ENHANCING OBSERVABILITY IN POWER DISTRIBUTION GRIDS

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ABSTRACT

Power distribution grids are currently challenged by observability issues due to limited metering infrastructure. On the other hand, smart meter data, including local voltage magnitudes and power injections, are collected at grid nodes with renewable generation and demand-response programs. A power flow-based approach using these data is put forth here to infer the unknown power injections at non-metered grid nodes. Exploiting the control capabilities of smart inverters and the relative time-invariance of conventional loads, the idea is to solve the non-linear power flow equations jointly over two system realizations. An intuitive condition pertaining to the graph of the underlying grid is shown to be necessary and sufficient for the local identifiability of this task. The derived graph theoretic criterion can be checked efficiently and is numerically verified under realistic scenarios on the IEEE 13-bus feeder.

Index Terms— Smart grid, power flow problem, generic rank, structural observability.

1. INTRODUCTION

Limited instrumentation, low investment interest in the past, and the sheer scale of residential electricity networks, all have resulted in partial observability of low-voltage grids. Utility operators used to collect load and voltage readings at only a few nodes on the major trunks of tree-like feeders. This setup has been functional so far due to the under-utilization of distribution grids and the stationarity of conventional demand. However, the integration of photovoltaic (PV) generation, electric vehicles, and demand response call for enhanced system identifiability. Data from smart meters and the power inverters found in PVs are readily available at high temporal resolution to operators, and could thus be engaged in advanced signal processing tasks.

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Given smart meter data and partial power injection specifications, the problem considered here is to recover the power injections at non-metered nodes. The unknown power injections can be inferred upon finding the related power system state, that is the vector of complex voltages across all grid nodes. Considering a single realization of the power grid and noiseless data, the latter problem constitutes the widely studied *power flow* (PF) problem [1]. The PF task involves a set of non-linear equations that are typically tackled using Newton-Raphson's method, its damped variants, and/or decoupled approximate schemes [2]. For a review on the number of PF solutions and the sensitivity of different PF solvers to initialization see [3], [4]. Recently, the power flow problem has been relaxed to a semidefinite program with analytical guarantees and numerical performance superior to the standard Newton-Raphson solver [5], [6]. A graph theoretic criterion for the identifiability of the PF problem under different types of node specifications has been derived for a linearized grid model in [7].

After reviewing the classic PF problem in Sec. 2, our contribution is twofold. Different from the aforementioned approaches, a PF-based approach coupling successive system states is first developed to recover the load at non-metered nodes (Section 3). Non-metered nodes are assumed to host relatively static loads, and hence their power injections remain unchanged over two grid states. The grid transitions between states either naturally due to fluctuations in solar generation and stochastic demand, or purposefully, by changing the power factor of PV inverters. Secondly, Section 4 characterizes the identifiability of the related non-linear equations based on the distribution network graph after ignoring the substation node and its incident edges: If non-metered nodes can be mapped to smart meter-enabled nodes via a set of vertex-disjoint paths, the novel coupled power flow problem is locally invertible. The numerical tests of Section 5 corroborate the validity of this criterion.

Notation: Column vectors (matrices) are represented using lower- (upper-) case boldface letters. Sets are denoted using calligraphic symbols, while $|\mathcal{X}|$ denotes the cardinality of set $\mathcal{X}$. The notation X_{nm} denotes the (n,m)-th entry of $\mathbf{X}$, while $(\cdot)^\top$ stands for transposition. The operator $\mathrm{dg}(\mathbf{x})$ defines a diagonal matrix having $\mathbf{x}$ on its main diagonal, and $\mathrm{rk}(\cdot)$ is the matrix rank.

2. CLASSIC POWER FLOW PROBLEM

Before presenting our coupled power flow approach, grid modeling preliminaries and the conventional power flow problem are reviewed. A radial single-phase distribution grid can be represented by a non-directed tree $\mathcal{T} = (\mathcal{N}^+, \mathcal{L})$, whose nodes $\mathcal{N}^+ := \{0, \ldots, N\}$ correspond to buses and its edges $\mathcal{L}$ to lines. The substation indexed by $n = 0$ serves as the root node. The complex voltage at bus n can be expressed in rectangular coordinates as $V_n = v_{r,n} + jv_{i,n}$. The $2(N+1)$-length vector $\mathbf{v} := [\mathbf{v}_r^\top \ \mathbf{v}_i^\top]^\top$ with $\mathbf{v}_r := [v_{r,0} \ \ldots \ v_{r,N}]^\top$ and $\mathbf{v}_i := [v_{i,0} \ \ldots \ v_{i,N}]^\top$ constitutes the system state.

Introduce the bus admittance matrix $\mathbf{Y} := \mathbf{G} + j\mathbf{B}$, which can be interpreted as a weighted graph Laplacian and thus satisfies $\mathbf{Y}\mathbf{1} = \mathbf{0}$ [1]. Then, the active power injection into bus n is a quadratic function of the state: $p_n(\mathbf{v}) = v_{r,n} \sum_{m=1}^N \left(v_{r,m}G_{nm} - v_{i,m}B_{nm}\right) + v_{i,n} \sum_{m=1}^N \left(v_{i,m}G_{nm} + v_{r,m}B_{nm}\right)$; and so is the reactive power injection $q_n(\mathbf{v}) = v_{i,n} \sum_{m=1}^N \left(v_{r,m}G_{nm} - v_{i,m}B_{nm}\right) - v_{r,n} \sum_{m=1}^N \left(v_{i,m}G_{nm} + v_{r,m}B_{nm}\right)$ for all $n \in \mathcal{N}^+$. Squared voltage magnitudes are apparently quadratic functions of $\mathbf{v}$ too since $|V_n(\mathbf{v})|^2 = v_{r,n}^2 + v_{i,n}^2$.

In the classic power flow problem, the system operator fixes two out of the three quantities $(p_n, q_n, |V_n|^2)$ for all $n \in \mathcal{N}^+$ to specified values, and solves the previous PF equations to recover $\mathbf{v}$. Bus n is deemed a PQ if (p_n, q_n) are specified; a PV bus if $(p_n, |V_n|^2)$ are specified; or it is the substation for which $(|V_0|^2, v_{i,0}) = (1, 0)$. In distribution grids, non-substation buses are typically modeled as PQ buses. Define the mapping $\mathbf{h} : \mathbb{R}^{2(N+1)} \to \mathbb{R}^{2(N+1)}$ from the system state to the power flow specifications $\mathbf{h}(\mathbf{v}) := \left[|V_0(\mathbf{v})|^2 \ v_{i,0}^2 \ p_1(\mathbf{v}) \ \ldots \ p_N(\mathbf{v}) \ q_1(\mathbf{v}) \ \ldots \ q_N(\mathbf{v})\right]^\top$. The grid state $\mathbf{v}$ satisfies

$$\mathbf{h}(\mathbf{v}) = \mathbf{z} \tag{1}$$

where $\mathbf{z} = [|\hat{V}_0|^2 \ 0 \ \hat{p}_1 \ \ldots \ \hat{p}_N \ \hat{q}_1 \ \ldots \ \hat{q}_N]^\top$ is the vector of specifications. Although the requirement of fixing two out of the three quantities $(p_n, q_n, |V_n|^2)$ for each bus is typically met in transmission grids; that is not the case in distribution grids where several buses do not communicate their local readings to the control center. Thus, the system in (1) becomes under-determined. Nevertheless, with the advent of smart metering infrastructure, the system operator may have access to all three quantities $(p_n, q_n, |V_n|^2)$ on a subset of metered buses. If sufficiently many additional specifications are provided, it may be possible to tackle this as explained next.

3. COUPLED POWER FLOW PROBLEM

Different from the classic PF setup, buses here are classified into four mutually exclusive and collectively exhaustive sets: the set $\mathcal{M}$ of metered buses for which all three quantities $(p_n, q_n, |V_n|^2)$ are specified; the set $\mathcal{O}$ of non-metered buses where no information is available; the singleton $\mathcal{S} = \{0\}$; and the remaining buses in $\mathcal{N}^+$ comprising the set of conventional PQ buses $\mathcal{C}$ for which (p_n, q_n) are specified. When the grid is at state $\mathbf{v}$, the aforementioned specifications read

$$|V_m(\mathbf{v})|^2 = |\hat{V}_m|^2 \qquad \forall m \in \mathcal{S} \cup \mathcal{M} \tag{2a}$$

$$q_m(\mathbf{v}) = \hat{q}_m \qquad \forall m \in \mathcal{C} \cup \mathcal{M} \tag{2b}$$

$$p_m(\mathbf{v}) = \hat{p}_m \qquad \forall m \in \mathcal{C} \cup \mathcal{M}. \tag{2c}$$

Solving (2) requires $3|\mathcal{M}| + 2|\mathcal{C}| \geq 2N$. Because $|\mathcal{C}| = N - |\mathcal{M}| - |\mathcal{O}|$, it follows $|\mathcal{M}| \geq 2|\mathcal{O}|$; i.e., the number of metered buses must be at least twice the number of non-metered buses.

The necessary condition $|\mathcal{M}| \geq 2|\mathcal{O}|$ may not be met because either there are not sufficiently many metered buses or the operator cannot communicate frequently enough with all of them. A coupled power flow (CPF) problem is proposed here to relax the condition on $|\mathcal{M}|$. The idea is to couple specifications corresponding to state $\mathbf{v}$ with others corresponding to a different state $\mathbf{v}'$. The grid can transition to $\mathbf{v}'$ either naturally due to variations in solar generation and demand; or by design via proper control of smart inverters. Upon collecting specifications for $\mathbf{v}'$, a set of equations identical in structure to those in (2) is obtained. If power injections at non-metered buses are assumed unchanged between the two time instances, the next additional specifications coupling $(\mathbf{v}, \mathbf{v}')$ are obtained:

$$p_m(\mathbf{v}) = p_m(\mathbf{v}') \qquad \forall m \in \mathcal{O} \tag{3a}$$

$$q_m(\mathbf{v}) = q_m(\mathbf{v}') \qquad \forall m \in \mathcal{O}. \tag{3b}$$

The assumption in (3) is reasonable if buses in $\mathcal{O}$ host conventional loads that are invariant over short intervals and buses in $\mathcal{M}$ have fast time-varying solar generation. The CPF problem can be now compactly written as

$$\mathbf{s}(\mathbf{v}, \mathbf{v}') := \begin{bmatrix} \mathbf{V}_\mathcal{M}(\mathbf{v}) - \hat{\mathbf{V}}_\mathcal{M} \\ \mathbf{q}_{\mathcal{C}\cup\mathcal{M}}(\mathbf{v}) - \hat{\mathbf{q}}_{\mathcal{C}\cup\mathcal{M}} \\ \mathbf{p}_{\mathcal{C}\cup\mathcal{M}}(\mathbf{v}) - \hat{\mathbf{p}}_{\mathcal{C}\cup\mathcal{M}} \\ \mathbf{p}_\mathcal{O}(\mathbf{v}) - \mathbf{p}_\mathcal{O}(\mathbf{v}') \\ \mathbf{q}_\mathcal{O}(\mathbf{v}) - \mathbf{q}_\mathcal{O}(\mathbf{v}') \\ \mathbf{V}_\mathcal{M}(\mathbf{v}') - \hat{\mathbf{V}}'_\mathcal{M} \\ \mathbf{q}_{\mathcal{C}\cup\mathcal{M}}(\mathbf{v}') - \hat{\mathbf{q}}'_{\mathcal{C}\cup\mathcal{M}} \\ \mathbf{p}_{\mathcal{C}\cup\mathcal{M}}(\mathbf{v}') - \hat{\mathbf{p}}'_{\mathcal{C}\cup\mathcal{M}} \end{bmatrix} = \mathbf{0} \tag{4}$$

where $\mathbf{a}_{\mathcal{B}}$ indicates the subvector of $\mathbf{a}$ indexed by set $\mathcal{B}$. A necessary condition for solving (4) is $6|\mathcal{M}| + 4|\mathcal{C}| + 2|\mathcal{O}| \geq 4N$ or $|\mathcal{M}| \geq |\mathcal{O}|$. Also, the system is deemed *locally solvable* if the related Jacobian matrix is invertible at $(\mathbf{v}, \mathbf{v}')$. The latter does not imply that (4) has a unique solution; it only guarantees that the mapping $\mathbf{s} : \mathbb{R}^{4N+2} \rightarrow \mathbb{R}^{4N+2}$ is locally invertible. The invertibility of this Jacobian matrix is studied next.

4. LOCAL IDENTIFIABILITY

To express the Jacobian matrix of (4) in a convenient form, consider the mappings from the system state to the squared voltage magnitudes and the power injections at all buses with corresponding Jacobian matrices $\mathbf{J}^v(\mathbf{v})$, $\mathbf{J}^p(\mathbf{v})$, and $\mathbf{J}^q(\mathbf{v})$. The Jacobian of $\mathbf{s}(\mathbf{v}, \mathbf{v}')$ is then

$$\mathbf{J}(\mathbf{v}, \mathbf{v}') = \left[\begin{array}{c:c} \mathbf{J}^v_{\mathcal{M}}(\mathbf{v}) & \mathbf{0} \\ \mathbf{J}^q_{\mathcal{C}\cup\mathcal{M}}(\mathbf{v}) & \mathbf{0} \\ \mathbf{J}^p_{\mathcal{C}\cup\mathcal{M}}(\mathbf{v}) & \mathbf{0} \\ \hdashline \mathbf{J}^p_{\mathcal{O}}(\mathbf{v}) & -\mathbf{J}^p_{\mathcal{O}}(\mathbf{v}') \\ \mathbf{J}^q_{\mathcal{O}}(\mathbf{v}) & -\mathbf{J}^q_{\mathcal{O}}(\mathbf{v}') \\ \hdashline \mathbf{0} & \mathbf{J}^v_{\mathcal{M}}(\mathbf{v}') \\ \mathbf{0} & \mathbf{J}^q_{\mathcal{C}\cup\mathcal{M}}(\mathbf{v}') \\ \mathbf{0} & \mathbf{J}^p_{\mathcal{C}\cup\mathcal{M}}(\mathbf{v}') \end{array}\right] \tag{5}$$

and can be partitioned into

$$\mathbf{J}(\mathbf{v}, \mathbf{v}') = \left[\begin{array}{c:c} \mathbf{J}_A(\mathbf{v}) & \mathbf{J}_B(\mathbf{v}') \\ \hdashline \mathbf{J}_C(\mathbf{v}) & \mathbf{J}_D(\mathbf{v}') \end{array}\right]. \tag{6}$$

Since characterizing $\text{rk}(\mathbf{J}(\mathbf{v}, \mathbf{v}'))$ for all $(\mathbf{v}, \mathbf{v}')$ is challenging, we study its *generic rank* instead [8]. Formally, the generic rank of a matrix is the maximum rank attained if its generally non-zero entries can take any real value while its zero entries are fixed. The next criterion determines the generic rank of $\mathbf{J}(\mathbf{v}, \mathbf{v}')$ through Th. 1.

Criterion 1. *Consider the graph derived from $\mathcal{T} = (\mathcal{N}^+, \mathcal{L})$ upon removing the substation and its incident edges. Assume there exists a set of node-disjoint paths connecting every node in $\mathcal{O}$ with a node in $\mathcal{M}$ in $\mathcal{G}$.*

Theorem 1. *The Jacobian matrix associated with the equations in* (4) *is invertible in general if and only if Criterion 1 holds.*

To appreciate Th. 1, let us study two scenarios on the IEEE 13-bus feeder in Fig. 1: Scenario A assumes $\mathcal{O} = \{2, 6, 12\}$ and $\mathcal{M} = \{3, 7, 9\}$. Since paths $(2, 1, 5, 7)$, $(6, 3)$, and $(12, 9)$ do not share any node, the related CPF problem is invertible. In Scenario B, metered bused switch to $\mathcal{M} = \{1, 7, 9\}$. Albeit bus 12 can be connected to bus 9, there is no way to pair $\{2, 6\}$ to $\{1, 7\}$ without passing via bus 1; hence, this case is not invertible. Criterion 1 can be checked by solving a maxflow problem [9].

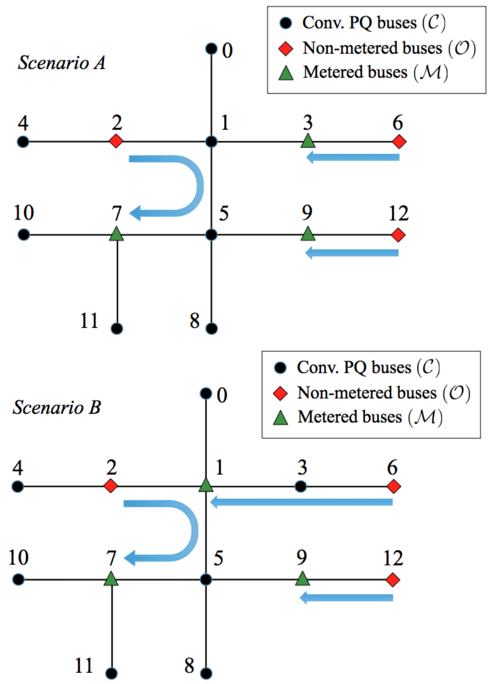

Fig. 1: Two scenarios on the IEEE 13-bus grid [10].

Before proving Th. 1, a result on structural observability is outlined [11], [12], [13]. Consider matrix

$$\mathbf{E} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \tag{7}$$

where $\mathbf{A} \in \mathbb{R}^{M\times M}$, $\mathbf{B} \in \mathbb{R}^{M\times K}$, $\mathbf{C} \in \mathbb{R}^{T\times M}$, and $\mathbf{D} \in \mathbb{R}^{T\times K}$ with $T \geq K$. To study the generic rank of $\mathbf{E}$, construct an associated directed graph $\mathcal{G}_p = (\mathcal{V}, \mathcal{E})$ as follows. Introduce the set of vertices $\mathcal{X} := \{x_1, \ldots, x_M\}$, $\mathcal{U} := \{u_1, \ldots, u_K\}$, and $\mathcal{Y} := \{y_1, \ldots, y_T\}$ so that $\mathcal{V} = \mathcal{X}\cup\mathcal{U}\cup\mathcal{Y}$. Define also the sets of directed edges $\mathcal{E}_{xx} = \{(x_{m_2}, x_{m_1}) : A_{m_1 m_2} \neq 0\}$, $\mathcal{E}_{ux} = \{(u_k, x_m) : B_{mk} \neq 0\}$, $\mathcal{E}_{xy} = \{(x_m, y_t) : C_{tm} \neq 0\}$, and $\mathcal{E}_{uy} = \{(u_k, y_t) : D_{tk} \neq 0\}$ such that $\mathcal{E} = \mathcal{E}_{xx} \cup \mathcal{E}_{ux} \cup \mathcal{E}_{xy} \cup \mathcal{E}_{uy}$.

Lemma 1 ([13])**.** *If $\mathbf{A}$ is invertible, the generic rank of the Schur complement $\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ is equal to the maximal number of node-disjoint paths from $\mathcal{U}$ to $\mathcal{Y}$ in $\mathcal{G}_p$.*

If blocks $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ and thus nodes $\mathcal{X}$ do not exist, matrix $\mathbf{D}$ is generically invertible if there is a matching between its column nodes $\mathcal{U}$ and its row nodes $\mathcal{Y}$ [8].

The rank additivity property asserts $\text{rk}(\mathbf{J}(\mathbf{v}, \mathbf{v}')) = \text{rk}(\mathbf{J}_A(\mathbf{v})) + \text{rk}(\mathbf{J}_D(\mathbf{v}') - \mathbf{J}_C(\mathbf{v})\mathbf{J}_A^{-1}(\mathbf{v})\mathbf{J}_B(\mathbf{v}'))$ assuming $\mathbf{J}_A(\mathbf{v})$ is non-singular [14, Ch. 13]. Therefore, $\mathbf{J}(\mathbf{v}, \mathbf{v}')$ is invertible if both $\mathbf{J}_A(\mathbf{v})$ and the Schur complement $\mathbf{J}_D(\mathbf{v}') - \mathbf{J}_C(\mathbf{v})\mathbf{J}_A^{-1}(\mathbf{v})\mathbf{J}_B(\mathbf{v}')$ are invertible.

Lemma 2. *Under Criterion 1, the partial Jacobian matrix $\mathbf{J}_A(\mathbf{v})$ is generically invertible.*

Proof of Lemma 2. Define the flat voltage profile $\mathbf{v}_{\text{fl}} := [\mathbf{1}_{N+1}^\top \ \mathbf{0}_{N+1}^\top]^\top$ occurring if no power is flowing on the grid [1], [15]. Matrix $\mathbf{J}_A(\mathbf{v}_{\text{fl}})$ captures the sparsity pattern of any $\mathbf{J}_A(\mathbf{v})$ and it can be partitioned as [5]:

$$\mathbf{J}_A(\mathbf{v}_{\text{fl}}) = \left[\begin{array}{c|c|c} 2\mathbf{I}_{\mathcal{S}\cup\mathcal{M},\mathcal{S}\cup\mathcal{M}} & \mathbf{0}_{\mathcal{S}\cup\mathcal{M},\mathcal{C}\cup\mathcal{O}} & \mathbf{0}_{\mathcal{S}\cup\mathcal{M},\mathcal{N}} \\ \hline \mathbf{B}_{\mathcal{C}\cup\mathcal{M},\mathcal{S}\cup\mathcal{M}} & \mathbf{B}_{\mathcal{C}\cup\mathcal{M},\mathcal{C}\cup\mathcal{O}} & \mathbf{G}_{\mathcal{C}\cup\mathcal{M},\mathcal{N}} \\ -\mathbf{G}_{\mathcal{N},\mathcal{S}\cup\mathcal{M}} & -\mathbf{G}_{\mathcal{N},\mathcal{C}\cup\mathcal{O}} & \mathbf{B}_{\mathcal{N},\mathcal{N}} \end{array}\right].$$

Because the top left block of $\mathbf{J}_A(\mathbf{v}_{\text{fl}})$ is an identity matrix and its top right block is zero, the rank additivity property asserts that $\mathbf{J}_A(\mathbf{v}_{\text{fl}})$ is full rank if and only if its bottom right block is full rank. Block $\mathbf{B}_{\mathcal{N},\mathcal{N}}$ is a symmetric submatrix of the Laplacian $\mathbf{B}$ and is invertible [5, Lemma 1]. Using the rank additivity property again, the bottom right block of $\mathbf{J}_A(\mathbf{v}_{\text{fl}})$ is invertible if and only if

$$\tilde{\mathbf{J}}_A := \mathbf{B}_{\mathcal{C}\cup\mathcal{M},\mathcal{C}\cup\mathcal{O}} + \mathbf{G}_{\mathcal{C}\cup\mathcal{M},\mathcal{N}}\mathbf{B}_{\mathcal{N},\mathcal{N}}^{-1}\mathbf{G}_{\mathcal{N},\mathcal{C}\cup\mathcal{O}}. \quad (8)$$

is full rank. If $\mathbf{A}$ is the reduced node-edge incidence matrix obtained upon removing the root node, it holds that $\mathbf{B} = \mathbf{A}^\top \operatorname{dg}(\mathbf{b})\mathbf{A}$ and $\mathbf{G} = \mathbf{A}^\top \operatorname{dg}(\mathbf{g})\mathbf{A}$, where $\mathbf{b}$ and $\mathbf{g}$ are the vectors of distribution line susceptances and conductances, respectively [15]. It is not hard to verify that $\mathbf{B}_{\mathcal{C}\cup\mathcal{M},\mathcal{C}\cup\mathcal{O}} = \mathbf{A}_{\mathcal{N},\mathcal{C}\cup\mathcal{M}}^\top \operatorname{dg}(\mathbf{b})\mathbf{A}_{\mathcal{N},\mathcal{C}\cup\mathcal{O}}$. Expressing all submatrices appearing in (8) in a similar fashion and exploiting the invertibility of $\mathbf{A}$ yields $\tilde{\mathbf{J}}_A = \mathbf{A}_{\mathcal{N},\mathcal{C}\cup\mathcal{M}}^\top \left(\operatorname{dg}(\mathbf{b}) + \operatorname{dg}^2(\mathbf{g})\operatorname{dg}^{-1}(\mathbf{b})\right)\mathbf{A}_{\mathcal{N},\mathcal{C}\cup\mathcal{O}}$. Introduce $\tilde{\mathbf{b}}$ with entries $\tilde{b}_n := b_n + \frac{g_n^2}{b_n}$ for $n = 1, \ldots, N$, and perform the column-wise partitions $\mathbf{A}_{\mathcal{N},\mathcal{C}\cup\mathcal{M}} = [\mathbf{A}_{\mathcal{N},\mathcal{C}} \ \mathbf{A}_{\mathcal{N},\mathcal{M}}]$ and $\mathbf{A}_{\mathcal{N},\mathcal{C}\cup\mathcal{O}} = [\mathbf{A}_{\mathcal{N},\mathcal{C}} \ \mathbf{A}_{\mathcal{N},\mathcal{O}}]$ to get

$$\tilde{\mathbf{J}}_A = \begin{bmatrix} \breve{\mathbf{J}}_A & \breve{\mathbf{J}}_B \\ \breve{\mathbf{J}}_C & \breve{\mathbf{J}}_D \end{bmatrix} \quad (9)$$

with its submatrices defined as $\breve{\mathbf{J}}_A := \mathbf{A}_{\mathcal{N},\mathcal{C}}^\top \operatorname{dg}(\tilde{\mathbf{b}})\mathbf{A}_{\mathcal{N},\mathcal{C}}$; $\breve{\mathbf{J}}_B := \mathbf{A}_{\mathcal{N},\mathcal{C}}^\top \operatorname{dg}(\tilde{\mathbf{b}})\mathbf{A}_{\mathcal{N},\mathcal{O}}$; $\breve{\mathbf{J}}_C := \mathbf{A}_{\mathcal{N},\mathcal{M}}^\top \operatorname{dg}(\tilde{\mathbf{b}})\mathbf{A}_{\mathcal{N},\mathcal{C}}$; and $\breve{\mathbf{J}}_D := \mathbf{A}_{\mathcal{N},\mathcal{M}}^\top \operatorname{dg}(\tilde{\mathbf{b}})\mathbf{A}_{\mathcal{N},\mathcal{O}}$. The generic invertibility of $\tilde{\mathbf{J}}_A$ can be determined now by Lemma 1. Heed that $\breve{\mathbf{J}}_A$ is a symmetric submatrix of the Laplacian $\mathbf{A}^\top \operatorname{dg}(\tilde{\mathbf{b}})\mathbf{A}$ and is thus invertible [5, Lemma 1]. If there exists a set of node-disjoint paths connecting every bus in $\mathcal{O}$ with a bus in $\mathcal{M}$ without passing through the substation, then Lemma 1 guarantees that $\breve{\mathbf{J}}_D - \breve{\mathbf{J}}_C\breve{\mathbf{J}}_A^{-1}\breve{\mathbf{J}}_B$ and consequently $\tilde{\mathbf{J}}_A$ enjoy full generic rank. $\square$

Lemma 3. *Under Criterion 1, the Schur complement* $\mathbf{J}_D(\mathbf{v}') - \mathbf{J}_C(\mathbf{v})\mathbf{J}_A^{-1}(\mathbf{v})\mathbf{J}_B(\mathbf{v}')$ *is invertible.*

Proof of Lemma 3. Applying Lemma 1 on (6), associate the directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ to $\mathbf{J}(\mathbf{v}, \mathbf{v}')$. Graph $\mathcal{G}$ has $6(N+1)$ nodes forming sets $\mathcal{X}, \mathcal{U}$, and $\mathcal{Y}$. Every node in $\mathcal{X} := \{x_1, \ldots, x_{2N+2}\}$ is related to an entry of $\mathbf{v}$; every node in $\mathcal{U} := \{u_1, \ldots, u_{2N+2}\}$ is related to an entry of $\mathbf{v}'$; and every node in $\mathcal{Y} := \{y_1, \ldots, y_{2N+2}\}$ is related to one of the specifications appearing in the bottom half of $\mathbf{s}(\mathbf{v}, \mathbf{v}')$ in (4). The directed edges in $\mathcal{E}$ capture the sparsity pattern of $\mathbf{J}(\mathbf{v}, \mathbf{v}')$.

Since $\mathbf{J}_A(\mathbf{v})$ and $\mathbf{J}_D(\mathbf{v}')$ exhibit the same sparsity pattern, Lemma 2 guarantees the generic invertibility of $\mathbf{J}_D(\mathbf{v}')$ too. From the degenerate version of Lemma 1, the invertibility of $\mathbf{J}_D(\mathbf{v}')$ implies that there exists a matching between the node sets $\mathcal{U}$ and $\mathcal{Y}$. This perfect matching constitutes a set of node-disjoint paths from $\mathcal{U}$ to $\mathcal{Y}$ in $\mathcal{G}$, and the claim follows from Lemma 1. $\square$

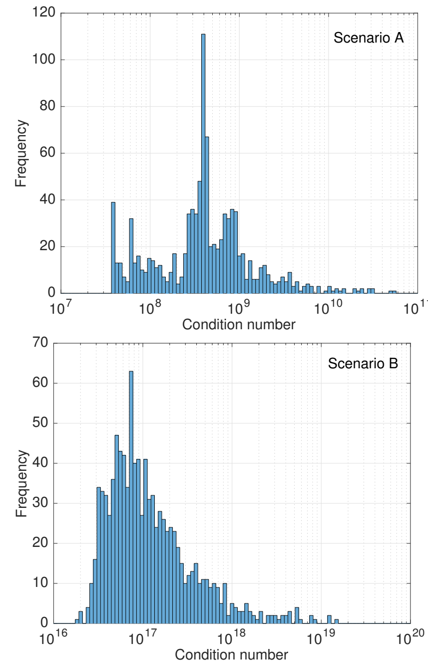

Fig. 2: Histograms of the condition number of $\mathbf{J}(\mathbf{v}, \mathbf{v}')$ under the scenarios of Fig. 1.

5. NUMERICAL TESTS

Although Th. 1 refers to the generic rank of $\mathbf{J}(\mathbf{v}, \mathbf{v}')$, its condition number was numerically validated at pairs $(\mathbf{v}, \mathbf{v}')$ obtained for the IEEE 13-bus feeder [10]. The latter was modified to a single-phase grid as described in [16]. At each zero-injection bus, a load equal to the load of its parent node was inserted. A PV with capacity 4 times the related load was added on all metered buses. Matrix $\mathbf{J}(\mathbf{v}, \mathbf{v}')$ was evaluated at 1,000 random states $(\mathbf{v}, \mathbf{v}')$. For grid state $\mathbf{v}$, PV generators produced their maximum active power at 0.9 lagging power factor. For state $\mathbf{v}'$, PV generation was uniformly drawn within its capacity range while fixing the power factor to 0.9 leading or lagging. Figure 2 shows the histograms of the condition numbers recorded for the two scenarios of Fig. 1. As evidenced by the plots, the Jacobian matrix for Scenario A that was deemed invertible exhibits a condition number much smaller than Scenario B.

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