EFFICIENT POSTCODING FILTER IN LU-BASED BEAMFORMING SCHEME

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ABSTRACT

This paper considers time-domain beamforming in MIMO convolutive systems, using the classical LU-based (Gauss elimination) polynomial matrix decomposition. As the pre- and postfilters are not paraunitary, the output signal-to-noise ratio is degraded compared to QR-based beamforming. We investigate the role of the postfilter on the noise enhancement and propose a simple and efficient solution. We show that a simple normalisation of the postfilter, using row balancing, is sufficient to significantly improve the conditioning of the system and thereby, the performance of the system. With the resulting performance increase, the LU-based polynomial matrix decomposition for MIMO beamforming becomes competitive as compared to its QR-based counterparts, in terms of bit error rate, for medium input SNR.

Index Terms— Polynomial Matrix Decomposition, MIMO Beamforming

1. INTRODUCTION

Beamforming is a well established bandwidth and power efficient method of communication over multipath wireless channels, using multiple transmit and receive antennas [1]. In a multipath propagation environment, several delayed and scaled versions of the transmitted signal arrive at the receiver [2]. This produces intersymbol interference (ISI), which is well known to degrade the bandwidth and power efficiency. The signal recorded at each receive antenna is also the superposition of the outputs of the subchannels originating from the different transmit antennas. This causes co-channel interference (CCI), which is another severe limitation in addition to ISI. Beamforming can mitigate the CCI.

For a system of p transmit antennas and q receive antennas, the $p \times q$ wideband MIMO channel can be represented by its matrix-valued transfer function $H(z)$. Consider the factorisation of $H(z)$ as in

$$H(z) = U(z)D(z)V(z), \tag{1}$$

where $U(z)$ and $V(z)$ are square matrices of sizes p and q respectively. If the inverses of $V(z)$ and $U(z)$, provided they exist, are inserted into the transmission chain respectively as pre- and post-filters, then the original MIMO channel becomes equivalent to $D(z)$. Diagonalization of $H(z)$, that is when $D(z)$ in (1) is diagonal, therefore reduces the MIMO wideband channel to $N = min(p, q)$ independent single input single output (SISO) subchannels, thereby cancelling the CCI. Such decomposition is most commonly achieved using the popular Singular Value Decomposition (SVD) method, leading to para-unitary factors $U(z)$ and $V(z)$.

From a practical point of view, it is natural to assume that the MIMO channel has finite duration. Then, $H(z)$ can be modeled as a polynomial matrix. As a consequence, one seeks for (1) a polynomial matrix decomposition, meaning that all three factors are also polynomial matrices, with $D(z)$ diagonal [3], [4]. Unfortunately, polynomial matrix SVD does not exist in general, although the rational counterpart is always feasible using, *e.g.*, the classical QR decomposition [5]. Since the presence of poles in the decomposition can cause instability, a common solution is to consider, instead, a Laurent polynomial matrix decomposition. Several Laurent polynomial matrix SVD algorithms, based on the QR factorisation [7], [8], [9] or Jacobi method [10] are now available. These are often considered in an OFDM context in order to mitigate both CCI and ISI. Alternatively, the beamforming can be performed in the frequency domain, on the constant channel matrix of each OFDM subcarrier [6]. We mention that the decomposition in (1) is difficult because the factors $V(z)$ and $U(z)$ are required to be paraunitary. The paraunitaryness assures that the power distributions of the signal and noise remain unaltered after pre- and post-filtering, respectively. But, this requirement can be released, since the diagonalisation of $H(z)$ is the main issue. Then, the decomposition (1) is always feasible with polynomial factors, since it is well known that in a Bezout ring [11], any matrix can be triangularised by a unimodular transformation. In this regard, a MIMO beamforming scheme based on a combination of the classical Smith canonical form and LU (Gauss elimination) was presented recently in [12], [13]. The resulting factors $U(z)$ and $V(z)$ are unimodular. The loss of the paraunitary property for the post-filter induces a serious limitation for this beamforming scheme. Indeed, the simulation in [12], [13]

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show that the noise power is enhanced after the post-filtering stage. However, the LU-based beamforming presents at the same time many interesting advantages over its QR-based counterparts. First, it is not necessary to consider Laurent matrix polynomial models which introduce artificial delays just for technical reasons. Also, compared to *e.g.* [7], the decomposition algorithm in [13], called UU-decomposition (Unimodular-Upper), is effective and does not require any iteration: The algorithm ends up after a finite and prescribed number of steps, with a matrix $D(z)$ which is exactly diagonal. Moreover, it was shown in [13] that except for some unprobable original MIMO channel, all but the last resulting independent SISO subchannels reduce to simple additive noise channels. In addition to completely cancelling the CCI, this decomposition also inheritly avoids the ISI problem. Because of its very interesting features, the LU-based factorisation is a good candidate for time domain beamforming, provided the noise enhacement problem is solved. This is precisely the aim of the present paper.

The rest of the paper is organized as follows. Section 2 is devoted to the problem description and investigates the causes of the noise enhancement. A solution is presented in section 3. Finally, simulation results showing that the proposed solution significantly reduces the noise enhancement are given in section 4.

2. PROBLEM DESCRIPTION

The UU-decomposition of $H(z) \in \mathbb{C}^{p \times q}$ follows the same steps as the classical LU factorization. However, a Bezout equation is solved in each step, in order to reduce the pivot element to a constant. Assume without any loss of generality that $p \geqslant q$. We first obtain

$$H(z) = U(z)R(z), \tag{2}$$

where $U(z)$ and $R(z)$ are respectively $p \times p$-unimodular and $p \times q$-upper triangular polynomial matrices. Recall that a polynomial matrix is said to be *unimodular* if it is 1) invertible and 2) the inverse is also a polynomial matrix. Next, the same decomposition is applied to obtain $R(z)^T = V(z)^T D(z)^T$, where the superscript ^{T} denotes the transpose operator and where $V(z)$ is $q \times q$-unimodular as $U(z)$ in (2). Then, the factorization (1) follows, where

$$D(z) = \begin{bmatrix} \widetilde{D}(z) \\ \cdots\cdots \\ O_{p-q,q} \end{bmatrix} \tag{3}$$

with $\widetilde{D}(z)$ $q \times q$-diagonal and polynomial and where $O_{i,j}$ denotes the zero matrix of size $i \times j$. Let us now consider the communication system through $H(z)$,

$$\boldsymbol{y}(z) = H(z)\boldsymbol{x}(z) + \boldsymbol{n}(z), \tag{4}$$

where $\boldsymbol{n}(z)$ stands for the z-transform of a sample realisation of the noise corruption $\boldsymbol{n} \in \mathbb{C}^p$. In the beamforming scheme using (1), the pre-filtered version, $\widetilde{\boldsymbol{x}}(z) = V(z)^{-1}\boldsymbol{x}(z)$, is transmitted instead of the original input signal $\boldsymbol{x}$. The corresponding channel's output $\widehat{\boldsymbol{y}}(z) = H(z)\widetilde{\boldsymbol{x}}(z) + \boldsymbol{n}(z)$ is then post-filtered as in $\widetilde{\boldsymbol{y}}(z) = U(z)^{-1}\widehat{\boldsymbol{y}}(z)$, which yields the final equivalent system

$$\widetilde{\boldsymbol{y}}(z) = D(z)\boldsymbol{x}(z) + U(z)^{-1}\boldsymbol{n}(z) \triangleq D(z)\boldsymbol{x}(z) + \widetilde{\boldsymbol{n}}(z).$$

2.1. Noise power amplification

Assuming a spatial-temporal unitary white noise $\boldsymbol{n}$, the output noise power after post-filtering reads as

$$\mathsf{E}(\|\widetilde{\boldsymbol{n}}\|^2) = \|U(z)^{-1}\|_2^2 \triangleq \frac{1}{2\pi}\int_0^{2\pi} \mathsf{Tr}\left[U(e^{i\omega})^* U(e^{i\omega})\right]^{-1} d\omega, \tag{5}$$

where $\mathsf{E}(\cdot)$ denotes the mathematical expectation, $\mathsf{Tr}(\cdot)$ the trace operator and the superscript *, the transpose-conjugation. The noise component is thus amplified whenever the L_2-norm of the post-filter is high. Fig. 1 displays the performance (in terms of bit error-rate vs SNR) of the LU-based beamforming for four different MIMO channels, each corrupted by a unit-variance spatial-temporal white noise. The performance significantly degrades as the noise amplification increases. Of course, this performance loss cannot be explained only by the noise power enhancement since the output signal $\widetilde{\boldsymbol{y}}$ also undergoes the same post-filtering.

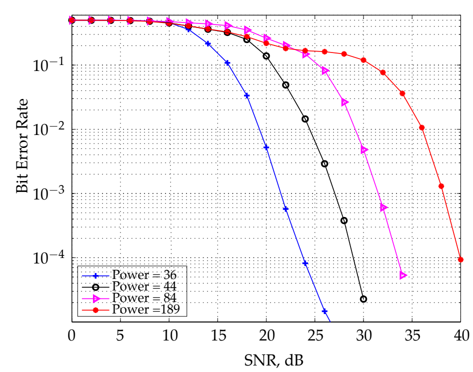

Fig. 1. Effect of the BER comparison for various values of the matrix postcoder's power.

2.2. Matrix conditioning

We observe that the post-filtering operation mentioned above can be seen as the resolution of the linear perturbed system $U(z)\widetilde{\boldsymbol{y}}(z) = H(z)\widetilde{\boldsymbol{x}}(z) + \boldsymbol{n}(z)$, with the error term $\boldsymbol{n}(z)$. As it is well known from classical perturbation analysis [14], the relative error between the computed output $\widetilde{\boldsymbol{y}}(z)$ and the ideal noise-free solution is bounded by:

$$\frac{\|\widetilde{\boldsymbol{y}}(z) - D(z)\boldsymbol{x}(z)\|_2}{\|D(z)\boldsymbol{x}(z)\|_2} \leqslant \kappa(U(z))\frac{\|\boldsymbol{n}(z)\|_2}{\|H(z)\widetilde{\boldsymbol{x}}(z)\|_2}, \tag{6}$$

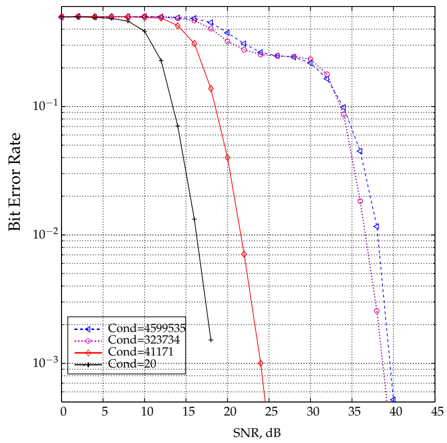

Fig. 2. BER comparison for various values of the matrix post-coder's conditioning values.

where we set $\|\boldsymbol{w}(z)\|_2^2 \triangleq \frac{1}{2\pi}\int_0^{2\pi} \mathsf{E}[\boldsymbol{w}(e^{i\omega})^*\boldsymbol{w}(e^{i\omega})]d\omega$ for the usual L_2 norm for a random vector-valued signal $\boldsymbol{w}$ and where $\kappa(U(z)) \triangleq \|U(z)\|\|U(z)^{-1}\|$ is the condition number of the unimodular matrix $U(z)$ with respect to the norm defined in (5). Note that $\kappa(U(z))$ does not depend on z. In the sequel, we write $\kappa(U)$.

If the post-filter $U(z)$ is ill-conditioned, *i.e.* $\kappa(U) \gg 1$, then the inequality (6) shows that the noise-to-signal ratio after post-filtering (left hand side) may be very high as compared to before post-filtering (factor of $\kappa(U)$). Fig. 2 shows the bit error rate versus SNR for four different systems. The performance is significantly affected when $\kappa(U)$ increases.

3. POST-FILTER SELECTION

Given a noisy input-output communication system (4), the decomposition (1) using paraunitary pre- and post-filters is optimal in the sense of minimum relative output error bound. This is a direct consequence of (6) and the well-known property that $\kappa(U) = 1$ for $U(z)$ paraunitary. In the present section, we propose a simple solution to improve the properties of the post-filter in a beamforming scheme using the UU-decomposition. The aim is to obtain a post-filter with properties as close as possible to those mentioned above for paraunitary post-filters.

To begin, assume that the preceding post-filter $U(z)^{-1}$ is now replaced by some polynomial matrix $S(z)$ of appropriate size. In the sequel, we set $W(z) = S(z)U(z)$. Then, the channel's output signal, after post-filtering (still denoted by $\widetilde{\boldsymbol{y}}(z)$) would read as:

$$\widetilde{\boldsymbol{y}}(z) = W(z)D(z)\boldsymbol{x}(z) + S(z)\boldsymbol{n}(z).$$

Obviously, the selection of $S(z)$ should not introduce CCI nor ISI, unless the later is controllable. Besides, note that row (or column) balancing is a simple trick that can be very efficient for improving the conditioning of a matrix [15]. Therefore, we propose to select the post-filter such that

1. $W(z) = W$ is a diagonal constant matrix. No CCI, no ISI are introduced by the post-filter $S(z)$.
2. Row balancing: $\|S(z)\|_2^2 = p$.

The corresponding post-filter then immediately follows as:

$$S(z) = WU(z)^{-1}, \tag{7}$$

where W is the $p\times p$ diagonal matrix with elements

$$W_{i,i} = \frac{1}{\|[U(z)^{-1}]_i\|_2}$$

where $[A]_i$ stands for the i^{th} row of matrix A.

4. PERFORMANCE COMPARISON

4.1. Performance analysis

The ability of the proposed solution to meet the objectives of low norm and low condition number is illustrated below. We simulate a complete beamforming scheme based on the UU-decomposition. The condition number of the post-filter $U(z)^{-1}$ and that of its modified version $S(z)$ are computed. Also, the output noise power, after post-filtering is estimated for each post-filter. Table 1 displays the results obtained with three different and randomly selected 3×3 MIMO Rayleigh fading channels $H(z)$.

Table 1. Power and conditioning

	Postfilter Power		Postfilter Conditioning	
Channels	$U(z)^{-1}$	$S(z)$	$U(z)^{-1}$	$S(z)$
Channel 1	568.57	1.47	4599535.	37.02
Channel 2	46.82	1.20	41171.	11.41
Channel 3	126.64	1.33	323734.	20.57

The results displayed in table 1 show a very significant improvement. It is therefore expected that this translates into enhanced MIMO-OFDM performance.

We now investigate the effect of this modification in terms of bit error rate performance. For the simulation we consider two channel models: indoor and outdoor ITU Pedestrian with parameters with 40MHz of bandwidth, $Ns = 512$ subcarriers, with a 4-QAM modulation.

Fig. 3 and Fig. 4 depict the bit error rates vs SNR, for the beamformers using the two post-filters $U(z)^{-1}$ and $S(z)$. Significant improvement are obtained in both contexts indoor (Fig. 3) and outdoor (Fig. 4). The performance gain is very important in the more severe outdoor context (Fig. 4).

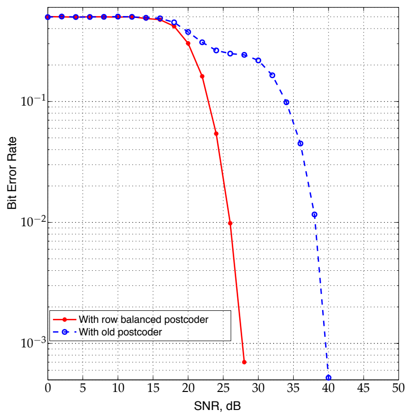

Fig. 3. BER comparison of the two beamformers: Outdoor ITU channel model

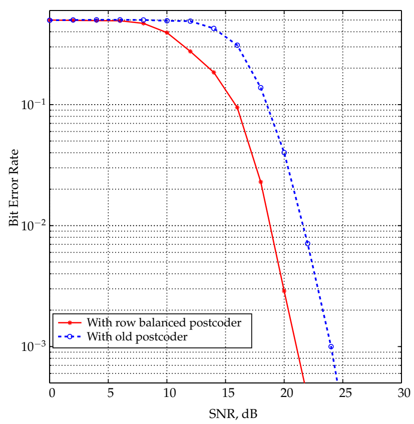

Fig. 4. BER comparison ofthe two beamformers: Indoor ITU channel model

4.2. Comparison with QR-based Beamforming

In order to better observe the impact of this improvement, we compare the obtained results with the performance of QR-Based beamforming [8] in MIMO-OFDM system. For the QR decomposition, we have set $\varepsilon = 10^{-3}$ for the off diagonal elements' tolerance parameter. With this value, the residual CCI is small. The truncation parameter is selected as $\mu = 10^{-3}$ to limit the growth of the degrees of the Laurent polynomials in $D(z)$. We refer to [8] for more details on the meaning and roles of these parameters. The comparisons are done through an outdoor pedestrian ITU MIMO 3×3 channel. Fig. 5 shows BER comparison of the two versions of the UU-based beamforming with QR-based beamforming. The comparison is performed on a complete transmission chain, taking into account all aspects of the UU-decomposition and the QR decomposition. With the UU-based beamformer using the original postfilter $U(z)^{-1}$, the combined negative effect of the noise amplification and ill-conditioning is so severe that this beamforming scheme is not suitable except for

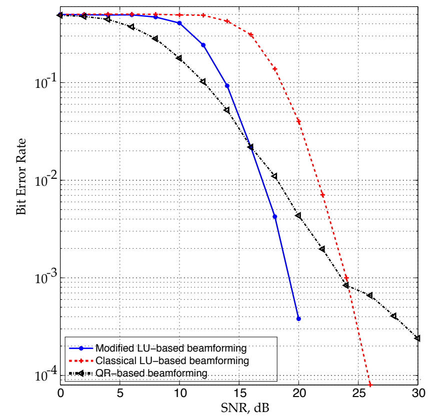

Fig. 5. BER comparison of UU-based beamforming and QR-based beamforming.

high SNR. Meanwhile, the proposed modification of the postfiltering stage is efficient enough to limit the negative effect of the noise postfiltering. The interesting properties of the UU-decomposition (in terms of CCI and ISI) now become apparent.

5. CONCLUSION

A solution is proposed to the problem of output noise enhancement, observed in MIMO beamforming systems using the LU-based polynomial matrix decomposition. The role of the postfilter in the performance degration is clarified: performance decreases as the L_2-norm and/or the condition number of the postfilter matrix-valued transfer function increases. A row balancing of the postfilter is a simple solution for both problem at once: it allows one to bring both the L_2-norm and the condition number close to unity. Significant improvement of the performance, in terms of bit error rate, are observed. For example, simulation using the outdoor IUT channel model shows that the same BER level of 10^{-3} is reached with the proposed solution with about $12dB$ drop in SNR compared to the original setting. The interesting properties (no CCI, no ISI) of the LU-based polynomial matrix decomposition for MIMO beamforming thus become exploitable.

6. REFERENCES

[1] A. Gorokhov, D. A. Gore and A. J. Paulraj, "Receive antenna selection for MIMO spatial multiplexing: theory and algorithms", *IEEE Trans. on Sig. Proc.*, vol. 51, pp. 2796-2807, 2005.

[2] P. Dighe, R. Mallik and S. Jamuar, "Analysis of Transmit-Receive Diversity in Rayleigh Fading", *IEEE Trans. on Com.*, vol. 51, pp. 694703, 2003.

[3] S. M. Razavi and T. Ratnarajah, "Subspace beamforming via block SVD for MIMO systems", *SPAWC*, 2012.

[4] H. Zamiri-Jafarian and M. Rajabzadeh, “A Polynomial Matrix SVD Approach for Time Domain Broadband Beamforming in MIMO-OFDM Systems”, *IEEE VTC Spring*, May 2008.

[5] S. Icart and P. Comon, “Some Properties of Laurent Polynomial Matrices”, *9th IMA Intern. Conf. on Math. in Sig. Proc.*, 2012.

[6] D. Palomar, J. Cioffi, and M. Lagunas, “Joint Tx-Rx Beamforming Design for Multicarrier MIMO Channels: A Unified Framework for Convex Optimization”, *IEEE Trans. on Sig. Proc.*, vol. 51, pp. 23812401, 2003.

[7] J. A. Foster, J. G. McWhirter, M. R. Davies and J. A. Chambers, “An Algorithm for Calculating the QR and Singular Value Decompositions of Polynomial Matrices”, *IEEE Trans. on Sig. Proc.*, vol. 58, No. 3, pp. 1263-1274, 2010.

[8] J. A. Foster, J. G. McWhirter and J. A. Chambers, “A Novel Algorithm for Calculating the QR Decomposition of a Polynomial Matrix”, *ICASSP’09*, Taipei, 2009.

[9] D. Cescato and H. Bölcskei, “QR Decomposition of Laurent Polynomial Matrices Sampled on the Unit Circle”, *IEEE Trans. on Inf. Theory*, vol 56, No. 9, pp. 4754-4761, 2010.

[10] M. Tohidian, H. Amindavar and A. M. Reza, “A DFT-based approximate eigenvalue and singular value decomposition of polynomial matrices”, *EURASIP J. Adv. on Sig. Proc.*, 2013:93.

[11] N. Bourbaki, *Commutative Algebra*, Springer, 1989.

[12] M. Mbaye, M. Diallo and M. Mboup, “Unimodular-Upper polynomial matrix decomposition for MIMO Spatial Multiplexing”, *IEEE SPAWC*, Toronto, Jun 2014.

[13] M. Mbaye, M. Diallo and M. Mboup, “LU based Beamforming schemes for MIMO systems”, *IEEE Trans. on Vehicular Technology*, doi: 10.1002/acs.2688, 2016 (to appear).

[14] R. A. Horn and C. R. Johnson, *Matrix analysis*, Cambridge University Press, 1990.

[15] E. E. Osborne, “On pre-conditioning of matrices”, *J. Assoc. Comput. Mach.*, vol. 7, pp. 338-345, 1960.