GRIDLESS COMPRESSED SENSING UNDER SHIFT-INVARIANT SAMPLING

Christian Steffens, Wassim Suleiman, Alexander Sorg, and Marius Pesavento

Communication Systems Group, Technische Universität Darmstadt, Germany

ABSTRACT

Parameter estimation has applications in many fields of signal processing, such as spectral analysis or direction-of-arrival estimation. Subspace-based methods like root-MUSIC and ESPRIT provide high parameter resolution at low computational complexity by exploiting specific sampling structure, namely uniform linear sampling and shift-invariant sampling, respectively. On the other hand, compressed sensing has been shown to outperform subspace-based methods in difficult scenarios such as low number of measurement vectors, high noise power or correlated signals. While it is well known that uniform sampling admits gridless compressed sensing methods, e.g., based on atomic norm minimization, no such approaches are known for shift-invariant sampling. In this paper we present a novel approach for gridless compressed sensing under shift-invariant sampling. We show by numerical experiments that the proposed method outperforms ESPRIT in difficult scenarios.

Index Terms— Joint Sparsity, Shift-Invariant Sampling, Gridless Parameter Estimation, Partly Calibrated Array

1. INTRODUCTION

A large number of samples with a large aperture are desirable in many signal processing applications to increase the number of identifiable parameters and to achieve a high parameter resolution [1]. In direction-of-arrival (DOA) estimation, for example, a large number of sensors allows to identify a large number of source signals and a large array aperture admits a high angular resolution. On the other hand, large sampling apertures become more difficult to calibrate. To overcome the difficulty of calibrating the entire sensor array the concept of partly calibrated arrays (PCAs) has been introduced, where the entire array is partitioned into subarrays which are easy to calibrate, while the overall array response may be unknown.

Various DOA estimation methods have been proposed for PCAs, such as the subspace-based methods RARE [2, 3] or ESPRIT [4, 5]. The RARE method [2] is applicable to arbitrary array topologies but requires an expensive spectrum search. For the special case of linear subarrays with common baseline the root-RARE method [3] admits search-free parameter estimation and provides improved estimation performance. The case of PCAs with identical subarrays has been considered, e.g., in [6, 7], and falls in the more general class of arrays with shift-invariances which admits application of the subspace-based ESPRIT method [4, 5]. Similar to the root-RARE method, the ESPRIT method provides search-free DOA estimates and, additionally, easy implementation in a decentralized fashion, as suggested, e.g., in [8, 9]. While subspace-based methods have been shown to perform asymptotically optimal, these methods suffer from severe performance degradation in the case of difficult scenarios such as high noise power, low number of measurement vectors or correlated source signals.

In recent years, compressed sensing (CS) techniques [10–13] have been shown to provide an attractive alternative to subspace-based methods, since CS exhibits good estimation performance even in difficult scenarios [14–16]. Similar to subspace-based methods, CS methods offer the superresolution property [17] at tractable computational performance. Most research on CS for DOA estimation has focused on fully calibrated arrays, e.g., [14–16], and is based on spectrum search. More recently it has been shown that special sampling structure admits search-free CS methods, e.g., uniform sampling [16, 18–21]. A grid-based CS method for PCAs of arbitrary topology has been presented in [22]. However, gridless Cs methods in the fashion of root-RARE or ESPRIT are not available for PCAs.

In this work we present a gridless CS method to provide search-free parameter estimates under shift-invariant sampling, with application to DOA estimation in PCAs. Our method is based on the recently proposed SPARROW formulation for joint sparse reconstruction from multiple measurement vectors (MMVs). We show by numerical experiments that our proposed shift-invariant SPARROW formulation outperforms the ESPRIT method in difficult scenarios.

2. SIGNAL MODEL

Consider a linear array of $M = PM_0$ omnidirectional, identical sensors partitioned into P identical subarrays of M_0 sensors, as displayed in Figure 1. We assume that the relative sensor positions within each subarray are perfectly known, such that the subarray responses are available in analytic form. The displacements between the different subarrays are assumed to be unknown such that the overall array response is generally unknown and the array is referred to as a partly calibrated array (PCA). It is well known that PCAs composed of identical and identically oriented subarrays exhibit multiple shift-invariances that can be exploited for parameter estimation [4–9]. For illustration of different kinds of shift-invariances consider the array topology given in Figure 1 and the corresponding shift-invariant sensor groups given in Figure 2. Let $r_{p,m} \in \mathbb{R}$ denote the global position of the mth sensor in the pth subarray in half signal wavelength, for $p = 1, \ldots, P$ and $m = 1, \ldots, M_0$, then the shift-invariance property is expressed as

$$r_{p,m} = r_{1,m} + \Delta_p^{(1)}, \quad p = 2, \ldots, P, \ m = 1, \ldots, M_0 \tag{1a}$$

$$r_{p,m} = r_{p,1} + \Delta_m^{(2)}, \quad p = 1, \ldots, P, \ m = 2, \ldots, M_0, \tag{1b}$$

where the shifts $\Delta_p^{(1)}$, $p = 2, \ldots, P$ denote the unknown inter-subarray displacements while the shifts $\Delta_m^{(2)}$, $m = 2, \ldots, M_0$ denote the perfectly known relative sensor positions within the subarrays. We remark that other array topologies might exhibit additional

The authors acknowledge the financial support of the Seventh Framework Programme for Research of the European Commission under grant number ADEL-619647 and the EXPRESS project within the DFG priority program CoSIP (DFG-SPP 1798).

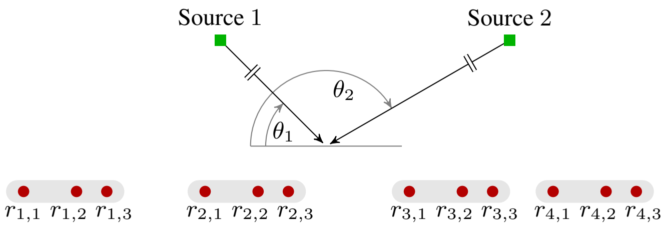

Fig. 1. Example of a linear PCA composed of $P = 4$ identical subarrays with $M_0 = 3$ sensors per subarray, and $L = 2$ sources

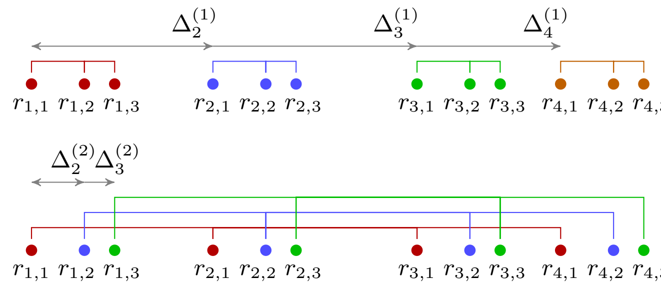

Fig. 2. Example of groups which are identical under linear shifts

shift-invariances that can be exploited, such as shift-invariances with overlapping groups or centro-symmetry. For ease of presentation we limit our discussion to the example in Figure 2.

Furthermore, consider L narrowband far-field sources in stationary angular directions $\theta_1, \ldots, \theta_L$, relative to the array axis, and define the spatial frequencies

$$\mu_l = \cos\theta_l \in [-1, 1) \tag{2}$$

for $l = 1, \ldots, L$, summarized in $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_L]$.

The array output for N time instants is stored in the measurement matrix $\boldsymbol{Y} \in \mathbb{C}^{M \times N}$, where $[\boldsymbol{Y}]_{m,n}$ is the output at sensor m and time instant n. The measurement matrix is modeled as

$$\boldsymbol{Y} = \boldsymbol{A}(\boldsymbol{\mu})\boldsymbol{\Psi} + \boldsymbol{W}, \tag{3}$$

where $\boldsymbol{\Psi} \in \mathbb{C}^{L \times N}$ denotes the source signal matrix, with $[\boldsymbol{\Psi}]_{l,n}$ denoting the signal transmitted by source l at time instant n. The matrix $\boldsymbol{W} \in \mathbb{C}^{M \times N}$ represents independent and identically distributed circular and spatio-temporal white Gaussian noise with covariance matrix $\mathrm{E}\{\boldsymbol{W}\boldsymbol{W}^{\mathsf{H}}\}/N = \sigma^2 \boldsymbol{I}_M$, where $\boldsymbol{I}_M$ denotes the $M \times M$ identity matrix and σ^2 the noise power. The $M \times L$ array steering matrix

$$\boldsymbol{A}(\boldsymbol{\mu}) = [\boldsymbol{a}(\mu_1), \ldots, \boldsymbol{a}(\mu_L)], \tag{4}$$

is composed of the normalized array steering vectors

$$\boldsymbol{a}(\mu) = \frac{1}{\sqrt{M}} \left[\mathrm{e}^{j\pi\mu r_{1,1}}, \ldots, \mathrm{e}^{j\pi\mu r_{1,M_0}}, \mathrm{e}^{j\pi\mu r_{2,1}}, \ldots, \mathrm{e}^{j\pi\mu r_{P,M_0}}\right]^{\mathsf{T}}, \tag{5}$$

i.e., the steering vector contains the sensor responses arranged by subarray and sensor index according to $r_{1,1}, \ldots, r_{1,M_0}, r_{2,1}, \ldots, r_{2,M_0}, r_{3,1}, \ldots, r_{P,M_0}$.

Let $\boldsymbol{J}_m^{(1)}$ and $\boldsymbol{J}_p^{(2)}$, for $m = 1, \ldots, M_0$ and $p = 1, \ldots, P$, denote a set of selection matrices to assign the sensors to the various shift-invariant groups, i.e.,

$$\boldsymbol{J}_p^{(1)} = \boldsymbol{e}_{P,p} \otimes \boldsymbol{I}_{M_0} \tag{6a}$$

$$\boldsymbol{J}_m^{(2)} = \boldsymbol{I}_P \otimes \boldsymbol{e}_{M_0,m} \tag{6b}$$

where $\boldsymbol{e}_{P,p} = [0, \ldots, 0, 1, 0, \ldots, 0]^{\mathsf{T}}$ denotes the P-dimensional basis vector with a single one in the pth element and zero entries elsewhere, and with $\otimes$ denoting the Kronecker product. By the shift-invariance property (1) the steering vector fulfills the identities

$$\boldsymbol{J}_p^{(1)\mathsf{T}}\, \boldsymbol{a}(\mu) = \mathrm{e}^{j\pi\mu\Delta_p^{(1)}}\, \boldsymbol{J}_1^{(1)\mathsf{T}}\, \boldsymbol{a}(\mu) \tag{7a}$$

$$\boldsymbol{J}_m^{(2)\mathsf{T}}\, \boldsymbol{a}(\mu) = \mathrm{e}^{j\pi\mu\Delta_m^{(2)}}\, \boldsymbol{J}_1^{(2)\mathsf{T}}\, \boldsymbol{a}(\mu) \tag{7b}$$

and correspondingly the steering matrix fulfills

$$\boldsymbol{J}_p^{(1)\mathsf{T}}\, \boldsymbol{A}(\boldsymbol{\mu}) = \boldsymbol{J}_1^{(1)\mathsf{T}}\, \boldsymbol{A}(\boldsymbol{\mu})\, \boldsymbol{\Phi}^{\Delta_p^{(1)}}(\boldsymbol{\mu}) \tag{8a}$$

$$\boldsymbol{J}_m^{(2)\mathsf{T}}\, \boldsymbol{A}(\boldsymbol{\mu}) = \boldsymbol{J}_1^{(2)\mathsf{T}}\, \boldsymbol{A}(\boldsymbol{\mu})\, \boldsymbol{\Phi}^{\Delta_m^{(2)}}(\boldsymbol{\mu}) \tag{8b}$$

where the $L \times L$ unitary diagonal matrix

$$\boldsymbol{\Phi}(\boldsymbol{\mu}) = \mathrm{diag}(\mathrm{e}^{j\pi\mu_1}, \ldots, \mathrm{e}^{j\pi\mu_L}) \tag{9}$$

contains the phase shifts for frequencies $\boldsymbol{\mu}$ on its main diagonal.

3. JOINT SPARSE COMPRESSED SENSING AND EQUIVALENT SPARROW FORMULATION

Before devising an algorithm that exploits the shift-invariance property for compressed sensing in partly calibrated arrays, in this section we revise the standard approach of grid-based compressed sensing for fully calibrated arrays.

Based on the signal model in (3) we introduce a sparse representation $\boldsymbol{A}(\boldsymbol{\mu})\boldsymbol{\Psi} = \boldsymbol{A}(\boldsymbol{\nu})\boldsymbol{X}$. In the sparse representation $\boldsymbol{A}(\boldsymbol{\nu})$ is an $M \times K$ overcomplete dictionary matrix obtained by sampling the field-of-view in $K \gg L$ spatial frequencies $\boldsymbol{\nu} = [\nu_1, \ldots, \nu_K]^{\mathsf{T}}$. The matrix $\boldsymbol{X} = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_K]^{\mathsf{T}} \in \mathbb{C}^{K \times N}$ is a sparse representation of the signal waveform matrix $\boldsymbol{\Psi}$, which has non-zero rows $\boldsymbol{x}_k$ only if the corresponding sampled spatial frequencies ν_k are contained in the spatial frequencies in $\boldsymbol{\mu}$. Using the sparse representation $\boldsymbol{A}(\boldsymbol{\nu})\boldsymbol{X}$, the DOA estimation problem can be formulated as the well-known convex mixed-norm minimization problem [15, 16, 23, 24]

$$\min_{\boldsymbol{X}} \frac{1}{2} \|\boldsymbol{A}(\boldsymbol{\nu})\, \boldsymbol{X} - \boldsymbol{Y}\|_F^2 + \lambda\sqrt{N}\|\boldsymbol{X}\|_{2,1}, \tag{10}$$

where $\lambda > 0$ is the regularization parameter determining the sparsity, i.e., the number of non-zero rows in the minimizer $\hat{\boldsymbol{X}}$, and

$$\|\boldsymbol{X}\|_{2,1} = \sum_{k=1}^{K} \|\boldsymbol{x}_k\|_2 \tag{11}$$

denotes the $\ell_{2,1}$ mixed-norm. The mixed-norm term (11) induces a coupling among the elements in each row $\boldsymbol{x}_k$, $k = 1, \ldots, K$, of the matrix $\boldsymbol{X}$ such that the ℓ_1 norm, i.e., the nonnegative summation, is performed on the ℓ_2 norms of the rows in $\boldsymbol{X}$.

Given a minimizer $\hat{\boldsymbol{X}} = [\hat{\boldsymbol{x}}_1, \ldots, \hat{\boldsymbol{x}}_K]^{\mathsf{T}}$ the DOA estimation problem reduces to the identification of the union support set, i.e., the indices of the non-zero rows, and assigning the corresponding frequency grid points to the set $\{\hat{\mu}\}$ of estimated spatial frequencies

$$\{\hat{\mu}\} = \left\{\nu_k \,\middle|\, \left\|\hat{\boldsymbol{x}}_k\right\|_2 \neq 0;\ k = 1, \ldots, K\right\}. \tag{12}$$

As stated by Theorem 1 of [16], the $\ell_{2,1}$ mixed-norm minimization problem (10) can equivalently be formulated as the SPARse ROW-norm reconstruction (SPARROW) problem

$$\min_{\boldsymbol{S}\in\mathbb{D}_+} \operatorname{Tr}\big(\big(\boldsymbol{A}(\boldsymbol{\nu})\,\boldsymbol{S}\,\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\nu})+\lambda\boldsymbol{I}_M\big)^{-1}\hat{\boldsymbol{R}}\big)+\operatorname{Tr}\big(\boldsymbol{S}\big), \tag{13}$$

with $\hat{\boldsymbol{R}}=\boldsymbol{Y}\boldsymbol{Y}^{\mathsf{H}}/N$ denoting the sample covariance matrix and $\mathbb{D}_+$ describing the set of nonnegative diagonal matrices. The equivalence holds in the sense that minimizers $\hat{\boldsymbol{X}}$ and $\hat{\boldsymbol{S}}$ for problems (10) and (13), respectively, are related by

$$\hat{\boldsymbol{X}}=\hat{\boldsymbol{S}}\,\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\nu})\big(\boldsymbol{A}(\boldsymbol{\nu})\,\hat{\boldsymbol{S}}\,\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\nu})+\lambda\boldsymbol{I}_M\big)^{-1}\boldsymbol{Y}. \tag{14}$$

In addition to (14), it holds that

$$\hat{s}_k=\frac{1}{\sqrt{N}}\|\hat{\boldsymbol{x}}_k\|_2, \tag{15}$$

for $k=1,\dots,K$, i.e., the matrix $\hat{\boldsymbol{S}}=\operatorname{diag}(\hat{s}_1,\dots,\hat{s}_K)$ contains the row-norms of the row sparse signal matrix $\hat{\boldsymbol{X}}=[\hat{\boldsymbol{x}}_1,\dots,\hat{\boldsymbol{x}}_K]^{\mathsf{T}}$ on its main diagonal such that the union support of $\hat{\boldsymbol{X}}$ is equivalently represented by the support of the sparse vector of row-norms $[\hat{s}_1,\dots,\hat{s}_K]$. By the assumption of unit norm steering vectors as defined in (5), it holds that

$$\operatorname{Tr}(\boldsymbol{S})=\operatorname{Tr}\big(\boldsymbol{A}(\boldsymbol{\nu})\,\boldsymbol{S}\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\nu})\big), \tag{16}$$

such that we can further reformulate the SPARROW problem (13) as

$$\min_{\boldsymbol{S}\in\mathbb{D}_+} \operatorname{Tr}\big((\boldsymbol{Q}+\lambda\boldsymbol{I}_M)^{-1}\hat{\boldsymbol{R}}\big)+\operatorname{Tr}(\boldsymbol{Q}) \tag{17a}$$

$$\text{s.t. } \boldsymbol{Q}=\boldsymbol{A}(\boldsymbol{\nu})\,\boldsymbol{S}\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\nu}). \tag{17b}$$

In the reformulation (17) the objective function (17a) only depends on the matrix variable $\boldsymbol{Q}$ with the dictionary-specific structure defined in the constraint (17b). An additional low rank structure in the minimizer $\hat{\boldsymbol{Q}}$ is encouraged by the trace-term $\operatorname{Tr}(\boldsymbol{Q})$ in (17a), since for $\boldsymbol{Q}\succeq\boldsymbol{0}$ it is equivalent to the nuclear norm of $\boldsymbol{Q}$ [25].

4. EXPLOITING THE SHIFT-INVARIANT STRUCTURE

Let us investigate the constraint (17b) in more detail. The shift-invariance of the steering matrix $\boldsymbol{A}(\boldsymbol{\mu})$ as expressed in (8a) directly applies also to the dictionary matrix $\boldsymbol{A}(\boldsymbol{\nu})$, such that we have the following identity

$$\begin{aligned}\boldsymbol{J}_p^{(1)\mathsf{T}}\boldsymbol{Q}\boldsymbol{J}_p^{(1)}&=\boldsymbol{J}_p^{(1)\mathsf{T}}\boldsymbol{A}(\boldsymbol{\nu})\,\boldsymbol{S}\,\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\nu})\,\boldsymbol{J}_p^{(1)}\\&=\boldsymbol{J}_1^{(1)\mathsf{T}}\boldsymbol{A}(\boldsymbol{\nu})\,\boldsymbol{\Phi}^{\Delta_p^{(1)}\mathsf{H}}(\boldsymbol{\nu})\,\boldsymbol{S}\,\boldsymbol{\Phi}^{\Delta_p^{(1)}}(\boldsymbol{\nu})\,\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\nu})\,\boldsymbol{J}_1^{(1)}\\&=\boldsymbol{J}_1^{(1)\mathsf{T}}\boldsymbol{A}(\boldsymbol{\nu})\,\boldsymbol{S}\,\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\nu})\,\boldsymbol{J}_1^{(1)}\\&=\boldsymbol{J}_1^{(1)\mathsf{T}}\boldsymbol{Q}\boldsymbol{J}_1^{(1)},\end{aligned} \tag{18}$$

for $p=2,\dots,P$, where the unitary diagonal matrix $\boldsymbol{\Phi}(\boldsymbol{\nu})$ is defined in correspondence to (9), such that $\boldsymbol{\Phi}^{\Delta_p^{(1)}\mathsf{H}}(\boldsymbol{\nu})\,\boldsymbol{S}\,\boldsymbol{\Phi}^{\Delta_p^{(1)}}(\boldsymbol{\nu})=\boldsymbol{S}$. From (18) it can be observed that for any two shift-invariant sensor groups the corresponding submatrices in $\boldsymbol{Q}$ are identical. The same observation holds true for the other groups of shift-invariances expressed in (8b), i.e.,

$$\boldsymbol{J}_m^{(2)\mathsf{T}}\boldsymbol{Q}\boldsymbol{J}_m^{(2)}=\boldsymbol{J}_1^{(2)\mathsf{T}}\boldsymbol{Q}\boldsymbol{J}_1^{(2)}, \tag{19}$$

for $m=2,\dots,M_0$. As mentioned in Section 2, other types of structure, such as overlapping shift-invariant groups or centro-symmetry, may be available in the array topology, which yield identical submatrices in $\boldsymbol{Q}$ in a similar way as (18) and (19).

We return to the SPARROW formulation (17) and replace the grid-based constraint $\boldsymbol{Q}=\boldsymbol{A}(\boldsymbol{\nu})\boldsymbol{S}\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\nu})$ in (17b) by the structural constraints in (18) and (19), to formulate the gridless SPARROW problem

$$\min_{\boldsymbol{Q}\succeq\boldsymbol{0}} \operatorname{Tr}\big((\boldsymbol{Q}+\lambda\boldsymbol{I}_M)^{-1}\hat{\boldsymbol{R}}\big)+\operatorname{Tr}(\boldsymbol{Q}) \tag{20a}$$

$$\text{s.t. } \boldsymbol{J}_p^{(1)\mathsf{T}}\boldsymbol{Q}\boldsymbol{J}_p^{(1)}=\boldsymbol{J}_1^{(1)\mathsf{T}}\boldsymbol{Q}\boldsymbol{J}_1^{(1)},\ p=2,\dots,P \tag{20b}$$

$$\boldsymbol{J}_m^{(2)\mathsf{T}}\boldsymbol{Q}\boldsymbol{J}_m^{(2)}=\boldsymbol{J}_1^{(2)\mathsf{T}}\boldsymbol{Q}\boldsymbol{J}_1^{(2)},\ m=2,\dots,M_0. \tag{20c}$$

The program in (20) does not require knowledge of the overall array response in form of the dictionary matrix $\boldsymbol{A}(\boldsymbol{\nu})$. In fact it does not even require knowledge of the shifts $\Delta_p^{(1)}$, $p=2,\dots,P$, and $\Delta_m^{(2)}$, $m=2,\dots,M_0$. The only information which is exploited is the precise mutual shift-invariance that applies to each group. By using semidefinite programmning the problem in (20) can be implemented as [16]

$$\min_{\boldsymbol{Q},\boldsymbol{U}\succeq\boldsymbol{0}} \operatorname{Tr}\big(\boldsymbol{U}\hat{\boldsymbol{R}}\big)+\operatorname{Tr}(\boldsymbol{Q}) \tag{21a}$$

$$\text{s.t. } \begin{bmatrix}\boldsymbol{U}_N & \boldsymbol{I}_M\\ \boldsymbol{I}_M & \boldsymbol{Q}+\lambda\boldsymbol{I}_M\end{bmatrix}\succeq\boldsymbol{0} \tag{21b}$$

$$\boldsymbol{J}_p^{(1)\mathsf{T}}\boldsymbol{Q}\boldsymbol{J}_p^{(1)}=\boldsymbol{J}_1^{(1)\mathsf{T}}\boldsymbol{Q}\boldsymbol{J}_1^{(1)},\ p=2,\dots,P \tag{21c}$$

$$\boldsymbol{J}_m^{(2)\mathsf{T}}\boldsymbol{Q}\boldsymbol{J}_m^{(2)}=\boldsymbol{J}_1^{(2)\mathsf{T}}\boldsymbol{Q}\boldsymbol{J}_1^{(2)},\ m=2,\dots,M_0. \tag{21d}$$

Given a minimizer $\hat{\boldsymbol{Q}}$ to problem (20) the underlying spatial frequencies can be recovered using different approaches, depending on the amount of shift-invariances and the exact knowledge that is available on the shifts $\Delta_p^{(1)}$, $p=2,\dots,P$, and $\Delta_m^{(2)}$, $m=2,\dots,M_0$. Assuming only a single shift structure, e.g. $\Delta_2^{(2)}$ in the case of a PCA composed of 2-element subarrays, the ESPRIT method [4] or the matrix pencil method [26,27] can be applied to the matrix $\hat{\boldsymbol{Q}}$ to estimate the spatial frequencies $\hat{\boldsymbol{\mu}}$ in a search-free fashion. If knowledge of multiple shifts $\Delta_p^{(1)}$, $p=2,\dots,P$, and $\Delta_m^{(2)}$, $m=2,\dots,M_0$ is available, more sophisticated methods such as Multiple Invariance ESPRIT [5] can be applied on $\hat{\boldsymbol{Q}}$ to obtain improved frequency estimates $\hat{\boldsymbol{\mu}}$.

5. REGULARIZATION PARAMETER SELECTION

To achieve good estimation performance in practical applications the SPARROW formulation requires an appropriate choice of the regularization parameter λ. While approximations for the regularization parameter have been derived, e.g., for uniform sampling with single measurement vectors [20], we follow a different approach and consider for the computation of the regularization parameter the asymptotic case of an infinite number of sensors $M\to\infty$ such that the dictionary matrix $\boldsymbol{A}(\boldsymbol{\nu})=[\boldsymbol{a}(\nu_1),\dots,\boldsymbol{a}(\nu_K)]$ in (13) becomes unitary, i.e. $\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\nu})\boldsymbol{A}(\boldsymbol{\nu})=\boldsymbol{I}_K$. In this case we can rewrite the Lagrangian function of the SPARROW problem (13) in a decoupled fashion as

$$f(s_1,\dots,s_k)=\sum_k\frac{\boldsymbol{a}^{\mathsf{H}}(\nu_k)\hat{\boldsymbol{R}}\boldsymbol{a}(\nu_k)}{s_k+\lambda}+(1-\gamma_k)s_k \tag{22}$$

where $\gamma_1,\dots,\gamma_K\geq 0$ are the Lagrangian multipliers accounting for the nonnegativity constraint $s_1,\dots,s_K\geq 0$. The minimum of $f(s_1,\dots,s_k)$ is attained for

$$\boldsymbol{a}^{\mathsf{H}}(\nu_k)\hat{\boldsymbol{R}}\,\boldsymbol{a}(\nu_k)=(1-\gamma_k)(s_k+\lambda)^2. \tag{23}$$

Consider the case that no signal is present, i.e., the sample covariance matrix only accounts for the noise according to $\hat{\boldsymbol{R}} = \boldsymbol{W}\boldsymbol{W}^{\mathsf{H}}/N$. Ideally, in this scenario the signal row-norms are estimated as $s_k = 0$, for $k = 1, \ldots, K$, which requires that (23) is fulfilled in the form

$$\|\boldsymbol{W}^{\mathsf{H}}\boldsymbol{a}(\nu_k)\|_2^2/N = \lambda^2(1-\gamma_k) \leq \lambda^2, \tag{24}$$

where the inequality stems from the nonnegativity of the Lagrange multipliers $\gamma_k \geq 0$, for $k = 1, \ldots, K$. From (24) it can be observed that λ must provide an upper bound on the spectral noise distribution according to,

$$\max_k \|\boldsymbol{W}^{\mathsf{H}}\boldsymbol{a}(\nu_k)\|_2/\sqrt{N} \leq \lambda. \tag{25}$$

In terms of statistical expectation we can compute an upper bound for the left-hand side of (25) as

$$\begin{aligned}\mathrm{E}\left\{\max_k \frac{\|\boldsymbol{W}^{\mathsf{H}}\boldsymbol{a}(\nu_k)\|_2}{\sqrt{N}}\right\} &\leq \mathrm{E}\left\{\max_k \frac{\|\boldsymbol{W}\|_2\|\boldsymbol{a}(\nu_k)\|_2}{\sqrt{N}}\right\} \\ &= \max_k \frac{\|\boldsymbol{a}(\nu_k)\|_2}{\sqrt{N}}\mathrm{E}\left\{\|\boldsymbol{W}\|_2\right\} \\ &\leq \sigma\frac{1}{\sqrt{N}}(\sqrt{M}+\sqrt{N}) \end{aligned}\tag{26}$$

where $\|\boldsymbol{a}(\nu_k)\|_2 = 1$, for $k = 1, \ldots, K$, and the expectation of the spectral norm $\|\boldsymbol{W}\|_2$ of the $M \times N$ complex Gaussian noise matrix $\boldsymbol{W}$ is upper bounded as $\mathrm{E}\{\|\boldsymbol{W}\|_2\} \leq \sigma(\sqrt{M}+\sqrt{N})$ for sufficiently large M, N [28, Theorem 5.32], [29]. Using (25) and (26) we compute the regularization parameter as

$$\lambda = \sigma\left(\sqrt{M/N}+1\right). \tag{27}$$

Even though the regularization parameter in (27) has been derived for the case of large number of sensors M and MMVs N, it provides satisfactory performance in the case of small numbers M and N.

6. SIMULATION RESULTS

For performance evaluation of our proposed SI-SPARROW method we compare to the conventional ESPRIT method [4] and the Cramer-Rao bound (CRB) for partly calibrated arrays [3]. Our simulation scenario includes a linear array consisting of $P = 4$ subarrays of $M_0 = 2$ sensors, respectively, at positions (0, 1), (7.3, 8.3), (18.7, 19.7) and (35.4, 36.4), where each pair $(r_{p,1}, r_{p,2})$ denotes the sensor positions of one calibrated subarray such that only the intra-subarray shift $\Delta_2^{(2)} = 1$ is known for frequency recovery. In our proposed SI-SPARROW method, frequency recovery is performed by application of the ESPRIT method on the reconstructed matrix $\hat{\boldsymbol{Q}}$, in contrast to the conventional approach [4] where the ESPRIT method is performed on the sample covariance matrix $\hat{\boldsymbol{R}}$. For all experiments we consider $L = 2$ complex Gaussian source signals which are correlated by a factor ρ, with spatial frequencies $\boldsymbol{\mu} = [0.4,\ 0.5]^{\mathsf{T}}$, and perform a number of 500 Monte Carlo runs to compute the root-mean-square error (RMSE) of the estimated spatial frequencies $\mathrm{RMSE}(\hat{\boldsymbol{\mu}})$.

In the first scenario we fix the correlation coefficient as $\rho = 0.99 \cdot \mathrm{e}^{j\pi/3}$ and the number of MMVs as $N = 10$, while varying the signal-to-noise ratio (SNR) defined as $\mathrm{SNR} = 1/\sigma^2$. Figure 3 depicts the RMSE as a function of the SNR. As can well be seen from Figure 3, the SI-SPARROW is superior to the conventional ESPRIT at low to medium SNR, since in this SNR regime the conventional ESPRIT can not always resolve the closely spaced signals.

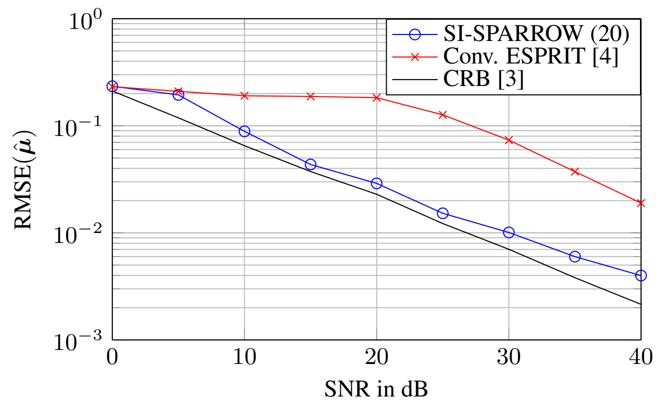

Fig. 3. RMSE for $L = 2$ source signals with correlation coefficient $|\rho| = 0.99$, spatial frequencies $\boldsymbol{\mu} = [0.4,\ 0.5]^{\mathsf{T}}$, and $N = 5$ MMVs

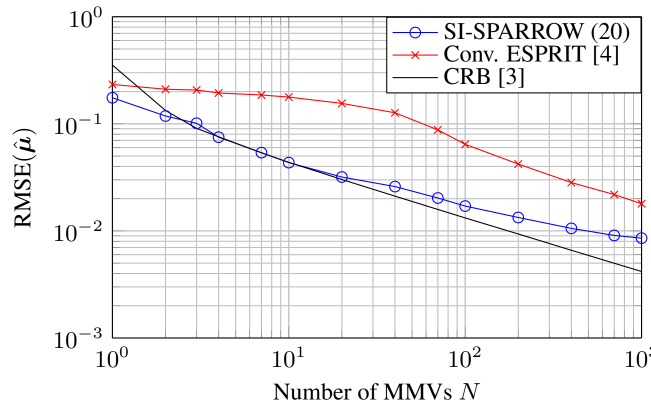

Fig. 4. RMSE for $L = 2$ source signals with correlation coefficient $|\rho| = 0.9$, spatial frequencies $\boldsymbol{\mu} = [0.4,\ 0.5]^{\mathsf{T}}$, and SNR = 10 dB

For the second scenario we reduce the correlation coefficient to $\rho = 0.9 \cdot \mathrm{e}^{j\pi/3}$ and fix the SNR as SNR = 10 dB, while varying the number of MMVs N. As before, in Figure 4 one can see that SI-SPARROW clearly outperforms the conventional ESPRIT for low number of MMVs N. We remark that for coherent signals, i.e., $|\rho| = 1$, the conventional ESPRIT fails, since the signal- and noise subspaces of the sample covariance matrix $\hat{\boldsymbol{R}}$ can not be properly separated, while SI-SPARROW still shows satisfactory results.

7. CONCLUSION

In this paper we have extended the SPARROW approach for joint sparse signal reconstruction, as introduced in [16], to shift-invariant sampling, with application to direction-of-arrival estimation in partly calibrated arrays. The proposed shift-invariant SPARROW (SI-SPARROW) admits gridless estimation of the underlying frequencies and thus avoids a computationally expensive spectrum-search. To the best of our knowledge SI-SPARROW is the first method for gridless compressed from non-uniform sampling structures with the shift-invariance property. Additionally, the proposed SI-SPARROW method only requires knowledge of the subarray responses which facilitates application in large sampling scenarios where, otherwise, perfect calibration would be required. We show by simulations that our proposed SI-SPARROW method outperforms the competing conventional ESPRIT algorithm in difficult scenarios, e.g. high noise power, low number of measurement vectors or correlated source signals.

8. REFERENCES

[1] H. Krim and M. Viberg, “Two decades of array signal processing research: the parametric approach,” *IEEE Signal Processing Magazine*, vol. 13, no. 4, pp. 67–94, Jul 1996.

[2] C. M. S. See and A. B. Gershman, “Direction-of-arrival estimation in partly calibrated subarray-based sensor arrays,” *IEEE Transactions on Signal Processing*, vol. 52, no. 2, pp. 329–338, 2004.

[3] M. Pesavento, A. Gershman, and K. M. Wong, “Direction finding in partly calibrated sensor arrays composed of multiple subarrays,” *IEEE Transactions on Signal Processing*, vol. 50, no. 9, pp. 2103–2115, 2002.

[4] R. Roy and T. Kailath, “Esprit-estimation of signal parameters via rotational invariance techniques,” *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 37, no. 7, pp. 984–995, 1989.

[5] A. L. Swindlehurst, B. Ottersten, R. Roy, and T. Kailath, “Multiple invariance esprit,” *IEEE Transactions on Signal Processing*, vol. 40, no. 4, pp. 867–881, Apr 1992.

[6] P. Parvazi, M. Pesavento, and A. Gershman, “Direction-of-arrival estimation and array calibration for partly-calibrated arrays,” in *2011 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2011, pp. 2552–2555.

[7] P. Parvazi and M. Pesavento, “A new direction-of-arrival estimation and calibration method for arrays composed of multiple identical subarrays,” in *2011 IEEE 12th International Workshop on Signal Processing Advances in Wireless Communications*, June 2011, pp. 171–175.

[8] W. Suleiman, M. Pesavento, and A. Zoubir, “Decentralized direction finding using partly calibrated arrays,” in *21st European Signal Processing Conference (EUSIPCO 2013)*, Sept 2013, pp. 1–5.

[9] W. Suleiman, M. Pesavento, and A. M. Zoubir, “Performance analysis of the decentralized eigendecomposition and esprit algorithm,” *IEEE Transactions on Signal Processing*, vol. 64, no. 9, pp. 2375–2386, May 2016.

[10] R. Tibshirani, “Regression shrinkage and selection via the lasso,” *Journal of the Royal Statistical Society. Series B (Methodological)*, vol. 58, pp. 267–288, 1996.

[11] S. S. Chen, D. L. Donoho, and M. A. Saunders, “Atomic decomposition by basis pursuit,” *SIAM Journal On Scientific Computing*, vol. 20, pp. 33–61, 1998.

[12] D. L. Donoho and M. Elad, “Optimally sparse representation in general (nonorthogonal) dictionaries via ℓ^1 minimization,” vol. 100, no. 5. National Acad Sciences, 2003, pp. 2197–2202.

[13] E. Candès, J. Romberg, and T. Tao, “Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information,” *IEEE Transactions on Information Theory*, vol. 52, no. 2, pp. 489–509, Feb 2006.

[14] M. M. Hyder and K. Mahata, “Direction-of-arrival estimation using a mixed $\ell_{2,0}$ norm approximation,” *IEEE Transactions on Signal Processing*, vol. 58, no. 9, pp. 4646–4655, Sept 2010.

[15] D. Malioutov, M. Çetin, and A. Willsky, “A sparse signal reconstruction perspective for source localization with sensor arrays,” *IEEE Transactions on Signal Processing*, vol. 53, no. 8, pp. 3010–3022, 2005.

[16] C. Steffens, M. Pesavento, and M. E. Pfetsch, “A Compact Formulation for the $\ell\_2, 1$ Mixed-Norm Minimization Problem,” *ArXiv e-prints, arXiv:1606.07231v1*, Jun. 2016.

[17] D. Donoho, “Superresolution via sparsity constraints,” *SIAM Journal on Mathematical Analysis*, vol. 23, p. 1309–1331, September 1992.

[18] E. J. Candès and C. Fernandez-Granda, “Towards a mathematical theory of super-resolution,” *Communications on Pure and Applied Mathematics*, vol. 67, no. 6, pp. 906–956, 2014.

[19] ——, “Super-resolution from noisy data,” *Journal of Fourier Analysis and Applications*, vol. 19, no. 6, pp. 1229–1254, 2013.

[20] B. N. Bhaskar and B. Recht, “Atomic norm denoising with applications to line spectral estimation,” in *Communication, Control, and Computing (Allerton), 2011 49th Annual Allerton Conference on*, Sept 2011, pp. 261–268.

[21] G. Tang, B. Bhaskar, P. Shah, and B. Recht, “Compressed sensing off the grid,” *IEEE Transactions on Information Theory*, vol. 59, no. 11, pp. 7465–7490, Nov 2013.

[22] C. Steffens, P. Parvazi, and M. Pesavento, “Direction finding and array calibration based on sparse reconstruction in partly calibrated arrays,” in *2014 IEEE 8th Sensor Array and Multichannel Signal Processing Workshop (SAM)*, June 2014, pp. 21–24.

[23] Y. L. Ming Yuan, “Model selection and estimation in regression with grouped variables,” *Journal of the Royal Statistical Society. Series B (Statistical Methodology)*, vol. 68, no. 1, pp. 49–67, 2006.

[24] M. Kowalski, “Sparse regression using mixed norms,” *Applied and Computational Harmonic Analysis*, vol. 27, no. 3, pp. 303 – 324, 2009.

[25] B. Recht, M. Fazel, and P. A. Parrilo, “Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization,” *SIAM Review*, vol. 52, no. 3, pp. 471–501, 2010.

[26] Y. Hua and T. K. Sarkar, “Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise,” *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 38, no. 5, pp. 814–824, May 1990.

[27] B. Himed and D. D. Weiner, “Application of the matrix pencil approach to direction finding,” DTIC Document, Tech. Rep., 1991.

[28] Y. C. Eldar and G. Kutyniok, Eds., *Compressed sensing : theory and applications*. Cambridge, New York: Cambridge University Press, 2012.

[29] D. P. Palomar and Y. C. Eldar, Eds., *Convex optimization in signal processing and communications*. Cambridge, UK, New York: Cambridge University Press, 2010.