SUPER-RESOLUTION DELAY-DOPPLER ESTIMATION FOR SUB-NYQUIST RADAR VIA ATOMIC NORM MINIMIZATION

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ABSTRACT

This paper studies the estimation of the delay and Doppler parameters of the sub-Nyquist radars. By formulating the delay-Doppler estimation as the low-rank matrix recovery, we propose an atomic norm minimization-based estimation approach. With the recovered low-rank matrix, we determine and pair the delay and Doppler parameters of the radar targets. Numerical simulations demonstrate the superior performance of the proposed approach, as compared to the state-of-the-art approaches.

***Index Terms*—** Delay-Doppler estimation, sub-Nyquist radar, compressive sensing, atomic norm, super-resolution

1. INTRODUCTION

In classical radar systems, we usually first sample the radar echoes by analog-to-digital conversion (ADC) systems and then process the sampled echoes in digital domain. When applied to wideband radar systems, these schemes require high-rate ADCs which in turn result in large dataflow and high cost and power consumption. To overcome these problems, radar scientists recently establish the concept of sub-Nyquist radars [1]. Different from the classical radars, the sub-Nyquist radars sample the radar echoes by analog-to-information conversion (AIC) systems [2–6]. Due to its low sampling rate, the sub-Nyquist radar is endowed with a number of implementation advantages. In particular, we can establish wide/ultra-wide band radar systems which are impossible by the current ADC techniques.

However, the sub-Nyquist samples obtained from AICs are different from the Nyquist samples. The classical processing methods cannot be directly applied for the sub-Nyquist radar processing and new methodology is expected to be established. Several methods have been developed recently. A large class of methods is based on the discretization of the delay-Doppler plane and the compressed sensing (CS) recovery theory is used to estimate the delay and Doppler parameters of the radar targets [7–9]. A problem with this class of methods is the basis mismatch effect [10], which greatly deteriorates the estimation performance. Another class of methods is to define parametrized dictionaries and compressive parametric estimation techniques are employed to estimate the target parameters [11, 12]. In general, the second class methods can solve the mismatch problem and has high estimation performance.

In this paper, we develop a new parametric estimation approach. The basic idea is to describe the radar target characteristics as a low-rank matrix and the AIC system as a linear mapping from the low-rank matrix to a sampling vector. The low-rank matrix, named as the delay-Doppler matrix (DDM) in this paper, is parameterized by the delay-Doppler parameters with its rank no more than the number of the targets. Therefore, the delay-Doppler estimation can be formulated as the low rank DDM recovery problem. By exploiting the inherent structure of DDM, we reveal that the estimation of the unknown delay-Doppler parameters can be regarded as a two-dimensional (2D) line spectrum estimation problem. Conventional 2D approaches [13, 14] can be used to find the estimates of the delay and Doppler parameters. Instead of doing by these approaches, this paper develops an atomic norm minimization-based approach [15–17] which has been shown to have high-resolution performance recently. Toward this, we first define an atom as a rank-one parametrized matrix and then recover the low-rank DDM via atomic norm minimization with a semi-definite matrix formulation. After recovering the DDM, we determine and pair the delay and Doppler parameters of the radar targets. Simulations demonstrate that the proposed method significantly improves the delay-Doppler estimation performance.

Before proceeding to the main context, we recall several related works. The work in [18] applies the atomic norm minimization-based theory to the delay-Doppler estimation, but it does not address the sub-Nyquist sampling and imposes constraints on the radar transmit waveforms. The recent works [19, 20] study the 2D line spectrum estimation under the framework of atomic norm. The differences in our work are that (1) a new atom is defined with which the optimization problem size is greatly reduced and (2) a pairing method to pair the delay and Doppler parameters is presented. Then the proposed method is computationally more efficient.

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2. SUB-NYQUIST RADAR MODEL AND PROBLEM FORMULATION

2.1. Signal Model

In this paper, we consider a radar transceiver that transmits a pulse train

$$s(t) = \sum_{n=0}^{N-1} g_n(t - nT), 0 \leq t \leq NT, \tag{1}$$

where N is the number of transmitted pulses, T is the pulse repetition interval (PRI), and $g_n(t)$ is a known pulsed signal with its pulse-width T_p ($T_p < T$) and bandwidth B.

The target scene is composed of K non-fluctuating point targets satisfying the stop-and-hop assumption [21], where K is generally unknown. Then the received signal can be written as:

$$x(t) = \sum_{n=0}^{N-1} x_n(t), \tag{2}$$

where $x_n(t)$ is the radar echo corresponding to the n-th pulse ($n = 1, 2, \cdots, N-1$),

$$x_n(t) = \sum_{k=1}^{K} \alpha_k g_n(t - nT - \tau_k) e^{j2\pi\nu_k nT}. \tag{3}$$

Throughout, we assume that the radar is operated in an unambiguous time-frequency region, *i.e.*, $|\nu_k| < 1/2T$ and $0 < \tau_k < T$.

2.2. Sub-Nyquist Radar

The AIC considered in this paper is shown in Fig.1, which consists of a random spectrum-spreading signal $p(t)$, a low-pass filter $h(t)$ and a low-rate ADC. The structure is a typical complex random demodulator [2] and the basic component in many other AICs [3, 4, 6].

During the n-th pulse interval, the AIC outputs the sample $y_n[m] = y_n(mT_{cs})$,

$$y_n(t) = \int_{nT}^{(n+1)T} h(\tau)p(t-\tau)x_n(t-\tau)d\tau, \tag{4}$$

where T_{cs} is the sampling interval. The aim of this paper is to detect and resolve the K delay-Doppler pairs $\{\tau_k, \nu_k\}_{k=1}^{K}$ from the set of sampling vectors $\{\mathbf{y}_n\}_{n=0}^{N-1}$ with $\mathbf{y}_n = [y_n[0], y_n[1], \cdots, y_n[M-1]]^T$.

To simplify the analysis, we make the following assumptions:

- The random spectrum-spreading signal is a T-periodic signal $p(t) = \sum_{l=-L_p}^{L_p} \rho_l e^{j\pi lt/T}$.
- The low-pass filter $h(t)$ is ideal with bandwidth $B_{cs}/2$.
- The sampling interval $T_{cs} = 1/B_{cs} = T/M$, where M is an integer.

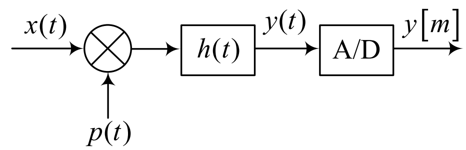

Fig. 1. The structure of the AIC.

3. DELAY-DOPPLER ESTIMATION VIA ATOMIC NORM MINIMIZATION

3.1. Frequency-Domain Representation of Sub-Nyquist Radar

Denote the continuous-time Fourier transforms of $y_n(t)$, $x_n(t)$ and $g_n(t)$ as $\tilde{y}_n(f)$, $\tilde{x}_n(f)$ and $\tilde{g}_n(f)$, respectively. Under the assumptions in 2.2, $\tilde{y}_n(f)$ can be computed as

$$\tilde{y}_n(f) = \sum_{l=-L_0}^{L_0} \rho_l \tilde{x}_n(f - lf_p), f \in [-B_{cs}/2, B_{cs}/2], \tag{5}$$

where $f_p = 1/T$, $L_0 = \lceil (B + B_{cs})T/2 \rceil$. With (3), we have $\tilde{x}_n(f)$ as

$$\tilde{x}_n(f) = \sum_{k=1}^{K} \alpha_k e^{j2\pi\nu_k nT} \tilde{g}_n(f) e^{-j2\pi f\tau_k}. \tag{6}$$

Here we remove the fixed phase due to the delay nT.

Define $\tilde{y}_n[m]$ as the spectrum samples of $\tilde{y}_n(f)$ at the sampling frequencies $f_{(m)} = B_{cs}/2 - mf_p$. Then we get

$$\begin{aligned} \tilde{y}_n[m] &= \sum_{l=-L_0}^{L_0} \rho_l \tilde{x}_n(f_{(m-l)}) \\ &= \sum_{l=L_1}^{L_2} \rho_{m-l} \tilde{x}_n(f_{(l)}), \end{aligned} \tag{7}$$

where $L_1 = \max\{-L_0, \lfloor (B_{cs} - B)T/2 \rfloor\}$ and $L_2 = \min\{M - 1 + L_0, \lfloor (B_{cs} + B)T/2 \rfloor\}$.

Let $\tilde{\mathbf{y}}_n = [\tilde{y}_n[0], \tilde{y}_n[1], \cdots, \tilde{y}_n[M-1]]^T$. Substituting (6) into (7), we have the following matrix representation:

$$\tilde{\mathbf{y}}_n = \mathbf{P}\mathbf{G}_n\mathbf{W}(\tau)\mathbf{B}\mathbf{V}^T(\nu)\mathbf{e}_n, \tag{8}$$

where $\mathbf{e}_n$ denotes the n-th column of the N-dimension identity matrix, $\mathbf{G}_n$ and $\mathbf{B}$ are two diagonal matrices,

$$\mathbf{G}_n = \text{diag}([\tilde{g}_n(f_{(L_1)}), \cdots, \tilde{g}_n(f_{(L_2)})]), \tag{9}$$

$$\mathbf{B} = \text{diag}([\beta_1, \cdots, \beta_K]), \tag{10}$$

with $\beta_k = \alpha_k e^{-j2\pi f_{(L_1)}\tau_k}$, the matrix $\mathbf{P} \in \mathbb{C}^{M \times L}$ is a partial Hankel matrix with $L = L_2 - L_1 + 1$,

$$\mathbf{P} = \begin{bmatrix} \rho_{1-L_1} & \rho_{2-L_1} & \cdots & \rho_{1+L_2} \\ \rho_{2-L_1} & \rho_{2-L_1} & \cdots & \rho_{2+L_2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{M-L_1} & \rho_{M+1-L_1} & \cdots & \rho_{M+L_2} \end{bmatrix}. \tag{11}$$

And $\mathbf{W}(\tau) = [\mathbf{w}(f_p\tau_1), \cdots, \mathbf{w}(f_p\tau_K)] \in \mathbb{C}^{L\times K}$ and $\mathbf{V}(\nu) = [\mathbf{v}(\nu_1 T), \cdots, \mathbf{v}(\nu_K T)] \in \mathbb{C}^{N\times K}$ are two Vandermonde matrices with

$$\mathbf{w}(\theta) = [1, e^{j2\pi\theta}, \cdots, e^{j2\pi(L-1)\theta}]^T, \tag{12}$$

$$\mathbf{v}(\theta) = [1, e^{j2\pi\theta}, \cdots, e^{j2\pi(N-1)\theta}]^T. \tag{13}$$

Let $\theta_k, \vartheta_k \in (-1/2, 1/2]$ with $\theta_k = f_p\tau_k - 1/2$ and $\vartheta_k = -\nu_k T$. Then (8) can be equivalently represented as

$$\begin{aligned}\tilde{\mathbf{y}}_n &= \mathbf{M}_n\mathbf{W}(\theta)\mathbf{B}\mathbf{V}^H(\vartheta)\mathbf{e}_n\\ &= \mathbf{M}_n\mathbf{X}\mathbf{e}_n,\end{aligned} \tag{14}$$

where $\mathbf{M}_n = \mathbf{P}\mathbf{G}_n\text{diag}([1, -1, \cdots, (-1)^{L-1}])$, $\mathbf{X} = \mathbf{W}(\theta)\mathbf{B}\mathbf{V}^H(\vartheta)$. Let $\mathbf{m}_{m,n}^H$ be the m-th row of the matrix $\mathbf{M}_n$. We can represent the m-th element of $\tilde{\mathbf{y}}_n$ as

$$\begin{aligned}\tilde{y}_n[m] &= \sum_{k=1}^{K} \beta_k \mathbf{m}_{m,n}^H \mathbf{w}(\theta_k)\mathbf{v}^H(\vartheta_k)\mathbf{e}_n\\ &= \text{Tr}\big(\mathbf{e}_n\mathbf{m}_{m,n}^H \sum_{k=1}^{K}\beta_k\mathbf{w}(\theta_k)\mathbf{v}^H(\vartheta_k)\big)\\ &= \langle \mathbf{X}, \mathbf{m}_{m,n}\mathbf{e}_n^H\rangle,\end{aligned} \tag{15}$$

where $\langle \mathbf{A}, \mathbf{B}\rangle \triangleq \text{tr}(\mathbf{B}^H\mathbf{A})$. Then the output of AIC system in N PRIs can be expressed as,

$$\tilde{\mathbf{y}} = \mathcal{B}(\mathbf{X}) \tag{16}$$

where $\tilde{\mathbf{y}} = [\tilde{\mathbf{y}}_0^T, \tilde{\mathbf{y}}_1^T, \cdots, \tilde{\mathbf{y}}_{N-1}^T]^T \in \mathbb{C}^{MN}$, and $\mathcal{B}$ is the linear operator mapping the matrix $\mathbf{X}$ to the vector $\tilde{\mathbf{y}}$.

It is noted that the delay and Doppler parameters of the radar targets are completely contained in the matrix $\mathbf{X}$. For convenience, we call the matrix $\mathbf{X}$ as the delay-Doppler matrix (DDM). In sparse radar scenarios, *i.e.*, $K \ll \min\{L, N\}$, $\mathbf{X}$ is a low-rank matrix. From (16), we can know that the linear map $\mathcal{B}$ depicts the AIC system, which maps the DDM to the AIC output vector. Therefore, our problem is equivalent to recovering the low-rank DDM $\mathbf{X}$ from the linear mapping (16). In the following subsection, we recover the low-rank matrix via the atomic norm minimization.

3.2. Delay-Doppler Matrix Recovery via Atomic Norm Minimization

To recover the DDM, we define a set of atoms as:

$$\mathcal{A} \triangleq \{\text{A}(\boldsymbol{\theta}, \phi) = e^{j\phi}\mathbf{w}(\theta)\mathbf{v}^H(\vartheta) : \theta, \vartheta \in \mathbb{T}, \phi \in \mathbb{S}\},$$

where $\boldsymbol{\theta} = \{\theta, \vartheta\}$, $\mathbb{T} \triangleq (-1/2, 1/2]$ and $\mathbb{S} \triangleq (0, 2\pi]$. Then the atomic l_0 norm of the matrix $\mathbf{X}$ is defined as the smallest number of atoms in $\mathcal{A}$ that can express $\mathbf{X}$:

$$\|\mathbf{X}\|_{\mathcal{A},0} = \inf\left\{\mathcal{K} : \mathbf{X} = \sum_{k=1}^{\mathcal{K}} c_k\text{A}(\boldsymbol{\theta}_k, \phi_k), c_k > 0\right\} \tag{17}$$

The following theorem states that the atomic l_0 norm $\|\mathbf{X}\|_{\mathcal{A},0}$ can be cast as an equivalent rank minimization problem.

Theorem 1 $\|\mathbf{X}\|_{\mathcal{A},0}$ *defined in (17) equals the optimal value of the following rank minimization problem:*

$$\begin{aligned}\min_{\mathbf{u}_1\in\mathbb{C}^L, \mathbf{u}_2\in\mathbb{C}^N} \quad & \text{rank}(\mathbf{M}),\\ \textit{s.t.} \quad & \mathbf{M} = \begin{bmatrix}\text{toep}(\mathbf{u}_1) & \mathbf{X}\\ \mathbf{X}^H & \text{toep}(\mathbf{u}_2)\end{bmatrix} \geq 0,\end{aligned} \tag{18}$$

where $\text{toep}(\mathbf{u})$ *denotes the Toeplitz matrix with* $\mathbf{u}^T$ *as its first row.*

The proof relies on the Vandermonde decomposition of Toeplitz matrix and is omitted here due to the space limitation.

It immediately follows from Theorem 1 that the DDM recovery can be cast as the following rank minimization problem:

$$\begin{aligned}\min_{\mathbf{u}_1\in\mathbb{C}^L, \mathbf{u}_2\in\mathbb{C}^N, \mathbf{X}} \quad & \text{rank}(\mathbf{M}),\\ \text{s.t.} \quad & \mathbf{M} = \begin{bmatrix}\text{toep}(\mathbf{u}_1) & \mathbf{X}\\ \mathbf{X}^H & \text{toep}(\mathbf{u}_2)\end{bmatrix} \geq 0,\\ & \tilde{\mathbf{y}} = \mathcal{B}(\mathbf{X}).\end{aligned} \tag{19}$$

It is noted that the rank minimization is nonconvex. One way is to approximately compute it through the reweighted strategy [22]. In this paper, we relax $\text{rank}(\mathbf{M})$ to $\text{trace}(\mathbf{M})$ and solve it by convex optimization toolbox CVX [23]. In solving the optimization problem, we have to optimize an $L+N$-dimensional vector $[\mathbf{u}_1^T, \mathbf{u}_2^T]^T$ and an $L \times N$-dimensional matrix $\mathbf{X}$. Compared with the atomic norm-based approaches in [19, 20], which generally require to optimize an LN-dimensional vector along with the matrix $\mathbf{X}$, the problem size is largely reduced. Therefore, the problem here is more computationally efficient.

3.3. Delay-Doppler Parameters Estimation and Pairing

By solving the relaxed version of problem (19), we get the optimal matrix $\mathbf{M}^*$ and two Toeplitz matrices $\text{toep}(\mathbf{u}_1^*)$ and $\text{toep}(\mathbf{u}_2^*)$. At the same time, the optimal K^*, *i.e.*, the number of delay-Doppler pairs, can be also determined from the rank of $\mathbf{M}^*$. By performing the Vandermonde decomposition of $\text{toep}(\mathbf{u}_1^*)$ and $\text{toep}(\mathbf{u}_2^*)$, we can acquire two sets of frequencies $\{\theta_1^*, \theta_2^*, \cdots, \theta_{K_1}^*\}$ and $\{\vartheta_1^*, \vartheta_2^*, \cdots, \vartheta_{K_2}^*\}$ ($K_1, K_2 \leq K^*$), which corresponds to different delay and Doppler parameters, respectively. Now our problem is to determine the K^* delay-Doppler pairs from the two sets.

It is noted that the rank-K^* matrix $\mathbf{M}^*$ can be decomposed as

$$\mathbf{M}^* = \begin{bmatrix}\mathbf{U}_1\\ \mathbf{U}_2\end{bmatrix}\begin{bmatrix}\mathbf{U}_1\\ \mathbf{U}_2\end{bmatrix}^H, \tag{20}$$

where $\mathbf{U}_1 \in \mathbb{C}^{L\times K^*}$ and $\mathbf{U}_2 \in \mathbb{C}^{N\times K^*}$. On the other hand, the Toeplitz matrices $\text{toep}(\mathbf{u}_1^*)$ and $\text{toep}(\mathbf{u}_2^*)$ can also be uniquely decomposed as:

$$\text{toep}(\mathbf{u}_1^*) = \mathbf{W}_1\boldsymbol{\Sigma}_1\mathbf{W}_1^H, \tag{21}$$

$$\operatorname{toep}(\mathbf{u}_2^*) = \mathbf{V}_2\mathbf{\Sigma}_2\mathbf{V}_2^H, \tag{22}$$

where $\mathbf{W}_1 = [\mathbf{w}(\theta_1^*), \mathbf{w}(\theta_2^*), \cdots, \mathbf{w}(\theta_{K_1}^*)]$ and $\mathbf{V}_1 = [\mathbf{v}(\vartheta_1^*), \mathbf{v}(\vartheta_2^*), \cdots, \mathbf{v}(\vartheta_{K_2}^*)]$. Since $\operatorname{toep}(\mathbf{u}_1^*) = \mathbf{U}_1\mathbf{U}_1^H$ and $\operatorname{toep}(\mathbf{u}_2^*) = \mathbf{U}_2\mathbf{U}_2^H$, there exist $\mathbf{O}_1 \in \mathbb{C}^{K_1^* \times K^*}$ and $\mathbf{O}_2 \in \mathbb{C}^{K_2^* \times K^*}$ satisfying $\mathbf{U}_1 = \mathbf{W}_1\mathbf{\Sigma}_1^{1/2}\mathbf{O}_1$ and $\mathbf{U}_2 = \mathbf{V}_2\mathbf{\Sigma}_2^{1/2}\mathbf{O}_2$. Then $\mathbf{X}$ can be represented as

$$\mathbf{X} = \mathbf{W}_1\mathbf{\Sigma}_1^{1/2}\mathbf{O}\mathbf{\Sigma}_2^{1/2}\mathbf{V}_2^H, \tag{23}$$

where the matrix $\mathbf{O} = \mathbf{O}_1\mathbf{O}_2^H \in \mathbb{C}^{K_1^* \times K_2^*}$ establishes the relationship between the two frequency sets. Therefore, we use it to pair the delays and Doppler frequencies. According to (23), the matrix $\mathbf{O}$ can be estimated as $\mathbf{O} = \mathbf{\Sigma}_1^{-1/2}\mathbf{W}_1^\dagger\mathbf{X}\mathbf{V}_2^\dagger\mathbf{\Sigma}_2^{-1/2}$, where $(\cdot)^\dagger$ denotes the Moore-Penrose pseudoinverse. In the ideal case, a nonzero element o_{ij} in $\mathbf{O}$ demonstrates there exits a pair of $\{\theta_i^*, \vartheta_j^*\}$. In practice, we can select the positions of the K^* largest elements in $\mathbf{O}$ to determine the K^* delay-Doppler pairs.

4. SIMULATIONS

We now present simulation results of the proposed delay-Doppler estimation algorithm for sub-Nyquist radar system. The radar transmits a linear frequency modulated signal with bandwidth $B = 25$MHz and pulse-width $T_p = 2\mu$sec and samples the radar echoes at one quarter of the Nyquist rate, *i.e.*, $B_{cs} = 6.25$MHz. Other parameters used are PRI $T = 4\mu$sec and number of pulses $N = 50$. Target delays and Doppler frequencies are randomly spread in the unambiguous region with uniform distribution. We compared the proposed method (denoted as "ANM") with the Doppler focusing (DF) [8] and the discretization-based method (denoted as "HiRes") in [7] (where the discrete grid is chosen as one-fifth of the Nyquist bin). We take the relative root mean square error (RMS), which is normalized to the Nyquist bin, as a metric to evaluate the performance.

Figs.2 and 3 demonstrate the RMS error performance of the different methods for various K and SNR values, respectively. From Fig.2 we see that the proposed method is superior to the other methods for the different K values. As shown in Fig.3, the proposed method improves the delay and Doppler estimation performance significantly when SNR is above 5dB and the RMS error decreases as SNR increases. However, due to the off-the-grid effect, the performance of the other methods keeps unchanged as SNR increases from 0dB to 50dB.

5. CONCLUSION

In this paper, we propose a sub-Nyquist radar delay-Doppler estimation approach in the recent super-resolution framework [15]. With the recovered DDM via the atomic norm minimization, the proposed approach can effectively resolve the delay and Doppler parameters with high accuracy. Simulations show the performance advantages of the proposed approach. As noted in (1), we do not impose any constraint on the waveforms and therefore, the proposed approach is also applicable to the pulse diversity radar systems.

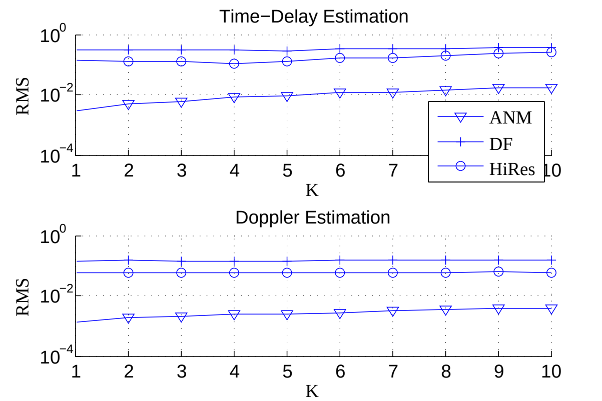

Fig. 2. RMS error of delay and Doppler estimation versus the number of targets, K, with SNR=20dB.

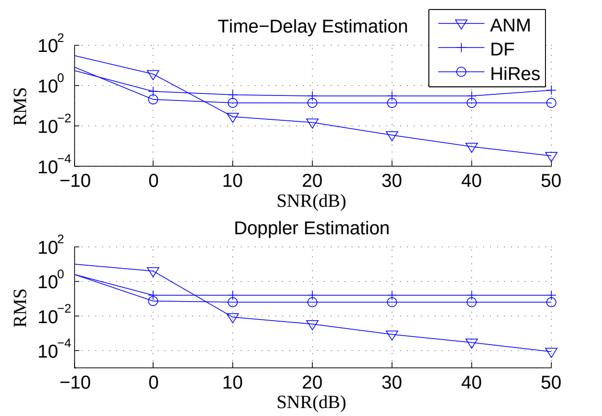

Fig. 3. RMS error of delay and Doppler estimation versus SNR with $K = 5$.

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