}$, $\omega \in [-\pi, \pi]$. Then $\{e_n\}_{n\in\mathbb{N}}$ is a basic sequence in $C[-\pi, \pi]$ [16, p. 247], i.e., a Schauder basis for $B_1 = \overline{\operatorname{span}(\{e_n\}_{n\in\mathbb{N}})}^{C[-\pi,\pi]}$. It follows that every $f \in D_{\text{sig}}$ has the representation $f = \sum_{r=1}^{\infty} \alpha_r(f) f_r$ with unique coefficients $\{\alpha_r(f)\}_{r\in\mathbb{N}}$. For the coefficient functionals $\alpha_r \colon D_{\text{sig}} \to \mathbb{C}$ we have $\|\alpha_r\| = 1$, $r \in \mathbb{N}$. Further, every $\hat{h} \in B_1$ has the representation $\hat{h} = \sum_{n=1}^{\infty} \beta_n(\hat{h}) e_n$ with unique coefficients $\{\beta_n(\hat{h})\}_{n\in\mathbb{N}}$, satisfying [16, p. 247]

$$\sum_{n=1}^{\infty} |\beta_n(\hat{h})| < \infty. \qquad (16)$$

Further, there exists a constant $C_6 > 0$ such that

$$C_6 \sum_{n=1}^{\infty} |\beta_n(\hat{h})| \le \|\hat{h}\|_{C[-\pi,\pi]} \le \sum_{n=1}^{\infty} |\beta_n(\hat{h})|.$$

Hence $\sum_{n=1}^{\infty} \beta_n(\hat{h}) e_n$ converges if and only if (16) is satisfied. For the coefficient functionals $\beta_n \colon B_1 \to \mathbb{C}$ we have $\|\beta_n\| = 1$, $n \in \mathbb{N}$. Let

$$\hat{u}_n = e_n + \frac{1}{C_4 2^{n+1}} \hat{h}_n,$$

where C_4 is the upper bound on $\|\hat{h}_n\|_{C[-\pi,\pi]}$ given in (13). Then we have

$$\sum_{n=1}^{\infty} \|\beta_n\| \|\hat{u}_n - e_n\|_{C[-\pi,\pi]} < 1.$$

It follows that $\{\hat{u}_n\}_{n\in\mathbb{N}}$ is a basic sequence for $C[-\pi, \pi]$ that is equivalent to $\{e_n\}_{n\in\mathbb{N}}$ [17, p. 46]. Hence, $\sum_{n=1}^{\infty} b_n \hat{u}_n$ converges if and only if

$$\sum_{n=1}^{\infty} |b_n| < \infty.$$

Let

$$\hat{D}_{\text{sys}} = \overline{\operatorname{span}(\{\hat{u}_n\}_{n\in\mathbb{N}})}^{C[-\pi,\pi]}.$$

Next, we will show divergence for any pair $(f, \hat{h}) \in D_{\text{sig}} \times \hat{D}_{\text{sys}}$ with $f \not\equiv 0$ and $\hat{h} \not\equiv 0$. Let $f \in D_{\text{sig}}$, $f \not\equiv 0$, and $\hat{h} \in \hat{D}_{\text{sys}}$, $\hat{h} \not\equiv 0$, be arbitrary. f can be written as $f = \sum_{r=1}^{\infty} \alpha_r(f) f_r$ and $\hat{h}$ as $\hat{h} = \sum_{n=1}^{\infty} b_n(\hat{h}) \hat{u}_n$. Let r_0 be the smallest natural number for which $\alpha_{r_0}(f) \neq 0$, and n_0 the smallest natural number for which $b_{n_0}(\hat{h}) \neq 0$. Then we have

$$f(t) = \sum_{r=r_0}^{\infty} \alpha_r(f) f_r(t)$$

for $t \in \mathbb{R}$, and

$$\hat{h}(\omega) = \sum_{n=n_0}^{\infty} b_n(\hat{h}) e_n(\omega) + \sum_{n=n_0}^{\infty} \frac{1}{C_4 2^{n+1}} b_n(\hat{h}) \hat{h}_n(\omega)$$

for $\omega \in [-\pi, \pi]$. From

$$\begin{aligned} &\frac{1}{2\pi} \int_{I_{r_0}} |\hat{f}_{r_0}(\omega)|^2 |(F_{M_s(n_0)} \hat{h}_{n_0})(\omega)|^2 \, \mathrm{d}\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}_{r_0}(\omega)|^2 \left| \sum_{k=-M_s(n_0)}^{M_s(n_0)} h_{n_0}(k) \, \mathrm{e}^{-i\omega k} \right|^2 \, \mathrm{d}\omega \\ &= \int_{-\infty}^{\infty} \left| \sum_{k=-M_s(n_0)}^{M_s(n_0)} f_{r_0}(t-k) h_{n_0}(k) \right|^2 \, \mathrm{d}t \end{aligned}$$

and (15), it follows that

$$\lim_{s\to\infty} \frac{1}{2\pi} \int_{I_{r_0}} |\hat{f}_{r_0}(\omega)|^2 |(F_{M_s(n_0)} \hat{h}_{n_0})(\omega)|^2 \, \mathrm{d}\omega = \infty. \qquad (17)$$

Using (17) it can be shown that

$$\lim_{s\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(\omega)|^2 |F_{M_s(n_0)} \hat{h}(\omega)|^2 \, \mathrm{d}\omega = \infty.$$

Since

$$\begin{aligned} &\int_{-\infty}^{\infty} \left| \sum_{k=-M_s(n_0)}^{M_s(n_0)} f(t-k) h(k) \right|^2 \, \mathrm{d}t \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(\omega)|^2 |(F_{M_s(n_0)} \hat{h})(\omega)|^2 \, \mathrm{d}\omega, \end{aligned}$$

and $f \in D_{\text{sig}}$, $f \not\equiv 0$, and $\hat{h} \in \hat{D}_{\text{sys}}$, $\hat{h} \not\equiv 0$, were chosen arbitrarily, this completes the proof. □

6. RELATION TO PRIOR WORK

Although the approximation of LTI systems by sampling series is well-studied [7–12], the convergence behavior of the series (2) is less well understood. In this paper we considered the Paley–Wiener space $\mathcal{PW}_\pi^2$ of bandlimited signals with finite energy, i.e., L^2-norm. Concerning the maximum norm, the series (1) and (2) are equiconvergent, however, as shown by Theorem 1, the L^2-norm of (2) diverges unboundedly for certain signals and systems. That is, the system approximation process (2) does not converge in the norm of the considered signal space.

Further, it was proved that the sets of signals and systems creating divergence are jointly spaceable. Spaceability is a relatively recent concept to describe linear structures in subsets of normed spaces. In [13] it was shown that the set of continuous nowhere differentiable functions on $C[0, 1]$ is spaceable.

The approximation processes (1) and (2) and their pointwise approximation behaviors, were analyzed in [18] for the signal space $\mathcal{PW}_\pi^1$, and joint spaceability of the sets of signals and systems creating divergence was proved. For further results please also see [19]. The significant differences to the present paper are the considered signal spaces ($\mathcal{PW}_\pi^1$ versus $\mathcal{PW}_\pi^2$) and the modes of convergence (pointwise convergence versus convergence in the L^2-norm).

An extended version of this paper is in preparation [20], where the full proofs are given and further discussion is provided.

7. REFERENCES

[1] Claude E. Shannon, "Communication in the presence of noise," in *Proceedings of the IRE*, Jan. 1949, vol. 37, pp. 10–21.

[2] Abdul J. Jerri, "The Shannon sampling theorem–its various extensions and applications: A tutorial review," *Proc. IEEE*, vol. 65, no. 11, pp. 1565–1596, Nov. 1977.

[3] John R. Higgins, "Five short stories about the cardinal series," *Bull. Amer. Math. Soc.*, vol. 12, no. 1, pp. 45–89, 1985.

[4] P. L. Butzer, W. Splettstößer, and R. L. Stens, "The sampling theorem and linear prediction in signal analysis," *Jahresber. d. Dt. Math.-Verein.*, vol. 90, no. 1, pp. 1–70, Jan. 1988.

[5] John R. Higgins, *Sampling Theory in Fourier and Signal Analysis – Foundations*, Oxford University Press, 1996.

[6] Farokh Marvasti, Ed., *Nonuniform Sampling: Theory and Practice*, Kluwer Academic / Plenum Publishers, 2001.

[7] Henry P. Kramer, "The digital form of operators on band-limited functions," *J. Math Anal. and Appl.*, vol. 44, no. 2, pp. 275–287, Nov. 1973.

[8] Athanasios Papoulis, "Generalized sampling expansion," *IEEE Trans. Circuits Syst.*, vol. 24, no. 11, pp. 652–654, Nov. 1977.

[9] Rudolf L. Stens, "A unified approach to sampling theorems for derivatives and Hilbert transforms," *Signal Process.*, vol. 5, pp. 139–151, 1983.

[10] Alan V. Oppenheim and Ronald W. Schafer, *Discrete-Time Signal Processing*, Prentice Hall, 3 edition, 2009.

[11] Muhammad K. Habib, "Digital representations of operators on band-limited random signals," *IEEE Trans. Inf. Theory*, vol. 47, no. 1, pp. 173–177, Jan. 2001.

[12] Paul L. Butzer, Gerhard Schmeisser, and Rudolf L. Stens, "Shannon's sampling theorem for bandlimited signals and their Hilbert transform, Boas-type formulae for higher order derivative—the aliasing error involved by their extensions from bandlimited to non-bandlimited signals," *Entropy*, vol. 14, no. 11, pp. 2192–2226, Nov. 2012.

[13] V. P. Fonf, V. I. Gurariy, and M. I. Kadets, "An infinite dimensional subspace of $C[0,1]$ consisting of nowhere differentiable functions," *C. R. Acad. Bulgare Sci.*, vol. 52, no. 11-12, pp. 13–16, 1999.

[14] Richard Aron, V. I. Gurariy, and J. B. Seoane, "Lineability and spaceability of sets of functions on $\mathbb{R}$," *Proc. Amer. Math. Soc.*, vol. 133, no. 3, pp. 795–803, 2005.

[15] Frédéric Bayart, "Linearity of sets of strange functions," *Michigan Math. J.*, vol. 53, no. 2, pp. 291–303, 2005.

[16] A. Zygmund, *Trigonometric Series*, vol. I, Cambridge University Press, 3 edition, 2002.

[17] Joseph Diestel, *Sequences and series in Banach spaces*, vol. 92, Springer Science & Business Media, 2012.

[18] Holger Boche and Ullrich J. Mönich, "Signal and system spaces with non-convergent sampling representation," in *Proceedings of European Signal Processing Conference (EUSIPCO)*, Aug. 2016, pp. 2131–2135.

[19] Holger Boche and Ullrich J. Mönich, "Spaceability for sets of bandlimited input functions and stable linear time-invariant systems with divergence behavior," *Probl. Inform. Transm.*, 2016, in press.

[20] Holger Boche and Ullrich J. Mönich, "System representations for the Paley–Wiener space PW_π^2," *J. Fourier Anal. Appl.*, 2016, in press.