GAME THEORETIC RESOURCE ALLOCATION FOR m-dependent channels with Application to ofdma

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ABSTRACT

In this paper we consider the problem of channel allocation for users who access a common channel using OFDMA. The spectrum is divided into subchannels and we assume that the bandwidth of each subchannel is smaller than the coherence bandwidth. This leads to correlations between the channel coefficients for each user. We model these correlated channels as an *m*-dependent sequence and generate an interference game at random, according to some marginal fading distribution. Performance is measured by the sum of achievable rates. Using a novel analysis of the random pure NE of the game, we prove that even for correlated channels the Mfrequency selective interference game, suggested in previous work, has only Nash equilibria that exhibit good performance with high probability, asymptotically with the number of users. This game is the basis for an asymptotically optimal and fully distributed OFDMA channel allocation algorithm, presented in simulations.

Index Terms— Resource Allocation, Game Theory, Nash Equilibrium, Random Games, Correlated Channels

I. INTRODUCTION

Channel allocation in large-scale networks is a major challenge [1]. The task of assigning each user a different channel is traditionally performed by a central entity in the network. In ad-hoc networks, or simply large-scale networks, this approach is infeasible. This task becomes even more involved when the designer wants to allocate each user a channel that specially suits him, because channel state information has to be acquired for all users. For this reason, many works have tried to solve the problem of channel allocation distributedly [2], [3], [4]. Game theory is an excellent choice for an analytical foundation for such distributed algorithms [5], [2], [6], [7], [8].

In previous work [9] we suggested a game formulation for the channel allocation problem called the M-frequency selective interference game (M-FSIG). We generated the interference game at random according to some fading distribution, and proved that the probability that the M-FSIG only exhibits optimal pure Nash Equilibria (NE), in the sum-rate sense, approaches one as the number of users approaches infinity. Using this game, any algorithm that can converge to some pure NE will have asymptotically optimal performance. We proposed a modified version of the fictitious play algorithm [10] as such a fully distributed algorithm that requires no communication at all between users.

Our previous results for the M-FSIG assumed i.i.d. channel coefficients. Although this model is convenient to analyze, it is

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inaccurate for many communication scenarios since the channel coefficients may be correlated. Correlations between channel coefficients of different frequency bands can occur if the distance between their carrier frequencies is smaller than the coherence bandwidth of the channel. If the frequency bands all have the same bandwidth, this distance is equal to this bandwidth. One such significant case is Orthogonal Frequency Division Multiplexing Access (OFDMA) [11], which is used in LTE in the allocation of resource blocks (see [12]).

This means that the channel coefficients of close enough frequency bands are indeed not statistically independent, but those of frequency bands that are separated by more than the coherence bandwidth can be considered independent. The notion of mdependent sequences captures this effect well [13]. Furthermore, it allows for a generalization of the term "channel" to time-slots in addition to frequency bands, since they can be thought of as highly-correlated (or identical) channels.

In this paper we extend the random pure NE analysis of [9] to the case of m-dependent channel coefficients.

This paper is organized as follows. In Section II we formulate the problem, and in section III we generalize the NE existence results of the M-FSIG to the case of *m*-dependent channels. This generalization enables the application of the M-FSIG as the basis for a fully distributed OFDMA channel allocation algorithm, and simulations of such an algorithm are presented in Section IV. Section V concludes this paper.

II. PROBLEM FORMULATION

Consider a wireless network consisting of N transmitter-receiver pairs (users) and N channels. Each user forms a link between his transmitter and receiver using a single channel. This assumption is for simplicity alone. Our game theoretic formulation allows for a straight forward generalization to multiple channels per user. One can separate the notions of user and player, by thinking of b distinct games that take place simultaneously. Each user can subscribe a player for each of these games, so we have b games of Nplayers and N channels in each. Our results apply to each of these games independently, and also suggest that this separation entails no asymptotic loss compared to a single game with N players and bN channels, as long as b is fixed with respect to N.

The channel between each transmitter and receiver is Gaussian frequency-selective and we assume that each frequency band is smaller than the coherence bandwidth of the channel. We also assume that the coherence time is significantly larger than the convergence time of our proposed algorithm, so that the channel gains can be considered static in our analysis.

The channel coefficients are modeled as N^3 random variables - one for each channel, each transmitter and each receiver. The coefficient between user *i*'s transmitter and user *j*'s receiver in channel *k* is denoted $h_{i,j,k}$. We assume that $h_{i,j,1}, ..., h_{i,j,N}$ are identically distributed for each i, j = 1, ..., N, but not independent. However, in order to obtain a non-vanishing transmission rate for each user we expect the bandwidth of a single frequency band to be non-decreasing with *N*. Hence, the coherence bandwidth of the channel is at most *m* times larger than the bandwidth of a frequency band, for some integer *m* that is fixed with respect to *N*. For this reason, only frequency bands with index k_2 such that $|k_2 - k_1| \le m$ are correlated with the k_1 -th frequency band. Based on this reasoning, we assume instead that $h_{i,j,1}, ..., h_{i,j,N}$ are *m*-dependent for some integer $m \ge 0$.

Definition 1. A random process $\{X_i\}$ is said to be m-dependent if and only if for each i, j such that |i - j| > m the variables X_i and X_j are statistically independent.

We also assume that $h_{i_1,j_1,k}$ and $h_{i_2,j_2,k}$ are independent for each k = 1, ..., N if $i_1 \neq i_2$ or $j_1 \neq j_2$. Note that N^2 of these coefficients serve as channel coefficients between a transmitter and receiver pair and are denoted for convenience by $h_{i,k}$ for user *i* in channel *k*. The other $N^2(N-1)$ coefficients serve as interference coefficients between transmitters and unintended receivers.

Each user has some preferred order of the N channels. Due to the independence of the channel coefficients between users, these preference lists are independent between users. Note that this preference order only considers the absolute value of the channel coefficient and not the interference. We denote by $h_{i,(N-q+1)}$ the q-th best channel coefficient for user i (so $h_{n,(1)}$ is the worst channel).

We assume that each user has perfect channel state information (CSI) of all his N channel coefficients, which he can achieve using standard estimation techniques. In addition, we assume that each user can sense the exact interference he experiences in each channel. Nevertheless, users do not have any knowledge concerning the channel coefficients of other users or the specific interference coefficients. There is no central entity of any sort that knows the channel gains of all users.

Our global performance metric is the sum of achievable rates while treating interference as noise. Denote by a the allocation vector (strategy profile), such that $a_i = k$ if user *i* is using channel *k*. We want to maximize the following performance function over all possible allocations

$$W(\mathbf{a}) = \sum_{i=1}^{N} \log_2 \left(1 + \frac{P_i |h_{i,a_i}|^2}{N_0 + I_{i,a_i}(\mathbf{a}_{-i})} \right)$$
(1)

where N_0 is the Gaussian noise variance which is assumed to be the same for all users, P_i is user *i*'s transmission power and $I_{i,k}(\mathbf{a}_{-i}) = \sum_{j|a_j=k} |h_{j,i,k}|^2 P_j$ is the interference user *i* experiences in channel *k*.

Our approach to the distributed channel allocation problem formulated above is game theoretic. This means we suggest a game formulation, and prove it only has asymptotically optimal NE. This guarantees that any algorithm that can converge to some pure NE will exhibit a close to optimal performance, obviating the need for equilibria selection. We suggest a modified version of fictitious play as such an algorithm that requires no communication between users to converge, and converges very fast.

III. THE M-FREQUENCY SELECTIVE INTERFERENCE GAME FOR CORRELATED CHANNELS

In this section we introduce the M-Frequency Selective Interference Game (M-FSIG). This game was shown to only have good NE for i.i.d channels, with a probability that approaches one as the number of users approaches infinity. Here we aim to generalize this result to the case of m-dependent channels. This game is a result of a utility design that is specially crafted for the problem of channel allocation [9]. The purpose of this design is to maximize the global performance (given by (1)) of the worst pure NE of the game. For each player, this designed utility is greater than zero only for his M best channels, and be equal for them.

Definition 2. The M-Frequency-Selective Interference Game (M-FSIG) is a normal-form game with parameter M > 0 and N players, where each has the set $A_i = \{1, 2, ..., N\}$ as a strategy space. The utility function for player i is

$$u_{i}\left(\mathbf{a}\right) = \begin{cases} \log_{2}\left(1 + \frac{P_{i}|h_{i,(N-M+1)}|^{2}}{N_{0}+I_{i,a_{i}}\left(\mathbf{a}-i\right)}\right) & \frac{|h_{i,a_{i}}|}{|h_{i,(N-M+1)}|} \geq 1\\ 0 & else \end{cases}$$
(2)

Note that maximizing u_i (a) over A_i is equivalent to minimizing I_{i,a_i} (\mathbf{a}_{-i}) over the set $\mathcal{M}_i = \left\{ k \mid \frac{|h_{i,k}|}{|h_{i,(N-M+1)}|} \ge 1 \right\}$. The following definition is needed for our proofs.

Definition 3. Define the tail quantile function as $\bar{q}_X(p) = F_X^{-1}(1-p) = \min \{x \mid F_X(x) \ge 1-p\}.$

Next we prove that even if the channel coefficients are *m*-dependent, there exists an increasing number of pure NE with asymptotically optimal performance.

Theorem 4. Assume that $\{h_{i,j,k}\}$ are independent for different i, jand $h_{i,1}, ..., h_{i,N}$ are identically distributed and m-dependent for each i, for some m > 0. Let $\mathcal{M}_i = \left\{k \mid \frac{|h_{i,k}|}{|h_{i,(N-M+1)}|} \ge 1\right\}$. If the M-FSIG parameter is chosen such that $M_N \ge (m+1)(3 + \varepsilon) \ln(N)$ for some $\varepsilon > 0$, then the probability there are at least $M_N!$ perfect matchings between users and channels, such that each user i gets a channel from \mathcal{M}_i , approaches one as $N \to \infty$.

Proof: Consider a random bipartite graph of the set of users and the set of channels. A user node *i* is connected to all channels in \mathcal{M}_i so his degree is exactly M_N . We want to lower bound the probability that a channel is a good channel for user *i*, and by so doing evaluate the minimum degree of a channel node. Denote $L = \bar{q}_X \left(\frac{M_N}{e(N+m+1)}\right)$ and $X_k = |h_{i,k}|$. We use the Fréchet inequality to obtain

$$\Pr\left(k \in \mathcal{M}_{i}\right) \geq \Pr\left(X_{(N-M_{N}+1)} \leq L, X_{k} \geq L\right) \geq (a)$$
$$\Pr\left(X_{(N-M_{N}+1)} \leq L\right) + \Pr\left(X_{k} \geq L\right) - 1 \quad (3)$$

where (a) follows because if $X_k \ge L$ and $X_{(N-M_N+1)} \le L$ then $k \in \mathcal{M}_N$. We want to find a lower bound for the first probability

in (3). First invoke $F_N(x) = \sum_{k=1}^N I(X_k \le x)$ on both sides of $X_{(N-M_N+1)} > L$ to obtain

$$\Pr\left(X_{(N-M_N+1)} > L\right) = \Pr\left(N - M_N + 1 > \sum_{k=1}^N I\left(X_k \le L\right)\right) = \Pr\left(\sum_{k=1}^N I\left(X_k > L\right) \ge M_N\right) \quad (4)$$

Now divide $X_1, ..., X_N$ into the m + 1 disjoint sets $\{X_1, X_{2+m}, ...\}, ..., \{X_{m+1}, X_{2+2m}, ...\}$ and define $Y_{i,j} = X_{j+(m+1)i}$ for j = 1, ..., m + 1 and all i such that $1 \leq j + (m+1)i \leq N$ for some j. So for large enough N we get

$$\Pr\left(\sum_{k=1}^{N} I\left(X_{k} > L\right) \ge M_{N}\right)$$

$$\leq \Pr\left(\bigcup_{j=1}^{m+1} \left\{\sum_{i=0}^{s_{j}} I\left(Y_{i,j} > L\right) \ge \frac{M_{N}}{m+1}\right\}\right)$$

$$\leq (m+1)\Pr\left(\sum_{i=0}^{\max s_{j}} I\left(Y_{i,1} > L\right) \ge \frac{M_{N}}{m+1}\right) \quad (5)$$

where

$$s_{j} = \begin{cases} \left\lfloor \frac{N}{m+1} \right\rfloor - 1 & j > N - \left\lfloor \frac{N}{m+1} \right\rfloor (m+1) \\ \left\lfloor \frac{N}{m+1} \right\rfloor & else \end{cases}$$

and (a) follows because some inner sum must be greater than $\frac{M_N}{m+1}$ in order for the total sum to be greater than M_N , and (b) from the union bound. We can apply a concentration upper bound on the last term, denoting the success probability of the corresponding Bernoulli process $Z_i = I(Y_{i,1} > L)$ by $p = 1 - F_X(L) = \frac{M_N}{e(N+m+1)}$ and $S = \max_j s_j + 1$. According Theorem A.1.12 in [14], for all N > 0 and $\beta > 1$

$$\Pr\left(\sum_{i=0}^{\max s_j} Z_i \ge \beta pS\right) \le \left(e^{\beta-1}\beta^{-\beta}\right)^{pS}.$$
 (6)

where here

$$\beta = \frac{\frac{M_N}{m+1}}{\frac{M_NS}{e(N+m+1)}} \ge \frac{\frac{M_N}{m+1}}{\frac{M_N(\frac{N}{m+1}+1)}{e(N+m+1)}} = e$$
(7)

where (a) is due $S = \max_{i} s_{i} + 1 \le \frac{N}{m+1} + 1$. We obtain

$$\Pr\left(\sum_{i=0}^{\max s_j} Z_i \ge \frac{M_N}{m+1}\right) \leq (e^{e-1}e^{-e})^{S\frac{M_N}{e(N+m+1)}} \leq e^{-\frac{M_N}{e(m+1)}\frac{1-\frac{m+1}{N}}{1+\frac{m+1}{N}}}$$
(8)

where (a) is due to (6) and (7) and (b) due to $S = \max_{j} s_{j} \ge \frac{N}{m+1} - 1$. If $M_{N} \ge (m+1)(3+\varepsilon)\ln(N)$ for some $\varepsilon > 0$ then by (4),(5) and (8) we conclude that for large enough N

$$\Pr\left(X_{(N-M_N+1)} > L\right) \le (m+1) e^{-\frac{M_N}{e(m+1)} \frac{1-\frac{m+1}{N}}{1+\frac{m+1}{N}}} \le (m+1) e^{-\frac{(3+\varepsilon)\ln(N)}{3}} = \frac{m+1}{N^{1+\frac{\varepsilon}{3}}} \quad (9)$$

where (a) follows since for large enough N the inequality $\frac{1-\frac{m+1}{N}}{1+\frac{m+1}{N}} \geq \frac{e}{3}$ holds. Using the above bound on (3) we conclude that for large enough N

$$\Pr\left(k \in \mathcal{M}_{i}\right) \geq \frac{M_{N}}{e\left(N+m+1\right)} - \frac{m+1}{N^{1+\frac{\varepsilon}{3}}} \geq \frac{M_{N}}{3N}$$
(10)

where (a) uses $\Pr(X_k \ge L) = \frac{M_N}{e(N+m+1)}$ and (9). Since channel coefficients of different users are independent, the probability that the degree of vertex k is less than two is given by the binomial distribution of the number of users who prefer channel k, with $p = \Pr(k \in \mathcal{M}_i) \ge \frac{M_N}{3N}$, so we obtain

$$\Pr\left(\deg(k) < 2\right) \le \left(1 - \frac{M_N}{3N}\right)^N + N \frac{M_N}{3N} \left(1 - \frac{M_N}{3N}\right)^{N-1}.$$
(11)

Using the union bound on the channel vertices we obtain

$$\Pr\left(\min_{k} \deg(k) < 2\right) \leq N\left(1 - \frac{M_{N}}{3N}\right)^{N} + N\frac{M_{N}}{3}\left(1 - \frac{M_{N}}{3N}\right)^{N-1} \leq \frac{1}{(a)}$$

$$Ne^{-\frac{M_{N}}{3}} + \frac{3}{2}N\frac{M_{N}}{3}e^{-\frac{M_{N}}{3}} \leq \frac{2(m+1)\left(1 + \frac{\varepsilon}{3}\right)\ln(N) + 1}{N^{(m+1)\left(1 + \frac{\varepsilon}{3}\right) - 1}}$$
(12)

Due to $\ln(1-x) \leq -x$ for all x < 1, the inequality $\ln\left(1-\frac{M_N}{N}\right)^N \leq -N\frac{M_N}{N}$ holds and so does $\left(1-\frac{M_N}{N}\right)^N \leq e^{-M_N}$ which is used in (a), together with $1-\frac{M_N}{3N} \geq \frac{2}{3}$. Inequality (b) follows by assuming $M_N \geq (m+1)(3+\varepsilon)\ln(N)$ with some $\varepsilon > 0$ and using the monotonicity of $Ne^{-\frac{M_N}{3}} + N\frac{M_N}{3}e^{-\frac{M_N}{3}}$.

We conclude that $\Pr\left(\min_{k} \deg(k) < 2\right) \to 0$ as $\stackrel{\circ}{N} \to \infty$. We know from [15, Theorem 1] that given $\min_{k} (\deg(k)) \ge 2$, the probability that a perfect matching exists approaches 1 as $N \to \infty$. This also guarantees that at least $M_N!$ such perfect matchings exist (see [9]).

Now we turn to generalize the non-existence of bad equilibria to m-dependent channels. This is based on the fact that all pure NE are asymptotically almost a permutation between users and channels.

Definition 5. A shared channel is a channel that is chosen by more than one user, and a sharing user is a user that chose a shared channel.

Theorem 6. Assume that $\{h_{i,j,k}\}$ are independent for different i, jand $h_{i,1}, ..., h_{i,N}$ are identically distributed and m-dependent for each i, for some m > 0. Suppose $M_N \ge (m+1)(3+\varepsilon)\ln(N)$ for some $\varepsilon > 0$. If \mathbf{a}^* is a pure NE of the M-FSIG with N_c sharing users, then $\frac{N_c}{N} \to 0$ in probability as $N \to \infty$. Furthermore, $\max_{\mathbf{a}^* \in \mathcal{P}_e} \frac{N_c(\mathbf{a}^*)}{N} \to 0$ in probability as $N \to \infty$, where \mathcal{P}_e is the set of pure NE. *Proof:* Denote the set of empty channels by \mathcal{E} and its cardinality by E. Note that E is large if the number of sharing users is large, which means that the given strategy profile is far from being a permutation. A necessary condition for a strategy profile to be a pure NE is that no user wishes to switch to an empty channel. Denote $L = \bar{q}_X \left(\frac{M_N}{e(N+m+1)}\right)$ and $X_k = |h_{i,k}|$. Now observe that, for user *i*, at least $\frac{E}{m+1}$ channels in \mathcal{E} are independent. Denote their set of indices by \mathcal{I} . Hence

$$\Pr\left(\bigcup_{k\in E} \{X_k \ge L\}\right) \ge \\\Pr\left(\bigcup_{k\in \mathcal{I}} \{X_k \ge L\}\right) = 1 - \left(1 - \frac{M_N}{e(N+m+1)}\right)^{\frac{E}{m+1}}$$
(13)

where (a) is due to $\Pr(X_k \ge L) = \frac{M_N}{e(N+m+1)}$. We lower bound the probability that user *i* desires no empty channel by

$$\Pr\left(\mathcal{M}_{i} \cap \mathcal{E} \neq \emptyset\right) \underset{(a)}{\geq} \\ \Pr\left(X_{(N-M+1)} \leq L, \bigcup_{k \in E} \{X_{k} \geq L\}\right) \underset{(b)}{\geq} \\ \Pr\left(X_{(N-M_{N}+1)} \leq L\right) + \left(\Pr\left(\bigcup_{k \in E} \{X_{k} \geq L\}\right) - 1\right) \underset{(c)}{\geq} \\ 1 - (m+1) e^{-\frac{M_{N}}{3(m+1)}} - \left(1 - \frac{M_{N}}{e(N+m+1)}\right)^{\frac{E}{m+1}}$$
(14)

where (a) is since if $X_k \ge L$ and $X_{(N-M_N+1)} \le L$ then $k \in \mathcal{M}_i$ and (b) is the Fréchet inequality. Inequality (c) follows, for large enough N, from the bounds in (9) and (13). Under the condition that $M_N \ge (m+1)(3+\varepsilon)\ln(N)$, the rest of this proof coincides with the equivalent proof for i.i.d channels, described in [16]. It is based on the fact that the probability that a strategy profile with too many sharing users is a pure NE goes to zero as $N \to \infty$.

IV. SIMULATION RESULTS

In our simulations we used a Rayleigh fading network; i.e. $\{|h_{i,j,k}|\}$ are Rayleigh random variables that are independent for each different pair of i, j. For a specific pair of i and j, $h_{i,j,1}, ..., h_{i,j,N}$ were generated using the Extended Pedestrian A model (EPA, see [17]) for the excess tap delay and the relative power of each tap. The parameter m of their dependency is roughly given by $m \geq \frac{T_s N}{50\sigma_\tau}$ where T_s is the duration of a symbol and σ_τ is the delay spread of the channel, which is 143[ns]. All coefficients were normalized such that their second moment was one, $E\{||h_{i,j,k}|^2|\} = 1$. Three realizations of the frequency bands coefficients of a specific user for different values of m are depicted in Fig. 1.

In Fig. 2 we present the convergence of the Modified Fictitious Play (see [9] for details) with $\alpha = 0.5$, in a single network realization, for N = 128 and M = 25. The transmission powers were chosen such that the mean SNR for each link, in the absence of interference, was 20[dB]. Here we used $T_s = 10^{-8}$ so m = 4. Clearly, that convergence was very fast and occurred within 30 iterations. The ratio of the sum of achievable rates to that of an optimal allocation was close to 1, and the ratio of the minimal



Fig. 1. Realizations of the channels coefficients for different values of m



Fig. 2. Sum-rate and min-rate compared to the optimal permutation allocation sum-rate for a single realization

achievable rate was also reasonable. Note that there were two sharing users in the resulting pure NE.

We also simulated the effect of M on the convergence to a pure NE. We generated 50 networks at random, and for each let the users play with $M = \lceil c \ln(K) \rceil$ for c = 2, 3, 4, 5, 6, 7. As expected by our proofs, there was a threshold phenomenon for M. For c = 2, 3, 4 none of the dynamics converged whereas for c = 5, 6, 7 all of them did, with fewer than 50 iterations. The meanrates for c = 5, 6, 7 were 4.03, 3.90, 3.88 and the minimal rates were 1.30, 2.09, 2.05. Correlations between channel coefficients decrease the multi-user diversity of the channels, which reduced the mean-rate. On the other hand, fairness was enhanced thanks to the similarity between channels.

V. CONCLUSION

We addressed the problem of channel allocation where each user experiences correlations between the available channels. We modeled these correlated channels as an m-dependent sequences. We proved that even for m-dependent channels, a previously suggested game, the M-FSIG, exhibits only asymptotically optimal pure Nash equilibria, with a probability that approaches one as the number of users approaches infinity. This makes it possible to use a modified version of fictitious play as a fully distributed and asymptotically optimal algorithm for OFDMA channel allocation.

VI. REFERENCES

- Qing Zhao and Brian M Sadler, "A survey of dynamic spectrum access," *Signal Processing Magazine*, *IEEE*, vol. 24, no. 3, pp. 79–89, 2007.
- [2] Kobi Cohen, Amir Leshem, and Ephraim Zehavi, "Game theoretic aspects of the multi-channel ALOHA protocol in cognitive radio networks," *Selected Areas in Communications, IEEE Journal on*, vol. 31, no. 11, pp. 2276–2288., 2013.
- [3] Amir Leshem, Ephraim Zehavi, and Yoav Yaffe, "Multichannel opportunistic carrier sensing for stable channel access control in cognitive radio systems," vol. 30, no. 1, pp. 82–95, January 2012.
- [4] Oshri Naparstek and A. Leshem, "Fully distributed optimal channel assignment for open spectrum access," *Signal Processing, IEEE Transactions on*, vol. 62, no. 2, pp. 283–294., 2014.
- [5] Nie Nie and Cristina Comaniciu, "Adaptive channel allocation spectrum etiquette for cognitive radio networks," in *New Frontiers in Dynamic Spectrum Access Networks*, 2005. DySPAN 2005. 2005 First IEEE International Symposium on, 2005.
- [6] Amir Leshem and Ephraim Zehavi, "Cooperative game theory and the Gaussian interference channel," *Selected Areas in Communications, IEEE Journal on*, vol. 26, no. 7, pp. 1078– 1088., 2008.
- [7] Lingyang Song, Dusit Niyato, Zhu Han, and Ekram Hossain, "Game-theoretic resource allocation methods for device-todevice communication," *Wireless Communications, IEEE*, vol. 21, no. 3, pp. 136–144, 2014.
- [8] Zhu Han, Zhu James Ji, and KJ Liu, "Fair multiuser channel allocation for OFDMA networks using Nash bargaining solutions and coalitions," vol. 53, no. 8, pp. 1366–1376, August 2005.
- [9] Ilai Bistritz and A. Leshem, "Asymptotically optimal distributed channel allocation: a competitive game-theoretic approach," in *Communication, Control, and Computing (Allerton), 2015 53nd Annual Allerton Conference on, 2015.*
- [10] Julia. Robinson, "An iterative method of solving a game," Annals of mathematics, vol. 54, no. 2, pp. 296–301, 1951.
- [11] Kibeom Seong, Mehdi Mohseni, and John M Cioffi, "Optimal resource allocation for OFDMA downlink systems," in 2006 IEEE International Symposium on Information Theory, 2006.
- [12] Arunabha Ghosh, Jun Zhang, Jeffrey G Andrews, and Rias Muhamed, *Fundamentals of LTE*, Pearson Education, 2010.
- [13] Wassily Hoeffding, Herbert Robbins, et al., "The central limit theorem for dependent random variables," *Duke math. J*, vol. 15, no. 3, pp. 773–780, 1948.
- [14] Noga Alon and Joel H Spencer, *The probabilistic method*, John Wiley & Sons, 2015.
- [15] Alan Frieze and Páll Melsted, "Maximum matchings in random bipartite graphs and the space utilization of Cuckoo hash tables," *Random Structures & Algorithms*, vol. 41, no. 3, pp. 334–364, 2012.
- [16] Ilai Bistritz and A. Leshem, "Game theoretic dynamic channel allocation for frequency-selective interference channels," Arxiv preprint.
- [17] ETSI LTE, "Evolved universal terrestrial radio access (e-utra); base station (bs) radio transmission and reception (3gpp ts 36.104 version 8.6. 0 release 8), july 2009," *ETSI TS*, vol.