# STRUCTURE OF THE SET OF SIGNALS WITH STRONG DIVERGENCE OF THE SHANNON SAMPLING SERIES

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## ABSTRACT

It is known that there exist signals in Paley–Wiener space  $\mathcal{PW}_{\pi}^{1}$  of bandlimited signals with absolutely integrable Fourier transform, for which the peak value of the Shannon sampling series diverges unboundedly. In this paper we analyze the structure of the set of signals which lead to strong divergence. Strong divergence is closely linked to the existence of adaptive methods. We prove that there exists an infinite dimensional closed subspace of  $\mathcal{PW}_{\pi}^{1}$ , all signals of which, except the zero signal, lead to strong divergence of the peak value of the Shannon sampling series.

Index Terms—Shannon sampling series, reconstruction process, strong divergence, spaceability, Paley–Wiener space

## 1. INTRODUCTION

Sampling theory plays a fundamental role in modern signal and information processing, because it is the basis for today's digital world. In his seminal work [1] Shannon started this theory. The fundamental initial result states that the Shannon sampling series

$$\sum_{k=-\infty}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \tag{1}$$

can be used to reconstruct bandlimited signals f with finite  $L^2$ -norm from their samples  $\{f(k)\}_{k\in\mathbb{Z}}$ .

Since Shannon's publication, the reconstruction of bandlimited signals from their samples has been widely used in numerous applications and theoretical concepts, and many different sampling theorems for various signal spaces have been developed [2–11]. In this paper we consider the Paley–Wiener space  $\mathcal{PW}_{\pi}^{1}$  of bandlimited signals with absolutely integrable Fourier transform. A precise definition of this space will be given Section 2.

For  $N \in \mathbb{N}$ , let

$$(S_N f)(t) = \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}, \quad t \in \mathbb{R},$$
(2)

denote the finite Shannon sampling series. A well-known result in sampling theory is Brown's theorem, which states that  $S_N f$  converges locally uniformly to f for all signals  $f \in \mathcal{PW}^1_{\pi}$  as N tends to infinity [7, 12, 13].

In many applications it is important to also control the peak value of  $S_N f$ . The appropriate measure in this case is the  $L^{\infty}$ -norm. A very strong requirement is

$$\lim_{N \to \infty} \|f - S_N f\|_{\infty} = 0, \tag{3}$$

i.e., global uniform convergence. A much weaker requirement is

$$\lim_{N \to \infty} \max_{t \in [-\tau,\tau]} |f(t) - (S_N f)(t)| = 0 \text{ for all } \tau > 0$$
together with
$$\sup_{N \in \mathbb{N}} ||S_N f||_{\infty} < \infty,$$
(4)

i.e., local uniform convergence in combination with global uniform boundedness. The control of the peak value  $||S_N f||_{\infty}$  is equivalent to the control of the peak approximation error  $||f - S_N f||_{\infty}$ .

For  $\mathcal{PW}_{\pi}^1$ , such a control of the peak value  $||S_N f||_{\infty}$  is not possible [14], because there exist signals  $f \in \mathcal{PW}_{\pi}^1$  such that

$$\limsup_{N \to \infty} \|f - S_N f\|_{\infty} = \infty.$$
(5)

The divergence in (5) is in terms of the lim sup. In a sense this is a weak notion of divergence, because it merely states the existence of a subsequence  $\{N_n\}_{n\in\mathbb{N}}$  of the natural numbers such that  $\lim_{n\to\infty} ||f - S_{N_n}f||_{\infty} = \infty$ . This leaves the possibility that there is a different subsequence  $\{N_n^*\}_{n\in\mathbb{N}}$  such that  $\limsup_{n\to\infty} ||f - S_{N_n^*}f||_{\infty} < \infty$  or even  $\lim_{n\to\infty} ||f - S_{N_n^*}f||_{\infty} = 0$ . Note that the subsequence  $\{N_n^*(f)\}_{n\in\mathbb{N}}$  can depend on the signal f. Thus, the reconstruction process  $S_{N_n^*(f)}$  would be adapted to the signal f and non-linear.

For weakly divergent reconstruction processes the following question is central: Does there exist a sequence  $\{N_n^*\}_{n\in\mathbb{N}}\subset\mathbb{N}$ , which can depend on the signal, such that we can control the peak value, i.e., have  $\sup_{n\in\mathbb{N}} ||S_{N_n^*}f||_{\infty} < \infty$ ? The answer is negative if and only if we have

$$\lim_{N \to \infty} \|S_N f\|_{\infty} = \infty.$$
(6)

This brings us to the notion of strong divergence. We say that a sequence  $\{a_n\}_{n\in\mathbb{N}}\subset\mathbb{R}$  diverges strongly if  $\lim_{n\to\infty}|a_n|=\infty$ . Clearly, this is a stronger statement than  $\limsup_{n\to\infty}|a_n|=\infty$ , because in case of strong divergence we have  $\lim_{n\to\infty}|a_{N_n}|=\infty$  for all subsequences  $\{N_n\}_{n\in\mathbb{N}}$  of the natural numbers.

In [15] it has been proved that there exists a signal  $f \in \mathcal{PW}_{\pi}^{1}$  such that the peak value of the Shannon sampling series diverges strongly, i.e., such that (6) is true. This result shows that adaptivity in the sequence  $\{N_{n}\}_{n\in\mathbb{N}}$  cannot be used to control the peak value of the Shannon sampling series.

While it is known that there exist signals  $f \in \mathcal{PW}_{\pi}^{1}$  such that (6) is true, little is known about the structure of the set of signals  $f \in \mathcal{PW}_{\pi}^{1}$  that satisfy (6). We will prove that there exists an infinite dimensional closed subspace of  $\mathcal{PW}_{\pi}^{1}$ , all signals of which, except the zero signal, satisfy (6), i.e., lead to strong divergence of the peak value of the Shannon sampling series.

#### 2. STRUCTURE OF SIGNAL SPACES

In applications, the employed signal sets need to satisfy certain requirements. A very basic requirement is a linear structure, which ensures that the signal set is closed under addition. Often, it is necessary to introduce, in addition to the linear structure, a distance measure or norm, to be able to compare two signals. Of particular importance are Banach spaces, i.e., linear spaces that are complete with respect to their norm.

Let  $\hat{f}$  denote the Fourier transform of a function f, where  $\hat{f}$  is to be understood in the distributional sense.

 $L^{p}(\mathbb{R}), 1 \leq p < \infty$ , is the space of all measurable, *p*th-power Lebesgue integrable functions on  $\mathbb{R}$ , with the usual norm  $\|\cdot\|_{p}$ , and  $L^{\infty}(\mathbb{R})$  the space of all functions for which the essential supremum norm  $\|\cdot\|_{\infty}$  is finite.  $L^{p}[-\sigma,\sigma], 1 \leq p < \infty, \sigma > 0$ , is the space of all measurable, *p*th-power Lebesgue integrable functions on the interval  $[-\sigma,\sigma]$ .

For  $\sigma > 0$  and  $1 \leq p \leq \infty$ , we denote by  $\mathcal{PW}_{\sigma}^{p}$  the Paley–Wiener space of signals f with a representation  $f(z) = 1/(2\pi) \int_{-\sigma}^{\sigma} g(\omega) e^{iz\omega} d\omega$ ,  $z \in \mathbb{C}$ , for some  $g \in L^{p}[-\sigma,\sigma]$ . The norm for  $\mathcal{PW}_{\sigma}^{p}$ ,  $1 \leq p < \infty$ , is given by  $\|f\|_{\mathcal{PW}_{\sigma}^{p}} = (1/(2\pi) \int_{-\sigma}^{\sigma} |\hat{f}(\omega)|^{p} d\omega)^{1/p}$ . Note that  $\mathcal{PW}_{\pi}^{2} \subset \mathcal{PW}_{\pi}^{1}$ .

The Shannon sampling series (1) is non-adaptive and its analysis leads to a sequence of linear operators  $\{S_N\}_{N\in\mathbb{N}}$ . The Banach– Steinhaus theory provides a unified tool for the analysis of sequences of linear operators. The theory answers when we have (3) as well as (4). We have (3) for all signals  $f \in \mathcal{PW}_{\pi}^{1}$  if and only if we have (4) for all  $f \in \mathcal{PW}_{\pi}^{1}$ . However, the Banach–Steinhaus theory does not provide a framework for the analysis of adaptive reconstruction processes, and a unified theory is missing.

For the Shannon sampling series, it was shown in [15] that (4) does not hold in general, by constructing a signal  $f \in \mathcal{PW}_{\pi}^{1}$  with

$$\lim_{N \to \infty} \|S_N f\|_{\infty} = \infty.$$
(7)

Here, we are interested in the structure of the set of signals  $f \in \mathcal{P}W^{1}_{\pi}$  satisfying (7). In particular, we want to know whether it this set contains a subset which exhibits a linear structure.

Note that it is significantly more difficult to show a linear structure in the set of signals with divergent Shannon sampling series, compared to showing a linear structure in the set of signals with convergent Shannon sampling series. If we have two signals  $f_1$  and  $f_2$ , for which  $S_N f_1$  and  $S_N f_2$  converge, it is clear that the sum of both signals is a signal for which we have convergence as well. However, for divergence this is not necessarily true, as can be seen by choosing  $w_1 = f_1 + g$  and  $w_2 = f_1 - g$ , where  $f_1$  is any signal with convergent Shannon sampling series and g any signal with divergent Shannon sampling series. Obviously, for the sum  $w_1 + w_2 = 2f_1$ we do not have divergence.

Spaceability, which has recently been used for example in [16–20], is a concept to study the existence of linear structures in general sets. A subset S of a Banach space X is said to be spaceable if  $S \cup \{0\}$  contains a closed infinite dimensional subspace of X. In [16] it was shown that the set of continuous nowhere differentiable functions on C[0, 1] is spaceable. The divergence of Fourier series was analyzed in [19], where it was shown that the set of functions in  $L^1(\partial \mathbb{D})$ , whose Fourier series diverges everywhere on  $\partial \mathbb{D}$  is spaceable.

#### 3. SPACEABILITY AND STRONG DIVERGENCE FOR THE SHANNON SAMPLING SERIES

The next theorem, the proof of which will be given in Section 4, shows that the set of signals with strong divergence of the peak value of the Shannon sampling series is spaceable.

The spaceability of this set is surprising because in a topological sense the set cannot be large. This is a crucial difference to weak divergence which occurs for almost all signals. In this sense adaptivity is useful. For further discussions please see [21], which will be published as part of the STSIP special issue in honor of Claude Shannon's centennial.

**Theorem 1.** The set of signals  $f \in \mathcal{PW}^1_{\pi}$  satisfying

$$\lim_{N \to \infty} \max_{t \in \mathbb{R}} \left| \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty$$
(8)

is spaceable. That is, there exists an infinite dimensional closed subspace  $\mathcal{D}_{Shannon} \subset \mathcal{PW}_{\pi}^{1}$  such that (8) holds for all  $f \in \mathcal{D}_{Shannon}$ ,  $f \neq 0$ .

The subspace  $\mathcal{D}_{\text{Shannon}}$ , which is constructed in the proof of Theorem 1, has interesting properties.  $\mathcal{D}_{\text{Shannon}}$  has an unconditional basis, i.e., there exists a sequence of functions  $\{\zeta_n\}_{n \in \mathbb{N}} \subset \mathcal{D}_{\text{Shannon}}$ such that for all  $f \in \mathcal{D}_{\text{Shannon}}$  there exists a unique sequence of coefficients  $\{a_n(f)\}_{n \in \mathbb{N}}$  such that

$$\lim_{N \to \infty} \left\| f - \sum_{n=1}^{N} a_n(f) \zeta_n \right\|_{\mathcal{PW}^{\frac{1}{n}}} = 0$$

The coefficient functionals  $f \mapsto a_n(f)$ ,  $n \in \mathbb{N}$ , are linear and continuous functionals on  $\mathcal{D}_{\text{Shannon}}$ . A further special property of the space  $\mathcal{D}_{\text{Shannon}}$  is expressed by the following theorem, which is a consequence of Paley's theorem [22, p. 104] and the open mapping theorem [23, p. 100].

**Theorem 2.** There exist two constants  $C_1, C_2 > 0$  such that

$$C_1 \left( \sum_{n=1}^{\infty} |a_n(f)|^2 \right)^{\frac{1}{2}} \le \|f\|_{\mathcal{PW}^1_{\pi}} \le C_2 \left( \sum_{n=1}^{\infty} |a_n(f)|^2 \right)^{\frac{1}{2}}$$
(9)

for all  $f \in \mathcal{D}_{Shannon}$ .

Theorem 2 shows that  $\mathcal{D}_{\text{Shannon}}$  is isomorphic to the Hilbert space  $l^2$ , or equivalently,  $\mathcal{PW}_{\pi}^2$ . Moreover, if we equip the space  $\mathcal{D}_{\text{Shannon}}$  with the norm  $||f||_{\mathcal{D}_{\text{Shannon}}} = \left(\sum_{n=1}^{\infty} |a_n(f)|^2\right)^{1/2}$  then it becomes a Hilbert space, and  $\{\zeta_n\}_{n\in\mathbb{N}}$  is a Riesz basis for the Hilbert space  $(\mathcal{D}_{\text{Shannon}}, ||\cdot||_{\mathcal{D}_{\text{Shannon}}})$ .

# 4. PROOF OF THEOREM 1

Proof. The problem can be reduced to showing that the set of signals

$$\mathcal{R} = \left\{ f \in \mathcal{PW}_{\pi}^{1} \colon \lim_{N \to \infty} \left| \sum_{k=-N}^{N} \frac{f(k)}{N + \frac{1}{2} - k} \right| = \infty \right\}$$

is spaceable. Due to space constraints we omit this problem reduction step. Next, we show that  $\mathcal{R}$  is spaceable. For  $N \in \mathbb{N}$  we define the functions

$$w_N(t) = \sum_{k=-\infty}^{\infty} w_N(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}, \quad t \in \mathbb{R},$$

where  $w_N(k)$  is given by

$$w_N(k) = \begin{cases} 1, & |k| \le N, \\ 1 - \frac{|k| - N}{N}, & N < |k| < 2N, \\ 0, & |k| \ge 2N. \end{cases}$$

Note that we have  $w_N \in \mathcal{PW}^1_{\pi}$  and  $||w_N||_{\mathcal{PW}^1_{\pi}} < 3$  for all  $N \in \mathbb{N}$ [24]. Further, for  $l \in \mathbb{N}$  let  $N_l = 2^{2^{2(l+1)}}$ . Based on  $w_N$ , we define

for  $n \in \mathbb{N}$  the functions

$$\phi_n(t) = \sum_{l=1}^{\infty} \frac{1}{l^{n-1} 2^{l+1}} w_{N_l}(t), \quad t \in \mathbb{R}.$$

Note that  $\phi_n(k) > 0$  for all  $k, n \in \mathbb{N}$ . Moreover, for arbitrary  $\hat{l} \in \mathbb{N}$ and  $k > N_{\hat{l}}$  we have  $w_{N_r}(k) = 0$  for all  $r < \hat{l}$ , and it follows that

$$\phi_{n+m}(k) = \sum_{l=\hat{l}}^{\infty} \frac{1}{l^{n+m-1}2^{l+1}} w_{N_l}(k)$$
  
$$\leq \frac{1}{\hat{l}^m} \sum_{l=\hat{l}}^{\infty} \frac{1}{l^{n-1}2^{l+1}} w_{N_l}(k) = \frac{1}{\hat{l}^m} \phi_n(k) \qquad (10)$$

for all  $m, n \in \mathbb{N}$ .

Next, we show that the set  $\{\phi_n\}_{n\in\mathbb{N}}$  is finitely linearly independent, i.e., that every finite subset is linearly independent. Assume that  $\{\phi_n\}_{n\in\mathbb{N}}$  is not finitely linearly independent. Then there exist a finite set  $\{\phi_{n_r}\}_{r=1}^R$  and numbers  $\{c_r\}_{r=1}^R$ , all different from zero, such that

$$c_1\phi_{n_1}(t) = \sum_{r=2}^R c_r\phi_{n_r}(t), \quad t \in \mathbb{R}$$

Without loss of generally we can assume that  $n_1 < n_2 < \cdots < n_R$ . Let  $\hat{l} \in \mathbb{N}$  be arbitrary and  $k \in (N_{\hat{l}}, N_{\hat{l}+1}]$ . Then we have

$$|c_1|\phi_{n_1}(k) \le \sum_{r=2}^R |c_r|\phi_{n_r}(k) \le \frac{1}{\hat{l}} \sum_{r=2}^R |c_r|\phi_{n_1}(k)$$

according to (10), which in turn implies that  $|c_1| \leq \frac{1}{\hat{l}} \sum_{r=2}^{R} |c_r|$ because  $\phi_{n_1}(k) > 0$ . Since  $\hat{l}$  was arbitrary, it follows that  $c_1 = 0$ , which is a contradiction. Hence, it follows that  $\{\phi_n\}_{n\in\mathbb{N}}$  is finitely linearly independent.

Let  $n \in \mathbb{N}$  be arbitrary but fixed. For each  $N \in \mathbb{N}$  there exists exactly one  $\hat{l}$  such that  $N \in (N_{\hat{l}}, N_{\hat{l}+1}]$ , and we have

$$\begin{split} &\sum_{k=0}^{N} \frac{\phi_n(k)}{N + \frac{1}{2} - k} \geq \frac{1}{(\hat{l}+1)^{n-1}2^{\hat{l}+2}} \sum_{k=0}^{N} \frac{w_{N_{\hat{l}+1}}(k)}{N + \frac{1}{2} - k} \\ &= \frac{1}{(\hat{l}+1)^{n-1}2^{\hat{l}+2}} \sum_{k=0}^{N} \frac{1}{N + \frac{1}{2} - k} > \frac{\log(2N+3)}{(\hat{l}+1)^{n-1}2^{\hat{l}+2}} \\ &> \frac{\log(N_{\hat{l}})}{(\hat{l}+1)^{n-1}2^{\hat{l}+2}} = \frac{2^{2(\hat{l}+1)}}{(\hat{l}+1)^{n-1}2^{\hat{l}+2}} \log(2) \\ &= \frac{2^{\hat{l}}}{(\hat{l}+1)^{n-1}} \log(2), \end{split}$$

which shows that

$$\lim_{N \to \infty} \sum_{k=M}^{N} \frac{\phi_n(k)}{N + \frac{1}{2} - k} = \infty$$
 (11)

for all  $M \in \mathbb{N}$ .

For  $n \in \mathbb{N}$ , let

$$q_n(t) = \frac{\sin(\pi(t-2^n))}{\pi(t-2^n)}, \quad t \in \mathbb{R}.$$

According to Paley's theorem [22, p. 104],  $\{q_n\}_{n\in\mathbb{N}}$  is a basic sequence in  $\mathcal{PW}_{\pi}^{1}$ . Let  $\{q_n^*\}_{n\in\mathbb{N}}$  denote the unique sequence of coefficient functionals. It is easy to see that  $||q_n^*|| = 1$ ,  $n \in \mathbb{N}$ . Further, for  $n \in \mathbb{N}$ , we define

$$\xi_n = q_n + \frac{1}{2^{n+2}3}\phi_n.$$

It follows that

$$\sum_{n=1}^{\infty} \|q_n^*\| \|\xi_n - q_n\|_{\mathcal{PW}_{\pi}^1} = \sum_{n=1}^{\infty} \frac{1}{2^{n+2}3} \|\phi_n\|_{\mathcal{PW}_{\pi}^1}$$
$$\leq \sum_{n=1}^{\infty} \frac{1}{2^{n+2}3} \sum_{l=1}^{\infty} \frac{\|w_{N_l}\|_{\mathcal{PW}_{\pi}^1}}{l^{n-1}2^{l+1}}$$
$$< \sum_{n=1}^{\infty} \frac{1}{2^{n+2}} \sum_{l=1}^{\infty} \frac{1}{2^{l+1}} = \frac{1}{8} < 1$$

Hence,  $\{\xi_n\}_{n\in\mathbb{N}}$  is a basic sequence for  $\mathcal{PW}_{\pi}^1$  that is equivalent to  $\{q_n\}_{n\in\mathbb{N}}$  [25, p. 46]. Let  $\mathcal{D}$  be the closure in the  $\mathcal{PW}_{\pi}^1$ -norm of the set

$$\left\{\sum_{n=1}^{M} a_n \xi_n \colon a_n \in \mathbb{R}, M \in \mathbb{N}\right\}.$$

 $\mathcal{D}$  is the desired infinite dimensional closed subspace of  $\mathcal{R} \cup \{0\}$ . It remains to show that

$$\lim_{N \to \infty} \left| \sum_{k=-N}^{N} \frac{f(k)}{N + \frac{1}{2} - k} \right| = \infty$$

for all  $f \in \mathcal{D}$ .

We have  $f \in \mathcal{D}$  if and only if  $\sum_{n=1}^{\infty} |a_n|^2 < \infty$  [22, p. 104]. Further, for every  $f \in \mathcal{D}$  there exists a unique  $l^2$ -sequence  $\{a_n\}_{n \in \mathbb{N}}$  such that

$$f = \sum_{n=1}^{\infty} a_n \xi_n.$$

Let  $f \in \mathcal{D}, f \not\equiv 0$ , be arbitrary but fixed. Then f has the expansion

$$f(t) = \sum_{n=1}^{\infty} a_n \xi_n(t), \quad t \in \mathbb{R}.$$

Let  $n_0$  denote the smallest natural number such that  $a_{n_0} \neq 0$ . We have

$$f(t) = \underbrace{\sum_{n=n_0}^{\infty} a_n q_n(t)}_{=G(t)} + \underbrace{\sum_{n=n_0}^{\infty} \frac{a_n}{2^{n+23}} \phi_n(t)}_{=H(t)}, \quad t \in \mathbb{R}.$$

Since  $\sum_{n=1}^{\infty} |a_n|^2 < \infty$ , it follows that  $G \in \mathcal{PW}^2_{\pi}$ , and we obtain

$$\begin{split} \left| \sum_{k=-N}^{N} \frac{G(k)}{N + \frac{1}{2} - k} \right| &\leq \left( \sum_{k=-\infty}^{\infty} |G(k)|^2 \right)^{\frac{1}{2}} \left( \sum_{k=-N}^{N} \frac{1}{(N + \frac{1}{2} - k)^2} \right)^{\frac{1}{2}} \\ &\leq \left( \sum_{k=-\infty}^{\infty} |G(k)|^2 \right)^{\frac{1}{2}} \left( \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2})^2} \right)^{\frac{1}{2}} \\ &= \|G\|_{\mathcal{PW}^2_{\pi}} \frac{\pi}{\sqrt{2}} = C_3, \end{split}$$

where  $C_3$  is a constant that is independent of N. Hence, we have

$$\left|\sum_{k=-N}^{N} \frac{f(k)}{N + \frac{1}{2} - k}\right| \ge \left|\sum_{k=-N}^{N} \frac{H(k)}{N + \frac{1}{2} - k}\right| - \left|\sum_{k=-N}^{N} \frac{G(k)}{N + \frac{1}{2} - k}\right|$$
$$\ge \left|\sum_{k=-N}^{N} \frac{H(k)}{N + \frac{1}{2} - k}\right| - C_3.$$

For  $M, N \in \mathbb{N}$ , M < N we have

$$\left|\sum_{k=-N}^{N} \frac{H(k)}{N + \frac{1}{2} - k}\right| \ge \left|\sum_{k=M}^{N} \frac{H(k)}{N + \frac{1}{2} - k}\right| - \left|\sum_{k=-N}^{-M} \frac{H(k)}{N + \frac{1}{2} - k}\right| - \left|\sum_{k=-M}^{M} \frac{H(k)}{N + \frac{1}{2} - k}\right|.$$
(12)

The second expression on the right hand side of (12) can be bounded from above by

$$\begin{split} & \left| \sum_{k=-N}^{-M} \frac{H(k)}{N + \frac{1}{2} - k} \right| \le \|H\|_{\mathcal{PW}^{\frac{1}{n}}} \sum_{k=M}^{N} \frac{1}{N + \frac{1}{2} + k} \\ & < \|H\|_{\mathcal{PW}^{\frac{1}{n}}} \sum_{k=1}^{N} \frac{1}{N + \frac{1}{2}} = \|H\|_{\mathcal{PW}^{\frac{1}{n}}} \frac{N}{N + \frac{1}{2}} < \|H\|_{\mathcal{PW}^{\frac{1}{n}}}. \end{split}$$

And for the third expression on the right hand side of (12) we have

$$\left| \sum_{k=-M}^{M} \frac{H(k)}{N + \frac{1}{2} - k} \right| \le \|H\|_{\mathcal{PW}_{\pi}^{1}} \left| \sum_{k=-M}^{M} \frac{1}{N + \frac{1}{2} - k} \right|$$
$$< (2M + 1) \|H\|_{\mathcal{PW}_{\pi}^{1}}.$$

Hence, for  $M, N \in \mathbb{N}$ , M < N, it follows that

$$\left|\sum_{k=-N}^{N} \frac{f(k)}{N + \frac{1}{2} - k}\right| \geq \left|\sum_{k=M}^{N} \frac{H(k)}{N + \frac{1}{2} - k}\right| - (2M + 2) \|H\|_{\mathcal{PW}^{\frac{1}{n}}} - C_3, \quad (13)$$

which shows that it suffices to analyze

$$\sum_{k=M}^{N} \frac{H(k)}{N + \frac{1}{2} - k}$$

in the following. For  $M, N \in \mathbb{N}$ , M < N, we have

$$\begin{split} &\sum_{k=M}^{N} \frac{H(k)}{N + \frac{1}{2} - k} \\ &= \left| \sum_{k=M}^{N} \frac{a_{n_0} \phi_{n_0}(k)}{2^{n_0 + 2} 3(N + \frac{1}{2} - k)} + \sum_{n=n_0+1}^{\infty} \frac{a_n}{2^{n+2} 3} \sum_{k=M}^{N} \frac{\phi_n(k)}{N + \frac{1}{2} - k} \right| \\ &\geq \frac{|a_{n_0}|}{2^{n_0 + 2} 3} \sum_{k=M}^{N} \frac{\phi_{n_0}(k)}{N + \frac{1}{2} - k} - \sum_{n=n_0+1}^{\infty} \frac{|a_n|}{2^{n+2} 3} \sum_{k=M}^{N} \frac{\phi_n(k)}{N + \frac{1}{2} - k} \\ &= \frac{|a_{n_0}|}{2^{n_0 + 2} 3} \sum_{k=M}^{N} \frac{\phi_{n_0}(k)}{N + \frac{1}{2} - k} - \sum_{\nu=1}^{\infty} \frac{|a_{n_0+\nu}|}{2^{n_0 + 2+\nu} 3} \sum_{k=M}^{N} \frac{\phi_{n_0+\nu}(k)}{N + \frac{1}{2} - k}. \end{split}$$

Let  $\tilde{l}$  be the smallest natural number such that

$$|a_{n_0}| - \frac{1}{\tilde{l}} \sum_{\nu=1}^{\infty} \frac{|a_{n_0+\nu}|}{2^{\nu}} > \frac{|a_{n_0}|}{2}.$$
 (14)

For  $m \in \mathbb{N}$  and all  $k \ge N_{\tilde{l}}$  we have

$$\phi_{n_0+m}(k) \le \frac{1}{\tilde{l}^m} \phi_{n_0}(k),$$

according to (10). Let  $M \ge N_{\tilde{l}}$  and N > M. Then it follows that

$$\begin{split} \sum_{k=M}^{N} \frac{H(k)}{N + \frac{1}{2} - k} &\geq \frac{|a_{n_0}|}{2^{n_0 + 2}3} \sum_{k=M}^{N} \frac{\phi_{n_0}(k)}{N + \frac{1}{2} - k} \\ &- \sum_{\nu=1}^{\infty} \frac{|a_{n_0 + \nu}|}{2^{n_0 + 2 + \nu}3} \sum_{k=M}^{N} \frac{\phi_{n_0}(k)}{\tilde{l}^{\nu} (N + \frac{1}{2} - k)} \\ &= \left(\sum_{k=M}^{N} \frac{\phi_{n_0}(k)}{N + \frac{1}{2} - k}\right) \left(\frac{|a_{n_0}|}{2^{n_0 + 2}3} - \sum_{\nu=1}^{\infty} \frac{|a_{n_0 + \nu}|}{2^{n_0 + 2 + \nu} \tilde{l}^{\nu}3}\right) \\ &> \left(\sum_{k=M}^{N} \frac{\phi_{n_0}(k)}{N + \frac{1}{2} - k}\right) \left(\frac{|a_{n_0}|}{2^{n_0 + 2}3} - \frac{1}{\tilde{l}} \sum_{\nu=1}^{\infty} \frac{|a_{n_0 + \nu}|}{2^{n_0 + 2 + \nu}3}\right) \\ &> \left(\sum_{k=M}^{N} \frac{\phi_{n_0}(k)}{N + \frac{1}{2} - k}\right) \frac{|a_{n_0}|}{2^{n_0 + 3}3}, \end{split}$$

where we used (14) in the last inequality. Using (11) we see that

$$\lim_{N \to \infty} \sum_{k=M}^{N} \frac{H(k)}{N + \frac{1}{2} - k} = \infty$$

for all  $M \ge N_{\tilde{i}}$ , which, according to (13) implies that

$$\lim_{N \to \infty} \left| \sum_{k=-N}^{N} \frac{f(k)}{N + \frac{1}{2} - k} \right| = \infty.$$
(15)

Since  $f \in D$  was arbitrary, we have (15) for all  $f \in D$ . According to our argumentation at the beginning of the proof, this implies that we have

$$\lim_{N \to \infty} \max_{t \in \mathbb{R}} \left| \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty$$

for all  $f \in \mathcal{D}_{\text{Shannon}}$ , where  $\mathcal{D}_{\text{Shannon}}$  is an infinite dimensional closed subspace that is computed from  $\mathcal{D}$  according to  $\mathcal{D}_{\text{Shannon}} = T\mathcal{D}$ , with T being an isomorphic isomorphism.  $\Box$ 

#### 5. RELATION TO PRIOR WORK

In this paper we studied the Shannon sampling series for the Paley–Wiener space  $\mathcal{PW}_{\pi}^{1}$  and analyzed the structure of the set of signals for which the peak value of the Shannon sampling series diverges strongly.

Some preliminary results have been achieved in [26], where the authors proved the lineability of this set. In the present paper we provide a strengthening of the result in [26], by showing that there exists an infinite dimensional closed subspace of  $\mathcal{PW}_{\pi}^{1}$  such that the peak value of the Shannon sampling series diverges strongly for all signals, except the zero signal, from this subspace.

This result is interesting because it shows that the divergence of these reconstruction processes is not a rare phenomenon occurring only for few signals, but rather a frequent event for infinitely many signals that form an infinitely dimensional vector space. The vector space property implies that any linear combination of signals from this vector space, that is not the zero signal, is again a signal that creates divergence.

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