A NEURAL FILTER-BASED SCHEME FOR SYNCHRONIZING CHAOTIC SYSTEMS

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ABSTRACT

Synchronization of chaotic systems and/or maps is a key step to implement secure communication schemes with chaos. If the process to synchronize chaotic systems is modeled stochastic, schemes based on extended Kalman filter (EKF) and unscented Kalman filter (UKF) have been studied in the past. However, such nonlinear filters are employed with assumptions of Gaussian noise processes and the Markov property. Further, EKF and UKF are suboptimal filtering methods, incurring unacceptable errors for high nonlinear systems. In this paper, neural filter (NF) is proposed for chaotic synchronization. This new approach requires no mentioned assumptions and achieves optimal filter. Numerical comparisons between the proposed approach and existing schemes are presented in this paper, showing the superiority of the proposed approach.

***Index Terms*—** Chaos, synchronization, nonlinear Kalman filter, neural filter, non-Gaussian noise

1. INTRODUCTION

Synchronization of chaotic systems is essential chaotic communication. The most important characteristic of a chaotic system is sensitivity to the initial condition, which makes the system ideal for secure communication. At the same time, the synchronization of the transmitter and receiver systems for different initial conditions is pivotal in reconstructing the signal in the chaotic communication process. In [1], Pecora and Carroll reported synchronization of chaotic system using a drive-response framework. They showed that if all the transversal Lyapunov exponents of the response system are negative, then the systems can be synchronized asymptotically. Following this work, many synchronization schemes have been developed [2–8]. A detailed review of the present state of synchronization of chaotic systems/maps is available in [9].

Among the various methods, coupled methods is a noteworthy approach. It utilizes a form of feedback control with proper feedback coefficients on a chaotic system to make its states synchronize with the transmitter. The coupling strength depends on the global transversal Lyapunov exponents of the system in noiseless situations and on the local transversal Lyapunov exponents in noisy situations [5]. Since the similarity between the coupled synchronization and the state estimation of nonlinear system, stochastic control methods are introduced from control theory for chaotic synchronization. Cuomoet al. [3] designed a synchronization scheme based on the extended Kalman filter (EKF). Cruz and Nijmeijer studied the performance of the EKF-based synchronization scheme for different chaotic maps [10]. The theoretical analysis of the EKF based scheme is reported by Leung and Zhu [11]. Ajeesh P. Kurian presented a scheme based on nonlinear predictive filter [5] and reported the theoretical analysis of this method [12]. Other nonlinear filtering methods are also applied for chaotic synchronization [7, 13, 14]. However, there are limitations of the methods derived from extensions of the Kalman filter. These methods rely on Gaussian approximation which do not hold in most applications. Moreover, nonlinear Kalman filters are suboptimal. Large errors may be introduced when such methods are applied to systems with higher-order nonlinearities. Such large errors in the state estimates cause the trajectories of the transmitter and receiver systems to diverge and result in eventual desynchronization [11].

In this paper, a scheme for chaotic synchronization that is based on a neural filter (NF) [15–17] is developed to overcome these limitations. The neural filter has the architecture of the recurrent neural network with interconnected hidden units (RNNWIHU) [16]. It is reported that an RNNWIHU exists that inputs the measurement process $y(t)$ and outputs an estimate of the signal process $x(t)$ such that the estimate approaches optimal estimate as the number of hidden neurons tends to infinity. Compared to nonlinear Kalman filters as applied to chaotic synchronization, NF has advantages: First, NF-based synchronization is data-driven, which has no such assumptions as Markov property, Gaussian distribution, and validity of linear approximation. Second, the NF-based synchronization approach applies even if a mathematical model of the chaotic system or transmitter process is unavailable. Third, the estimate of state signals by an NF-

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based scheme is virtually optimal in the sense that it approximates the minimum-variance estimate to any degree as the number of hidden neurons of the recurrent neural network is large enough.

The performance of the proposed method for chaotic synchronization is compared with those of the EKF and UKF based schemes. In our numerical tests, the Ikeda map with white Gaussian noise and colored noise is conducted for the three methods. The normalized mean square error (NMSE), total NMSE (TNMSE), normalized instantaneous squared error (NISE) and online operating time taken for synchronization are included in the performance comparison.

2. CHAOTIC SYNCHRONIZATION BASED ON NEURAL FILTER

2.1. Recursive Bayesian Filter based Synchronization

Bayes filter, is a general probabilistic approach for estimating an unknown probability density function recursively over time using receiving observation and a mathematical process model. For synchronization of chaotic system, nonlinear Bayes filtering algorithms can be applied as a coupled synchronization method [14], which can determine the coupling coefficient matrix adaptively instead of being constant in conventional coupled synchronization. For a nonlinear system, a closed form solution for Bayes filter is not available. However, suboptimal filtering methods, the EKF and UKF can be performed for chaotic synchronization.

2.1.1. Extended Kalman Filter

The Kalman filter is the optimal estimate algorithm for linear system models with additive independent white noise which is described in terms of state space model [18]. extended Kalman filter (EKF) is the nonlinear version of the Kalman filter which linearizes about an estimate of the current mean and covariance by first order Taylor series approximation [19–21].

2.1.2. Unscented Kalman Filter

The UKF algorithm was proposed by Julier and Uhlmann [22, 23]. It utilizes the unscented transform (UT) to give a Gaussian approximation to the filtering solutions of non-linear optimal filtering problem. UT is a method for calculating the statistics of a random variable which undergoes a nonlinear transformation. In UT the state distribution is represented using a minimal set of carefully chosen sample points, called sigma points. These sigma points are propagated through the nonlinear function and the resulting points are used to estimate the mean and covariance. It is shown that the UKF based approximation is equivalent to a third order Taylor series approximation. It is shown in [24] that the approximation introduced by the UKF has more number of Taylor series terms which promises an improved performance compared to the EKF. Furthermore, since no explicit Jacobian or Hession calculation is needed, computational complexity of UKF is comparable to EKF.

As shown in [7], UKF can also be applied for chaotic synchronization. UKF has better performance than EKF, since it has a smaller system approximation error.

2.2. Neural Filter based Synchronization

Neural filter first introduced by Lo in [15, 16]. The theorem states that an recurrent neural network with interconnected hidden units (RNNWIHU) exists that inputs the measurement process $y(t)$ and outputs an estimate of the signal process $x(t)$, where the estimate can be made as close as desired to the conditional expectation of the signal process given the past history of the measurement process that has been processed, said conditional expectation being defined on the empirical joint probability distribution represented by the training dataset. Then the theorem is extended in [17]. The neural filter has several advantages over nonlinear Kalman Filter: 1). It performs with no such assumption as Markov property, Gaussian distribution; 2) It is data-driven method even if a mathematical model of the signal is unavailable; 3)NF is trained in a offline manner and employed online without weights adjustment, which provides computational efficiency for practical applications; 4)It is proved to converge to the minimum-variance filter as the number of hidden units. High synchronization accuracy can be achieved based on the property of NF.

The fundamental theorem of NF with RNNWIHU is expressed as following:

Theorem 1 *Consider an n-dimensional stochastic process $x(t)$ and an m-dimensional stochastic process $y(t), t = 1, ..., T$ defined on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$. Assume that the range $\{y(t,\omega)|t = 1, ..., T, \omega \in \Omega\} \subset \mathcal{R}^m$ is compact. Let w and $\alpha(t)$ denote respectively the weights (including biases) and the output vector at time t of an RNNWIHU which has taken the inputs, $y(1), ..., y(t)$, in the given order.*

1. Given $\varepsilon > 0$, there exists an RNNWIHU with one hidden layer of fully interconnected neurons such that

$$\frac{1}{T}\sum_{t=1}^{T} E[\|\alpha(t) - E[x(t)|y^t]\|^2] < \varepsilon \qquad (1)$$

2. If the recurrent neural network has one hidden layer of N neurons, which are fully interconnected, and the output $\alpha(t)$ is written as $\alpha(t; w, N)$ here to indicate its dependency on N and the weights w of the RNNWIHU, then

$$r(N) := \min_{w} \frac{1}{T} \sum_{t=1}^{T} E[\|\alpha(t;w,N) - E[x(t)|y^t]\|^2] \tag{2}$$

is monotone decreasing and converges to 0 as N tends to infinity.

For proof details of this theorem, we refer to the related articles [16, 17].

3. EXPERIMENTS AND DISCUSSIONS

In our simulations, NMSEs and TNMSEs are used for evaluating relative performances of the three filters (EKF, UKF and NF). The NMSE^i between transmitter state x_k^i and receiver state $\hat{x}_k^i$ is defined as,

$$\text{NMSE}_i = \frac{\sum_{k=1}^{N} (x_k^i - \hat{x}_k^i)^2}{\sum_{k=1}^{N} (x_k^i)^2} \tag{3}$$

where N is the number of iterations and the superscript i represents the ith state variable. The total NMSE (TNMSE) is defined as the sum of all the NMSEs corresponding to individual states TNMSE $= \sum_{i=1}^{n}$ NMSE^i, where n is the number of states. To avoid the effect of initial transients, the initial few hundred samples are discarded . For each system, the experiments is evaluated on 100 Monte Carlo runs and the average of the NMSEs and the TNMSEs are computed. For the comparison of the time taken by the three different schemes for synchronization, the normalized instantaneous squared error (NISE) is computed, which is defined as,

$$\text{NISE} = \frac{1}{n} \sum_{i=1}^{n} (x_k^i - \hat{x}_k^i)^2 \tag{4}$$

For evaluation of computational efficiency, total operating time of three schemes on 100 Monte Carlo runs each with 2,000 iterations, are presented.

For NF based scheme, recurrent neural networks with 1 hidden layer containing 50 units are trained as neural filters. After training, the weights of NF are frozen without any adjustment when online operated.

Ikeda proposed a model of light going around across a nonlinear optical resonator [25]. The model is nonlinear, two-dimensional and deterministic. It was proven that for a certain set of parameters the system exhibits chaotic behavior. The following set of equations describes how the dynamical state of the Ikeda model evolves over time in a complex non-repeating pattern:

$$\begin{aligned} m_k &= c_1 - \frac{c_3}{1 + x_{1,k}^2 + x_{2,k}^2} \\ x_{1,k+1} &= c_4 + c_2(x_{1,k}\cos(m_k) - x_{2,k}\sin(m_k)) \\ x_{2,k+1} &= c_2(x_{1,k}\sin(m_k) + x_{2,k}\cos(m_k)) \end{aligned} \tag{5}$$

Table 1. Performance for Ikeda map with Gaussian noise.

	EKF	UKF	NF
NMSE$_1$	$4.77e^{-1}$	$2.60e^{-2}$	$1.43e^{-2}$
NMSE$_2$	$3.37e^{+1}$	$2.29e^{-1}$	$5.76e^{-2}$
TNMSE	$3.42e^{+1}$	$2.55e^{-1}$	$7.19e^{-2}$

where c_1, c_2, c_3 and c_4 are real-valued parameters. We set $c_1 = 0.4$, $c_2 = 0.84$ $c_3 = 6.0$, $c_4 = 1.0$. The states are initialized with $\mathbf{x}_0 = [1, 0]^T$ and generated iteratively. From this equation, it can be easily verified that the map has non-negligible higher order terms in the Taylor series approximation due to the term of sine and cosine components.

For Ikeda map, the state x_1 is transmitted for synchronization adding channel noise ω. Therefore, the received signal y denoted as:

$$y_k = x_{1,k} + \omega_k \tag{6}$$

Two different types of noise, Gaussian noise and colored noise, are added to y_k. The SNR of received signal y_k in our experiment is restricted to 10db.

Transmitted Signal with Gaussian Noise

In the first case, we assume stochastic sequences ω_k is following Gaussian distribution with zero mean. NISEs are computed for comparing the convergence property of each method. In Fig.1, it is shown the NISE curves of each synchronization scheme for Ikeda map with 100 Monte Carlo runs. It can be found that UKF and NF based schemes achieve faster synchronization than EKF. In the case of the EKF, it has no obvious convergence behavior. The EKF scheme suffers from desynchronization with larger NISE. A possible explanation for this phenomenon is when the SNR is low EKF method has a relatively large deviation from averaged CRLB for Type-II systems (chaotic systems with state-dependent gradient square) [11]. For NF-based scheme, it converges to a smaller value of NISE compared to UKF based scheme. It also can be observed that the NF scheme has a very small NISE at the beginning of synchronization process and then keeps synchronized. The actual NMSE and TNMSE values are presented in Table 1. It can be seen clearly that EKF can not provide good performance for synchronization especially for the second component of state signal. NF-based method performs at lower errors compared to EKF-based and UKF-based schemes.

Transmitted Signal with Colored Noise

In the second case, we set stochastic sequences ω_k to be colored noise, which is dependent to its past states. It can be stated as:

$$\omega_{k+1} = \alpha\omega_k + e_k \tag{7}$$

where α is a real constant and e_k is a Gaussian stochastic variable.

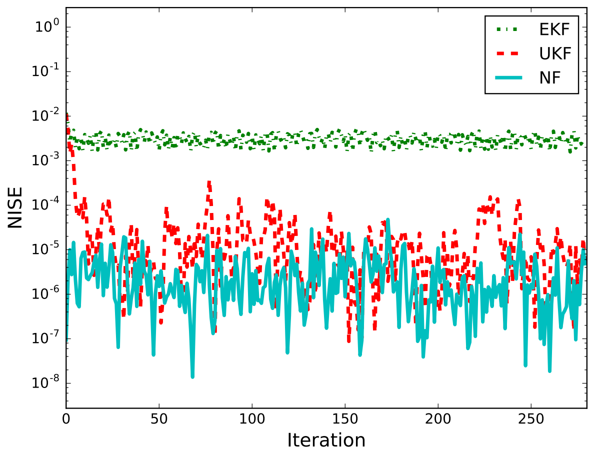

Fig. 1. NISE of Ikeda map with Gaussian noise of 100 Monte Carlo runs.

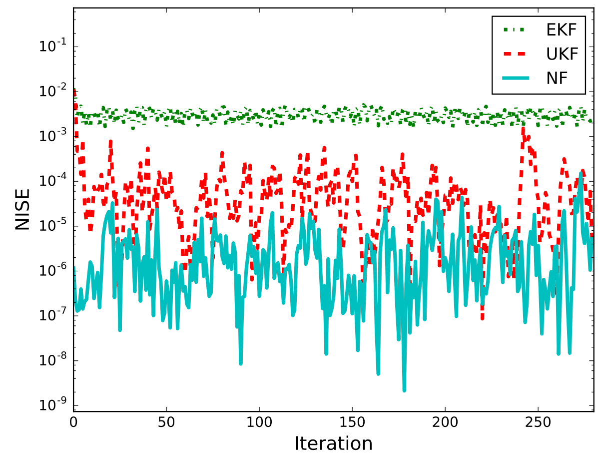

Fig. 2. NISE of Ikeda map with colored noise of 100 Monte Carlo runs.

Table 2. Performance for Ikeda map with colored noise.

	EKF	UKF	NF
NMSE_1	$5.53e^{-1}$	$1.30e^{-1}$	$1.47e^{-2}$
NMSE_2	$3.36e^{+1}$	$1.43e^{0}$	$5.21e^{-2}$
TNMSE	$3.41e^{+1}$	$1.56e^{0}$	$6.69e^{-2}$

Table 3. Online operating time of EKF, UKF and NF based schemes for Ikeda maps on 100 Monte Carlo runs.

Noise type	EKF	UKF	NF
Gaussian	18.143s	25.787s	1.498s
Colored	17.679s	25.910s	1.295s

Fig. 2 presents the NISE of NF-based scheme compared to EKF and UKF based schemes. It can be observed that trajectories of EKF-based scheme diverges and result in desynchronization as previous case. Both UKF and NF-based methods can achieve synchronization of the transmitter and the receiver trajectories. NF-based scheme provides a lower synchronization errors, whose NISE values settles around 10^{-5} irrespective of the iterations. In Table 2, NMSEs and T-NMSE for this case are provided. It is shown that EKF-based method presents the same dynamic behavior as preceding case. For UKF, it still keep synchronization but the performance is poorer than that in the Gaussian noise condition. In contrast, NF-based method synchronizes the state signal with higher accuracy, which doesn't suffer from non-Gaussian process.

Computational Performance for Online Operating

Although estimate accuracy is an important criterion for chaotic synchronization, online computational complexity is also be taken into account for performance evaluation in practical applications. An empirical analysis comparing the computational performances of EKF, UKF and NF with the same architecture as preceding experiments is presented, which is running on a laptop with Intel Core i5-2520M CPU and 4 GB of RAM. In Table 3, total operating time of 100 Monte Carlo runs for Ikeda maps and are shown. From this tables, it can be observed that EKF based scheme runs a little faster for Ikeda maps compared to UKF scheme. Meanwhile, NF based scheme shows the best computational performance, which is at least one order of magnitude lower than that of EKF and UKF. Benefit from the offline training, computational complexity of NF based scheme only relies on the scale of the hidden layer of a recurrent neural network. Hence, it can be highly efficient for practical applications.

4. CONCLUTION

In this work, a scheme for chaotic synchronization based on neural filtering is proposed. We report numerical results from comparing the scheme with those based on the EKF and UKF. The transmitter and receiver systems are the Ikeda map with white Gaussian noise in one experimental study and colored Gaussian noise in another. The normalized mean squared error (NMSE), total normalized mean square error (TNMSE), normalized instantaneous square error (NISE) and online running time are included in the performance comparison. Chaotic synchronization by EKF actually fails to converge for the Ikeda map example. NF-based scheme for chaotic synchronization outperforms UKF-based scheme in NMSE, TNMSE, and NISE by a significant margin. The numerical results further confirm that NF-based scheme is computationally more efficient.

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